CONSERVATION OF MAGNETIC MOMENT OF CHARGED PARTICLES IN STATIC ELECTROMAGNETIC FIELDS

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In the report, corrections to the magnetic moment invariant for a charged particle motion are calculated, and the equation defining magnetic moment variation in time is derived.
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1. INTRODUCTION

The magnetic moment of the charged particle moving in the electric and magnetic fields is

$$\mu = \frac{m\mathrm{v}_\perp^2}{2B}, \tag{1}$$

where $\mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel$ and $\mathbf{v}_\parallel = \dfrac{\mathbf{B}(\mathbf{B}\cdot\mathbf{v})}{B^2}$ are the perpendicular and parallel to the steady magnetic field components of the particle velocity.

The magnetic moment is an approximate invariant of motion, when the motion is adiabatic and the fields vary slowly at the particle gyro-center.

The accuracy of the expression for the magnetic moment is of low-order with respect to the adiabaticity parameter, i.e. the ratio $\lambda = \rho_L / L$ of the particle Larmor radius to the characteristic scale of non-uniformity, and a higher order invariant, which to leading order is the magnetic moment, needs to be consistent with other independent invariants.

Corrections for the invariant for several particular cases are calculated in Refs. [1-6]. However, the equation for the evolution of μ is not derived there.

2. ANALYTICAL TREATMENT

To describe the particle motion in static electric and magnetic fields which are slowly varying in space, the Newton's and Lorentz force equations are analyzed:

$$m\frac{d\mathbf{v}}{dt} = e\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{B})\right], \tag{2}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}. \tag{3}$$

The particle velocity and the equations of motion are projected onto the unitary orthogonal vector triplet aligned to the magnetic field:

$$\mathbf{e}_1 = \frac{(\mathbf{A}\times\mathbf{B})\times\mathbf{B}}{B|\mathbf{A}\times\mathbf{B}|}, \tag{4}$$

$$\mathbf{e}_2 = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|}, \tag{5}$$

$$\mathbf{e}_\parallel = \frac{\mathbf{B}}{B}, \tag{6}$$

where $\mathbf{A}$ is an arbitrary constant in space vector ($\nabla\mathbf{A} = 0$).

The electric field is equal to:

$$\mathbf{E} = -\nabla\varphi. \tag{7}$$

Here φ is the scalar electric potential.

Energy conservation reads:

$$\varepsilon = \frac{m\mathrm{v}^2}{2} + e\varphi = const. \tag{8}$$

The equation for parallel velocity can be written in the following form:

$$\frac{d\mathrm{v}_\parallel}{dt} = \mathbf{v}\cdot(\mathbf{v}\cdot\nabla)\frac{\mathbf{B}}{B} + \frac{e}{m}\mathbf{E}\cdot\frac{\mathbf{B}}{B}. \tag{9}$$

Using the above formula the equation for the variation of the magnetic moment (1) could be derived:

$$\frac{d\mu}{dt} = -\frac{m\mathrm{v}_\parallel}{B^2}\mathbf{v}_\perp\cdot(\mathbf{v}_\perp\cdot\nabla)\mathbf{B} - \frac{m\mathrm{v}_\parallel^2}{B^2}\mathbf{v}_\perp\cdot\left(\frac{\mathbf{B}}{B}\cdot\nabla\right)\mathbf{B} + \\ + \frac{m\mathrm{v}_\perp^2}{2}(\mathbf{v}\cdot\nabla) + e\frac{\mathbf{v}_\perp\cdot\mathbf{E}}{B}. \tag{10}$$

In non-uniform fields the motion is mainly Larmor rotation with slowly varying parallel and perpendicular guiding center velocities. For this reason the right-hand side of the equation (10) contains slowly varying parts and parts oscillating with the gyro-frequency and its harmonics. The particle perpendicular velocity could be represented as:

$$\mathbf{v}_\perp = \mathbf{v}_\perp^{(1)} + \delta\mathbf{v}_\perp^{(0)} + \delta\mathbf{v}_\perp^{(2)}, \tag{11}$$

where $\mathbf{v}_\perp^{(1)}$ is the term responsible for the Larmor rotation, $\delta\mathbf{v}_\perp^{(0)}$-drift in inhomogeneous magnetic field, $\delta\mathbf{v}_\perp^{(2)}$- unharmonicity, the term describing Larmor circle deformation. The upper index values could be explained in the following way: '0' is associated with the non-oscillating motion, '1' describes the fundamental cyclotron harmonic, '2' stands for second cyclotron harmonic terms.

Following (9), the parallel part of the particle velocity is

$$\mathbf{v}_\parallel = \mathbf{v}_\parallel^{(0)} + \delta\mathbf{v}_\parallel^{(1)}, \tag{12}$$

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where $\mathbf{v}_{\parallel}^{(0)}$ is the major part of the parallel velocity, and $\delta\mathbf{v}_{\parallel}^{(1)}$ is the first order correction which is the fundamental harmonic oscillation.

3. ORDERING TERMS

To provide the same order contribution to the magnetic moment corrections, the oscillating terms should be one order higher in the adiabaticity parameter than the slowly varying terms, since $\int \tilde{f}dt \sim \frac{f}{\omega_H}$, $\int \bar{f}dt \sim \frac{fL}{\mathrm{v}_{\parallel}}$, where $\tilde{f}$ is the oscillating function, $\bar{f}$ - the slowly varying function.

The expression (10) could be represented in following form:

$$\frac{d\mu}{dt} = T_1 + T_2 + T_3 + T_4, \tag{13}$$

where

$$T_1 = -\frac{m\mathrm{v}_{\parallel}}{B^2}\mathbf{v}_{\perp}\cdot(\mathbf{v}_{\perp}\cdot\nabla)\mathbf{B}, \tag{14}$$

$$T_2 = -\frac{m\mathrm{v}_{\parallel}^2}{B^2}\mathbf{v}_{\perp}\cdot\left(\frac{\mathbf{B}}{B}\cdot\nabla\right)\mathbf{B}, \tag{15}$$

$$T_3 = \frac{m\mathrm{v}_{\perp}^2}{2}(\mathbf{v}\cdot\nabla), \tag{16}$$

$$T_4 = e\frac{\mathbf{v}_{\perp}\cdot\mathbf{E}}{B}. \tag{17}$$

In each of these terms the contributions of different orders are separated:

$$T_i = T_{0i} - \frac{d}{dt}\delta\mu_i + \delta T_i, \tag{18}$$

where T_{0i} are zero-order slowly varying terms, $\delta\mu_i$ is associated with the contribution to the magnetic moment for oscillating terms, δT_i - slowly varying first-order terms. These terms are calculated using the energy conservation law (8) and the equation for the parallel velocity (9).

The zero-order slowly varying terms are:

$$T_{01} = -\frac{m\mathrm{v}_{\parallel}\mathrm{v}_{\perp}^2}{2B^2}\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right), \tag{19}$$

$$T_{02} = 0, \tag{20}$$

$$T_{03} = -\frac{m\mathrm{v}_{\parallel}\mathrm{v}_{\perp}^2}{2B^2}\frac{\partial B_z}{\partial z}, \tag{21}$$

$$T_{04} = 0. \tag{22}$$

The sum of the zero-order terms nullify because $\nabla\cdot\mathbf{B} = 0$. This is the sign of the magnetic moment conservation.

The slowly-varying first-order terms are:

$$\delta T_1 = 0, \tag{23}$$

$$\delta T_2 = -\frac{m\mathrm{v}_{\perp}^2\mathrm{v}_{\parallel}^2}{2B^3\omega_c}\left[2\frac{\partial B_z}{\partial x}\frac{\partial B_y}{\partial z} - 2\frac{\partial B_z}{\partial y}\frac{\partial B_x}{\partial z} - \right.$$
$$-\frac{\partial B_x}{\partial x}\frac{\partial B_y}{\partial x} - \frac{\partial B_y}{\partial x}\frac{\partial B_y}{\partial y} + \frac{\partial B_x}{\partial y}\frac{\partial B_x}{\partial x} + \tag{24}$$
$$\left. + \frac{\partial B_y}{\partial y}\frac{\partial B_x}{\partial y} - B_z\frac{\partial^2 B_y}{\partial x\partial z} + B_z\frac{\partial^2 B_x}{\partial y\partial z}\right],$$

$$\delta T_3 = \frac{m\mathrm{v}_{\perp}^2\mathrm{v}_{\parallel}^2}{B^3\omega_c}\left(\frac{\partial B_z}{\partial x}\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}\frac{\partial B_x}{\partial z}\right), \tag{25}$$

$$\delta T_4 = 0. \tag{26}$$

The slowly varying first-order terms could be written in coordinate-independent form:

$$\delta T = \sum_{i=1}^{4}\delta T_i =$$
$$= \frac{m\mathrm{v}_{\perp}^2\mathrm{v}_{\parallel}^2}{2B^3\omega_c}\left(-\frac{\partial B_z}{\partial z}(\nabla\times\mathbf{B})\big|_z + B_z\frac{\partial}{\partial z}(\nabla\times\mathbf{B})\big|_z\right) =$$
$$= \frac{m\mathrm{v}_{\perp}^2\mathrm{v}_{\parallel}^2}{2B^3\omega_c}\mathbf{B}\cdot\nabla\left(\frac{\mathbf{B}\cdot\nabla\times\mathbf{B}}{B^2}\right). \tag{27}$$

The contributions to the magnetic moment from the oscillating terms are:

$$\delta\mu_1 = \frac{m\mathrm{v}_{\parallel}\mathrm{v}_x\mathrm{v}_y}{2B^2\omega_c}\left(\frac{\partial B_y}{\partial y} - \frac{\partial B_x}{\partial x}\right) -$$
$$-\frac{m\mathrm{v}_{\parallel}(\mathrm{v}_x^2 - \mathrm{v}_y^2)}{4B^2\omega_c}\left(\frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y}\right), \tag{28}$$

$$\delta\mu_2 = \frac{m\mathrm{v}_{\parallel}^2}{B^2\omega_c}\left(\mathrm{v}_x\frac{\partial B_y}{\partial z} - \mathrm{v}_y\frac{\partial B_x}{\partial z}\right), \tag{29}$$

$$\delta\mu_3 = \frac{m\mathrm{v}_{\perp}^2}{B^2\omega_c}\left(\mathrm{v}_x\frac{\partial B_z}{\partial y} - \mathrm{v}_y\frac{\partial B_z}{\partial x}\right), \tag{30}$$

$$\delta\mu_4 = \frac{e}{B\omega_c}\left(\mathrm{v}_y E_x - \mathrm{v}_x E_y\right). \tag{31}$$

The cumulative expression for the contributions of the oscillating terms to the magnetic moment can be written as:

$$\delta\mu = \sum_{i=1}^{4}\delta\mu_i =$$
$$= -\frac{m}{2\omega_B B^2}\left[\mathrm{v}_x\mathrm{v}_y\mathrm{v}_z\frac{\partial B_x}{\partial x} - \frac{\mathrm{v}_z}{2}\left(\mathrm{v}_x^2 - \mathrm{v}_y^2\right)\frac{\partial B_y}{\partial x} + \right.$$
$$+ \mathrm{v}_y\left(\mathrm{v}_x^2 + \mathrm{v}_y^2\right)\frac{\partial B_z}{\partial x} - \frac{\mathrm{v}_z}{2}\left(\mathrm{v}_x^2 - \mathrm{v}_y^2\right)\frac{\partial B_x}{\partial y} -$$
$$\left. - \mathrm{v}_x\mathrm{v}_y\mathrm{v}_z\frac{\partial B_y}{\partial y} - \mathrm{v}_x\left(\mathrm{v}_x^2 + \mathrm{v}_y^2\right)\frac{\partial B_z}{\partial y}\right] - \tag{32}$$
$$-\frac{m\mathrm{v}_z^2}{\omega_B B^2}\left(\mathrm{v}_y\frac{\partial B_x}{\partial z} - \mathrm{v}_x\frac{\partial B_y}{\partial z}\right) -$$
$$-\frac{e}{\omega_B B}\left(\mathrm{v}_y\frac{\partial\varphi}{\partial x} - \mathrm{v}_x\frac{\partial\varphi}{\partial y}\right).$$

The terms containing the drift velocity may be separated:

$$\delta\mu + \frac{m\mathbf{v}\cdot\mathbf{v}_d}{B} = -\frac{m}{2\omega_B B^2}\Bigg[\mathrm{v}_x\mathrm{v}_y\mathrm{v}_z\frac{\partial B_x}{\partial x} - \frac{\mathrm{v}_z}{2}\left(\mathrm{v}_x^2-\mathrm{v}_y^2\right)\frac{\partial B_y}{\partial x} - \frac{\mathrm{v}_z}{2}\left(\mathrm{v}_x^2-\mathrm{v}_y^2\right)\frac{\partial B_x}{\partial y} - \mathrm{v}_x\mathrm{v}_y\mathrm{v}_z\frac{\partial B_y}{\partial y}\Bigg], \quad (33)$$

where

$$\mathbf{v}_d = -c\frac{\nabla\varphi\times\mathbf{B}}{B^2} + \frac{mc\mathrm{v}_\perp^2(\mathbf{B}\times\nabla B)}{2eB^3} + \frac{mc\mathrm{v}_\parallel^2\left[\mathbf{B}\times(\mathbf{B}\cdot\nabla)\mathbf{B}\right]}{eB^4} \quad (34)$$

is the drift velocity. The residual terms in (33), which stand for the second harmonic oscillations, can be grouped as:

$$\delta\mu + \frac{m\mathbf{v}\cdot\mathbf{v}_d}{B} = -\frac{m\mathrm{v}_z}{4\omega_B B^2}\Bigg\{\mathrm{v}_y\left[\frac{\partial}{\partial x}(\mathrm{v}_x B_x + \mathrm{v}_y B_y) + \frac{\partial}{\partial y}(\mathrm{v}_y B_x - \mathrm{v}_x B_y)\right] + \mathrm{v}_x\left[-\frac{\partial}{\partial y}(\mathrm{v}_x B_x + \mathrm{v}_y B_y) + \frac{\partial}{\partial x}(\mathrm{v}_y B_x - \mathrm{v}_x B_y)\right]\Bigg\}. \quad (35)$$

Finally, the expression for the corrected magnetic moment becomes:

$$\bar{\mu} = \Bigg\{\frac{m\mathrm{v}_\perp^2}{2B} - \frac{m\mathbf{v}\cdot\mathbf{v}_d}{B} + \frac{m\mathrm{v}_\parallel}{4\omega_B B^3}\mathbf{B}\cdot[\mathbf{v}\times\nabla(\mathbf{v}\cdot\mathbf{B}) + (\mathbf{v}\cdot\nabla)(\mathbf{v}\times\mathbf{B}) - \mathrm{v}_\parallel(\mathbf{v}\times\nabla B)]\Bigg\}. \quad (36)$$

Here is $\bar{\mu} = \mu + \delta\mu$ and the first term in (36) is the standard expression for the magnetic moment.

Finally, the equation for the corrected magnetic moment in coordinate-independent form reads:

$$\frac{d\bar{\mu}}{dt} = \frac{m\mathrm{v}_\perp^2\mathrm{v}_\parallel^2}{2B^2\omega_c}\mathbf{B}\cdot\nabla\left(\frac{\mathbf{B}\cdot\nabla\times\mathbf{B}}{B^2}\right). \quad (37)$$

The right-hand side of this equation determines the slow variation of the magnetic moment in time, and is associated with the magnetic field vorticity in current-carrying plasma.

4. CONCLUSIONS

In the report, the adiabatic motion of charged particles in static electromagnetic fields is analyzed. The equation for the corrected magnetic moment is obtained in coordinate-independent form. The derived local corrections to the magnetic moment invariant are oscillating and are associated with the particle drift. They have no influence on conservation of the magnetic moment in average, but they make the higher order magnetic moment invariant consistent with the other invariants such as the generalized momentum.

The right-hand side of the equation determines the slow variation of the magnetic moment in time, and is associated with the magnetic field vorticity in current-carrying plasma.

The corrections to the magnetic moment invariant are consistent with the standard expressions for the first order drift and parallel motion of the guiding center.

REFERENCES

1. M.D. Kruskal. *Prinston Univ. Rep.* PM-S-33, 1958.
2. C.S. Gardner // *Phys. Fluids*. 1966, v. 9, 1997.
3. A.M. Dykhne, V.L. Pokrovsky // *Soviet Physics JETP*. 1961, v. 12, p. 264.
4. A.M. Dykhne, A.V. Chaplik // *Soviet Physics JETP*. 1961, v. 13, p. 465.
5. V.M. Balebanov, N.N. Semashko // *Nuclear Fusion*. 1967, v. 7, p. 207.
6. R.J. Hastie, G.B. Hobbs, J.B. Taylor // *Plasma Physics and Controlled Nuclear Fusion Research*/ Vienna: «IAEA», 1969, v. 1, p. 389-401.

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СОХРАНЕНИЕ МАГНИТНОГО МОМЕНТА ЗАРЯЖЕННОЙ ЧАСТИЦЫ В СТАТИЧЕСКИХ ЭЛЕКТРОМАГНИТНЫХ ПОЛЯХ

В.Е. Моисеенко, М.А. Суркова, O. Ågren

Рассчитаны поправки к магнитному моменту заряженной частицы, а также получено уравнение для скорректированного магнитного момента.

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