`A THEORETICAL STUDY OF SURFACE LOCALIZED MODES IN FREE SPACE

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The azimuthally symmetrical surface localized modes in free space are analyzed. In the geometry aligned to the wave-field localization surface, a combination of WKB eikonal with the exponential- polynomial series is used to find approximate solutions. It is found that in axially symmetrical case the surface of wave-field localization is a hyperboloid. The shape of the reflecting surface for a single-mode resonator is a section of eccentric paraboloid.

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1. MAXWELL'S EQUATIONS IN CASE OF AXIAL SYMMETRY

System of time-harmonic Maxwell's equations in free space in cylindrical coordinates in axially symmetric case ($\frac{\partial}{\partial\varphi}=0$) reduces to equation for TE-mode.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}E_\varphi\right)+\frac{\partial^2}{\partial z^2}E_\varphi+\left(\frac{\omega^2}{c^2}-\frac{1}{r^2}\right)E_\varphi=0. \qquad (1)$$

By substitution $E_\varphi=\frac{\Phi}{\sqrt{r}}$ this equation transforms to two-dimensional Helmholtz-type equation

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{\partial^2\Phi}{\partial z^2}+G\Phi=0, \qquad (2)$$

where $G=\frac{\omega^2}{c^2}-\frac{3}{4}\frac{1}{r^2}$. (3)

2. GEOMETRY ALIGNED TO THE GUIDING SURFACE

We suggest that axially symmetrical localized wave may exist in free space in the vicinity of a guiding surface $z=f(r)$. The last equation determines the trajectory curve at the plane $\varphi=const$. Following [1], we introduce coordinates u across the trajectory curve and υ along it (see Fig.1). Below implicit formulas for them are given.

$$r=r_s+u\frac{f'(r_s)}{\sqrt{1+f'^2(r_s)}},\; z=f(r_s)-u\frac{1}{\sqrt{1+f'^2(r_s)}}, \qquad (4)$$

$$\upsilon=\int_0^{r_s}\sqrt{1+f'^2(r)}\,dr,$$

where $f'=\frac{df}{dr}$. Coordinate u is chosen to be the distance between point (r,z) and curve $z=f(r)$ with the appropriate sign. Then the trajectory curve equation could be written as $u=0$ .Coordinate υ is the length of segment of the curve between the initial point and the point (r_s,z_s) which is the cross of the trajectory curve and a straight line that originates from the current point (r,z) and is perpendicular to the trajectory curve (Fig. 1). In such coordinates equation (2) has the following form.

$$\frac{1}{B}\frac{\partial}{\partial u}B\frac{\partial}{\partial u}\Phi+\frac{1}{B^3}\frac{\partial}{\partial\upsilon}B\frac{\partial}{\partial\upsilon}\Phi+G\Phi=0, \qquad (5)$$

where $B=1+u\kappa$, $\kappa(r_s)=\frac{f''(r_s)}{\left(1+f'^2(r_s)\right)^{3/2}}$ is the curvature of the trajectory curve.

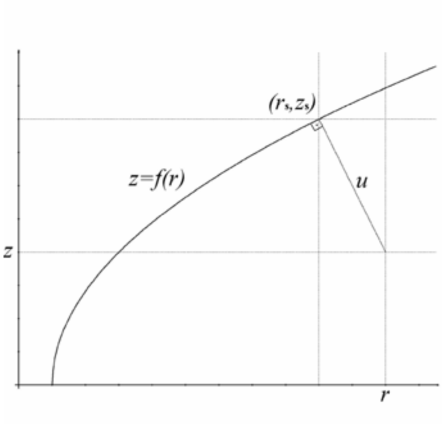

Fig. 1. Trajectory curve and coordinates u and υ

3. APPROXIMATE SOLUTION OF SOURCE EQUATION

As in [1], the approximate solution of equation (5) is suggested as a combination of WKB eikonal along the trajectory and the first radial mode of parabolic cylinder equation across it.

$$\Phi=\frac{\exp\left(-g(\upsilon)u^2+i\psi(\upsilon)\right)}{\sqrt{|\nabla\psi(\upsilon)|}}. \qquad (6)$$

Here g and ψ are functions of υ. Both of them vary in space with the characteristic space scale L_0 and the wavelength is much smaller than this scale ($\gamma^{-4}=GL_0^2>>1$ where γ is series expansion parameter). ψ determines the wave field oscillation along the trajectory curve while g controls the wave channel width across the trajectory.

After substitution of the expression (6) to equation (5), terms of different orders appear. Neglecting of terms which order is $G\gamma^0$ results in the following equation.

$$G-k^2=0. \qquad (7)$$

Here $\mathbf{k}=\nabla\psi$. Neglecting terms which order is $G\gamma^2$ results in the next equation.

$$4g^2u^2-2g-2iu^2\frac{1}{B}\frac{d\psi}{d\upsilon}\frac{dg}{d\upsilon}+B\left(G-k^2\right)=0. \qquad (8)$$

Substituting Taylor expansion of the $B(G-k^2)$ in u, using new variable τ which is determined by the equation $\frac{d\upsilon}{d\tau}=\frac{d\psi}{d\upsilon}$ and equating the coefficients before each power of u one can obtain four equations similar to ray-tracing equations (see, for example, [2] §2.1).

$$\frac{\partial \mathbf{R}}{\partial \tau}=\mathbf{k_0}\,, \tag{9}$$

$$\frac{\partial \mathbf{k_0}}{\partial \tau}=\frac{1}{2}\nabla\left(-2g+G\big|_{u=o}\right), \tag{10}$$

$$\frac{d\psi}{d\tau}=G\big|_{u=o}-2g\,, \tag{11}$$

$$\frac{dg}{d\tau}=-2ig^2-\frac{i}{4}\frac{\partial^2\left(G-k^2\right)}{\partial u^2}\Bigg|_{u=o}. \tag{12}$$

Here $\mathbf{k_0}=\mathbf{k}\big|_{u=0}$.

4. TRAJECTORY CURVES

Combining equations (9) and (10), the following equation for wave trajectory is obtained:

$$\frac{\partial^2 \mathbf{R}}{\partial \tau^2}=\frac{1}{2}\nabla G-\nabla g\,. \tag{13}$$

With account of the ratio of orders of the terms in the right-hand side of the equation the last term can be neglected since $\frac{G}{g}\sim\left(GL_0^{\;2}\right)^{\frac{3}{4}}>>1$.

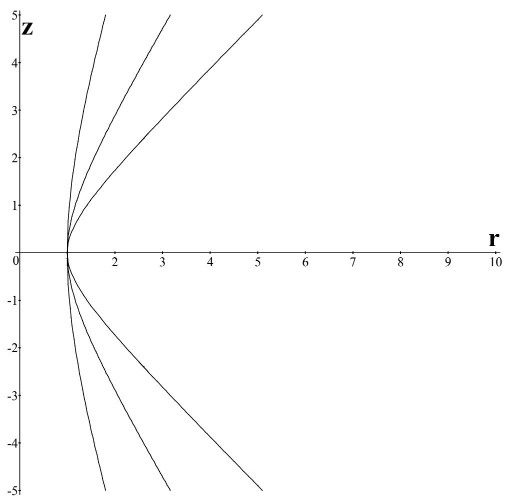

Fig. 2. Trajectory curves for $r_0=1$ and different values of parameter a, from left to right $a=10$, $a=5$, $a=3$

This equation is solved analytically. The solution is the family of hyperbolas (Fig. 2).

$$\begin{cases}\dfrac{r^2}{r_0^{\;2}}-\dfrac{z^2}{\left(a^2r_0^{\;4}/3\right)}=1\\[2ex] z=a\tau\end{cases}. \tag{14}$$

5. STRUCTURE OF THE WAVE FIELDS

With account of equations (3) and (14) the last term of equation (12) reads:

$$\frac{\partial^2\left(\operatorname{Re}G-(\nabla\psi)^2\right)}{\partial u^2}\Bigg|_{u=o}= -\frac{9a^2r_0^{\;6}\left(3\tau^2\left(3+a^2r_0^{\;2}\right)+a^2r_0^{\;6}+12r_0^{\;4}\right)}{2\left(3\tau^2+r_0^{\;4}\right)\left(3\tau^2\left(3+a^2r_0^{\;2}\right)+a^2r_0^{\;6}\right)^2}. \tag{15}$$

For a large values of z (or τ) the curvature of the trajectory curve becomes small and the localization of the wave worsens. Therefore, the small τ are of particular interest.

The equation (12) could be solved using iteration method with an initial assumption $\frac{dg}{d\tau}=0$. Such iterations converge fast for small τ. The first iteration results in the following approximate solutions:

$$\operatorname{Re}g=\operatorname{Re}g\big|_{\tau=0}+\left(\frac{\partial \operatorname{Re}g}{\partial\tau}\bigg|_{\tau=0}\right)\tau=\frac{3}{2r_0^2}\sqrt{1+\frac{12}{a^2r_0^2}}\,, \tag{16}$$

$$\operatorname{Im}g=-\frac{3\tau}{4r_0^4}\frac{\left(2a^4r_0^4+39a^2r_0^2+72\right)}{a^2r_0^2\left(a^2r_0^2+12\right)}. \tag{17}$$

Imaginary part of g determines the bending of the wave front. By substitution expressions (3), (16) and (17) equation (11) could be written as

$$\frac{d\psi}{d\tau}=\left(\nabla\psi\big|_{u=0}\right)^2=a^2\left(\frac{\dfrac{9\tau^2}{a^2r_0^{\;6}}+\dfrac{3\tau^2}{r_0^{\;4}}+1}{\dfrac{3\tau^2}{r_0^{\;4}}+1}\right). \tag{18}$$

With the initial condition $\psi\big|_{\tau=0}=0$ solution of this equation is

$$\psi=a^2\left(\frac{3}{a^2r_0^2}+1\right)\tau-\sqrt{3}arctg\frac{\sqrt{3}\tau}{r_0^2}\,. \tag{19}$$

So, expression for the azimuthal projection of the electric field reads as:

$$E_\varphi=E_{\varphi0}\frac{\exp\left(-\left(\dfrac{3}{2r_0^2}\sqrt{1+\dfrac{12}{a^2r_0^2}}-i\dfrac{3\tau}{4r_0^4}\dfrac{\left(2a^4r_0^4+39a^2r_0^2+72\right)}{a^2r_0^2\left(a^2r_0^2+12\right)}\right)u^2+i\left(a^2\left(\dfrac{3}{a^2r_0^2}+1\right)\tau-\sqrt{3}arctg\dfrac{\sqrt{3}\tau}{r_0^2}\right)\right)}{\sqrt{r}\sqrt[4]{\left(\dfrac{a}{1+u\kappa}\right)^2\left(\dfrac{\dfrac{9\tau^2}{a^2r_0^{\;6}}+\dfrac{3\tau^2}{r_0^{\;4}}+1}{\dfrac{3\tau^2}{r_0^{\;4}}+1}\right)}}. \tag{20}$$

6. REFLECTING SURFACES FOR RESONATOR

Since equation (2) is real, complex conjugate of solution (20) is also a solution linearly independent of the initial solution. Using these two solutions two real value expressions for the electric field of the standing wave can be constructed.

$$E_{1,2}=\frac{\exp\left(-u^2\operatorname{Re}g\right)}{\sqrt{r}\sqrt{\left|\nabla\psi(\upsilon)\right|}}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\left(\psi-u^2\operatorname{Im}g\right). \tag{21}$$

There is a family of surfaces on which the solution nullify. The equation for these surfaces comes from the condition of nullifying sine or cosine functions in (21).

$$a^2\left(\frac{3}{a^2 r_0^{\,2}}+1\right)\tau-u^2\,\mathrm{Im}\,g-\sqrt{3}arctg\frac{\sqrt{3}\tau}{r_0^{\,2}}=\frac{\pi n}{2}. \quad (22)$$

To form a resonator the conductive surfaces(metallic walls of the resonator) should coincide with those ones defined by Eq.(22). For small τ a shape of these surfaces can be found analytically. Let reflecting surface crosses the localization surface in the point $\tau=\hat{\tau}$ ($r=\hat{r}$, $z=\hat{z}$), which we can obtained from equation (22) setting $u=0$

$$\hat{\tau}=\frac{\pi n}{2a^2}. \quad (23)$$

Also

$$\begin{cases}\hat{r}=r_0\sqrt{1+\dfrac{3\pi^2 n^2}{4a^4 r_0^{\,4}}}\\ \hat{z}=a\,\hat{\tau}=\dfrac{\pi n}{2a}\end{cases}. \quad (24)$$

For small τ

$$\Delta\tau=-\pi n\frac{3}{8}\frac{\left(2a^4 r_0^{\,4}+39a^2 r_0^{\,2}+72\right)}{a^6 r_0^{\,6}\left(a^2 r_0^{\,2}+12\right)}u^2, \quad (25)$$

where $\Delta\tau=\tau-\hat{\tau}$. With account of formula $\frac{1}{a}\frac{f'(\hat{r})}{\sqrt{1+f'^2(\hat{r})}}\Delta\upsilon=\Delta\tau$, in coordinates (u,υ) equation for reflecting surface is:

$$\Delta\upsilon=-\pi n\frac{3}{8}\frac{\left(2a^4 r_0^{\,4}+39a^2 r_0^{\,2}+72\right)}{a^8 r_0^{\,9}\left(a^2 r_0^{\,2}+12\right)}\left(1+\frac{3\pi^2 n^2}{4a^4 r_0^{\,4}}\right)^{-\frac{1}{2}}\sqrt{\frac{9}{4}\pi^2 n^2+a^6 r_0^{\,6}\left(1+\frac{3\pi^2 n^2}{4a^4 r_0^{\,4}}\right)}u^2. \quad (26)$$

Transforming to coordinates (r,z) gives the following equation:

$$\left(r-r_0\sqrt{1+\frac{3\pi^2n^2}{4a^4r_0^{\,4}}}\right)\frac{2a^3r_0^{\,3}\sqrt{1+\frac{3\pi^2n^2}{4a^4r_0^{\,4}}}}{3\pi n}-\left(z-\frac{\pi n}{2a}\right)=\frac{1}{4}\frac{\left(2a^4r_0^{\,4}+39a^2r_0^{\,2}+72\right)}{a^8r_0^{\,9}\left(a^2r_0^{\,2}+12\right)\sqrt{1+\frac{3\pi^2n^2}{4a^4r_0^{\,4}}}}\left(\left(r-r_0\sqrt{1+\frac{3\pi^2n^2}{4a^4r_0^{\,4}}}\right)+\left(z-\frac{\pi n}{2a}\right)\frac{2a^3r_0^{\,3}\sqrt{1+\frac{3\pi^2n^2}{4a^4r_0^{\,4}}}}{3\pi n}\right)^2. \quad (27)$$

As follows from the last equation, the shape of the reflecting surface for the single-mode resonator is a section of eccentric paraboloid (Fig. 3) placed perpendicular to the wave channel.

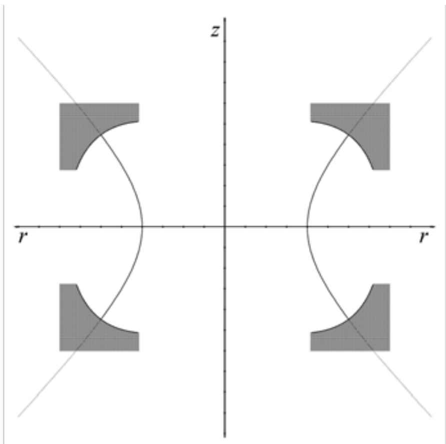

Fig. 3. Two-dimensional configuration of reflecting surfaces. Section $\varphi=\pm const$

CONCLUSIONS

In free space there exist modes which fields are concentrated in the vicinity of a certain (guiding) surface. In axial symmetrical case the guiding surfaces for such modes are hyperboloids. In the perpendicular direction of the guiding surface the fields decay exponentially. The reflecting surfaces forming a resonator on such a mode are calculated.

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ТЕОРЕТИЧЕСКОЕ ИЗУЧЕНИЕ ПОВЕРХНОСТНО ЛОКАЛИЗОВАННЫХ МОД В СВОБОДНОМ ПРОСТРАНСТВЕ

В.Е. Моисеенко, А.П. Ковтун

Проанализированы азимутально-симметричные поверхностно локализованные моды в свободном пространстве. Для нахождения приближенного решения использована комбинация метода ВКБ и экспоненциально-полиномиального разложения в геометрии, привязанной к поверхности локализации волны. Определено, что поверхностью локализации поля волны является гиперболоид вращения. Рассчитана форма отражающих поверхностей для одномодового резонатора. Это сегмент параболоида вращения со смещенной осью.

ТЕОРЕТИЧНЕ ДОСЛІДЖЕННЯ ПОВЕРХНЕВО ЛОКАЛІЗОВАННИХ МОД У ВІЛЬНОМУ ПРОСТОРІ

В.Є. Моісеєнко, А.П. Ковтун

Було проведено аналіз азимутально-симетричних поверхнево локалізованих мод у свободному просторі. Для знаходження наближеної розв’язки використано комбінацію методу ВКБ та експоненційно-поліноміального розкладання в геометрії, що прив’язана до поверхні локалізації хвилі. Визначено, що поверхнею локалізації поля хвилі є гіперболоїд ротації. Обрахована форма поверхні для одномодового резонатора. Це є сегмент параболоїда ротації зі зсуненою віссю.