NON-MARKOVIAN EFFECTS IN TURBULENT DIFFUSION: KINETIC THEORY AND NUMERICAL SIMULATION

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The theory of time-nonlocal random processes formulated in terms of non-Markovian Fokker-Planck equation is used to describe the results of numerical simulations of particle diffusion in the random longitudinal field with prescribed statistical properties.

Introduction

Studies of particle and heat transport in turbulent fields remains one of the key issues of plasma physics during the last decades. A noticeable progress in this field stimulates much interest to the specific aspects of this problem. In particular this concerns the study of time-nonlocal (memory) effects in transport phenomena in turbulent plasmas.

To outline the problem let us consider a model non-Markovian equation for distribution function $f(x,t)$ which describes the diffusion in x space

$$\frac{\partial}{\partial t} f(x,t) = \frac{\partial}{\partial x} \int\limits_0^t h(x,t-t') \frac{\partial}{\partial x} f(x,t') dt'. \qquad (1)$$

In such type of equation, which could be obtained under rather general assumption, the diffusion coefficient $h(x,\tau)$ tends to zero at $\tau > \tau_{\rm cor}$, where $\tau_{\rm cor}$ denotes the correlation time of random field.

In many problems diffusion processes are considered on time scales which much exceeds $\tau_{\rm cor}$, when the variation of $f(x,t)$ within the current time interval $(t-\tau_{\rm cor},t)$ become negligible, and transition to Markovian (time-local) description is substantiated

$$\frac{\partial}{\partial t} f(x,t) = \frac{\partial}{\partial x} D(x) \frac{\partial}{\partial x} f(x,t),$$
$$D(x) = \int\limits_0^\infty h(x,\tau) d\tau. \qquad (2)$$

However, if we are interested in diffusion on small time scales, which are less, or of the order of $\tau_{\rm cor}$ and/or if the variation of $f(x,t)$ on such scales is not small the transition to Markovian description is not valid.

The experiments on fast recovery of the gaussian shape of electron temperature profile after the pellet injection (on time scale one order smaller than electron energy confinement time) indicate the existence of two classes of transport [1]. Fast and slow transport in this example seems very likely corresponds to fast and slow variation of distribution function on time interval of the order of $\tau_{\rm cor}$, the first of which needs non-Markovian description.

Other example, where non-Markovian effects are important, concerns saturation of instabilities due to trajectory diffusion, like it was considered in the Dupree-Weinstock renormalization approach [2, 3]. Usually, the diffusion is supposed to be classical one and is determined by the time asymptotic of the diffusion coefficient. However, more precisely, the unstable wave dissipation depends on the rate of divergence of trajectories starting in the same point of phase space. The initial stage of this process, for which time non-local effects are strong, could not be ignored.

Recently much attention has been also paid to the turbulent diffusion which considerably deviate from classical one. In many cases such deviations are associated with the non-Markoviety [4–8].

Statistical derivation of the non-Markovian diffusion equation was done in [6]. Here we present the basic points of the statistical approach to the description of time-nonlocal effects, and apply it to calculation of particle diffusion in velocity space in the case of the Langmuir-type turbulence. The analytical estimates will be compared with the results obtained in numerical simulation.

Non-Markovian Fokker-Planck equation

Evolution equation for particle distribution function $f(x,v,t)$ in random (turbulent) field can be written in the form of the Fokker-Planck-type equation with time-nonlocal collision term

$$\left\{\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right\} f(x,v,t) = \\ = \frac{\partial}{\partial v}\int\limits_0^t dt' h(v,t-t')\frac{\partial}{\partial v} f(x,v,t'), \qquad (3)$$

where x and v are coordinate and velocity, t – time, $h(v,\tau)$ is the time-nonlocal diffusion coefficient in the velocity space which depends on turbulent electric field spectrum. Eq. (3) is obtained from the Klimontovich-Vlasov equation by its averaging over the ensemble of random electric field. Splitting of the distribution function on averaged and fluctuating parts is also used. The collision term which appear in course of such derivation is usually approximated by the time-local value, however we retain the time dependence as non-local.

Introducing the transition probability which is governed by the same equation and the initial condition

$$W(x-x', v, v'; \tau = 0) = \delta(x-x')\delta(v-v'), \qquad (4)$$

we can express time-nonlocal diffusion coefficient in terms of transition probability and correlation function of the potential of turbulent electric field $\langle\Phi^2\rangle_{k\tau}$

$$h(v,\tau) = \left(\frac{e}{m}\right)^2 \int dv' \int \frac{dk}{2\pi} k^2 \langle\Phi^2\rangle_{k\tau} W_k(v,v';\tau). \qquad (5)$$

Eqs. (3)–(5) are the close set of equations which determine the distribution function in turbulent electric field with regard to the trajectory diffusion (Dupree-Weinstock renormalization) as well as time non-local effects.

Simplified well-known versions of these equations could be obtained under three following assumption:
1. Time-local description, mentioned in introduction, is obtained if the variation of $f(x,v,t')$ is slow, so in right-hand side of Eq. (3) it could be replaced by $f(r,v,t)$ and further integration is fulfilled

$$\int\limits_0^t h(v,\tau) f(t-\tau) d\tau \to f(t) \int\limits_0^\infty h(v,\tau) d\tau = D(v) f(t).$$

2. Reduction to quasilinear theory (where trajectories are unperturbed) is obtained when right-hand side of Eq. (3) for the transition probability is neglected. Putting the such transition probability (which does not depend in this case on h) in Eq. (5) we obtain the diffusion coefficient in the explicit form. Otherwise, when the transition probability is dependent on h, Eq. (5) is the integral equation for h.
3. Even under previous two assumptions the solution of Eq. (3) could not be obtained in analytical form. Chandrasekhar [9] found the solution of such type equation rewritten in the Fokker-Planck form

$$\left\{\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{\partial}{\partial v}\left(\beta v + \eta\frac{\partial}{\partial v}\right)\right\} f(x,v,t) = 0$$

under the assumption that coefficients β, γ do not depend on v. This means $D(v)$ is approximated by two terms of Taylor expansion

$$\beta = -\frac{\partial}{\partial v} D(v)|_{v=v_0}, \qquad \eta = D(v)|_{v=v_0},$$

where v_0 corresponds to the phase velocity of the central harmonic in the spectrum of turbulent field.

It is interesting to note, that in the description of time-nonlocal effects we made only third assumption. The right-hand side of Eq. (3) is the convolution of two functions, so making Fourier transformation we obtain the product $h_\omega f_\omega$. This means the Chandrasekhar solution for the transition probability can be generalized to the case of time-nonlocal diffusion coefficient. Namely, instead of β and η we should substitute β_ω and η_ω into ω-representation of the Chaudrasekhar's solution. Here,

$$\beta_\omega = -\frac{\partial}{\partial v} h_\omega(v)|_{v=v_0}, \qquad \eta_\omega = h_\omega(v)|_{v=v_0}.$$

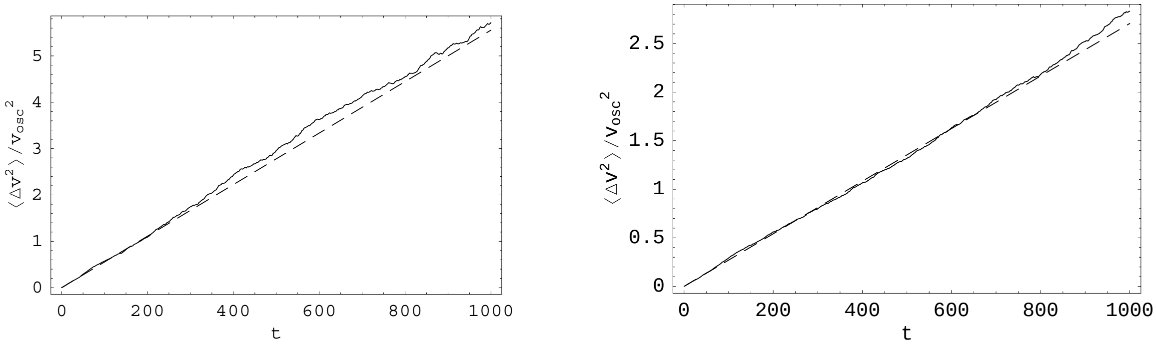
Such formal solution, in fact, gives the integral equations for the transition probability β_ω and η_ω. They could be solved by retaining the finite number of harmonics.

Particle diffusion in the set of Langmuir waves with random phases: non-Markovian effects

The approach presented above was used to describe particle diffusion in velocity space. To check its validity numerical simulations were made. Temporal dependence of mean-square velocity deviations calculated from the non-Markovian Fokker-Planck equation is compared with the results of numerical simulations.

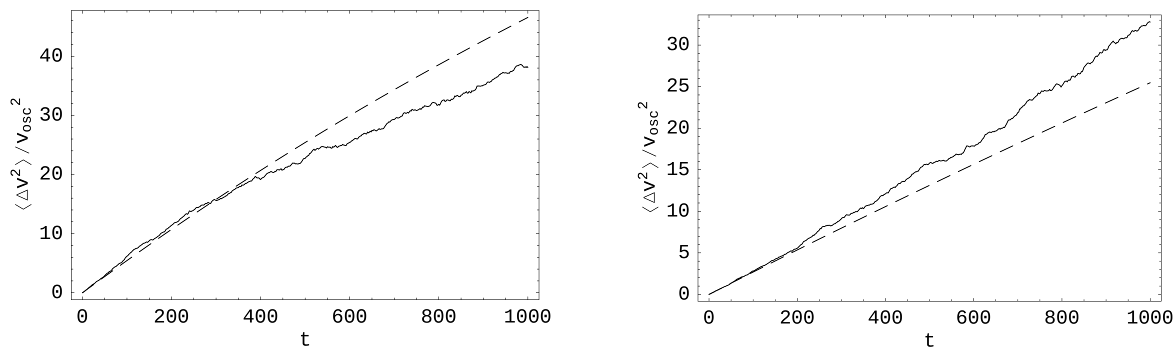
Turbulent field Φ was taken as superposition of N Langmuir type waves with eigenfrequency $\omega_0 = \text{const}$ and random phases α

$$\Phi(x,t) = \sum_{n=1}^{N} \Phi_n \cos(\omega_0 t - k_n x + \alpha_n).$$



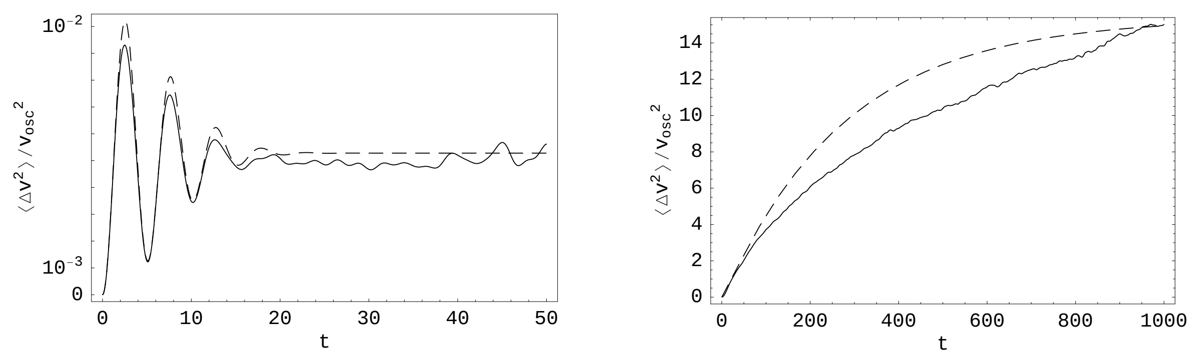
a) $k_0 v_0/\omega_0 = 1$ b) $k_0 v_0/\omega_0 = 1.2$

Figure 1: $\delta k = 0.4$, $\sigma = 10^{-4}$

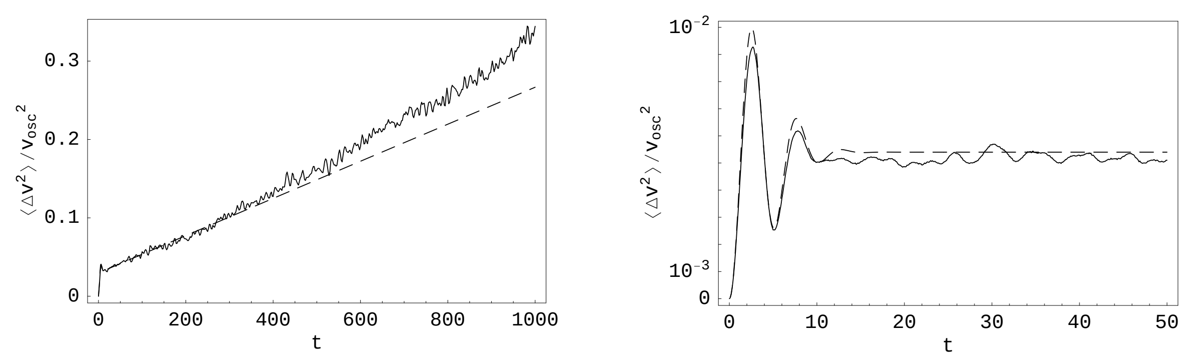


a) $k_0 v_0/\omega_0 = 1$ b) $k_0 v_0/\omega_0 = 1.2$

Figure 2: $\delta k = 0.4$, $\sigma = 10^{-3}$



a) $k_0 v_0/\omega_0 = 0.8$ b) $k_0 v_0/\omega_0 = 1$



c) $k_0 v_0/\omega_0 = 1.1$ d) $k_0 v_0/\omega_0 = 1.2$

Figure 3: $\delta k = 0.04$, $\sigma = 10^{-4}$

The amplitude of potential Φ_n is nonzero in the interval $(k_{\min}, k_{\max})$ and depends on current wave number k_n as the following

$$\begin{aligned} \Phi_n^2 &= \Phi_0^2 \frac{dk}{\sqrt{\pi}\Delta k} \exp\left[-\left(\frac{k_n - k_0}{\Delta k}\right)^2\right] \\ dk &= (k_{\max} - k_{\min})/N, \\ k_n &= k_{\min} + ndk, \\ n &= 1, \ldots, N. \end{aligned}$$

Here k_0 corresponds to the central harmonic, Δk is the width of the spectrum, and Φ_0 characterized the total field intensity.

The analytical (dashed line) and numerical (solid line) temporal dependencies of velocity dispersion are shown in Figs. 1–3 for various field intensities, spectrum widths and particle initial velocities v_0. The velocity dispersion is normalized on bounce velocity $v_{\rm osc} = (e\Phi_0/m)^{1/2}$, time is measured in units of period $2\pi/\omega_0$. The quantities $\sigma \equiv (k_0/\omega_0)^2(e\Phi_0/m)$ and $\delta k \equiv \Delta k/k_0$ are used as the dimensionless parameters of field intensity and spectrum width.

In the case of weak field ($\sigma = 10^{-4}, 10^{-3}$) and rather wide spectrum ($\delta k \sim 0.4$) the meansquare velocity displacement of resonant (nearly-resonant) particles is proportional to time interval (Figs. 1, 2), as it is predicted by quasilinear theory.

However, further increase of the field intensity at the same spectral width leads to the saturation of the meansquare velocity displacements; the value of the field intensity required for saturation of $\langle \Delta v^2 \rangle_\tau$ is dependent on the spectral width. For narrow spectrum $\delta k \sim 0.04$ the smaller field $\sigma = 10^{-4}$ provides the saturation of the meansquare velocity displacement of resonant particles (Figs. 3b).

The evolution of meansquare velocity displacement of non-resonant particles is accompanied by oscillations (Figs. 3a,d).

Conclusions

The conventional quasilinear theory in the approximation disregarding the time and velocity dependence of the diffusion coefficient in the velocity space can be used only in the case of small intensity and large width of turbulent field spectrum. The increase of the intensity as well as the decrease of the spectral width lead to considerable deviation of the results of simulations (such as saturation and frequent oscillation of the meansquare velocity displacement) from the predictions of the quasilinear theory.

The proposed non-Markovian (time non-local) description of particle diffusion in turbulent field gives a good agreement with results of numerical simulation for a moderate field intensities $\sigma \leq 10^{-3}$. It reproduce not only linear dependence of velocity meansquare deviation on time (Fig. 1,2), but also its saturation (Fig. 3b) as well as oscillations (Figs. 3a,d).

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