SOME APPROXIMATE SOLUTIONS OF THE BOLTZMANN EQUATION

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The process of interaction between two inhomogeneous flows in a gas of hard or rough spheres is approximately described by the bimodal distributions of a special form. Different remainders tend to zero with accordant asymptotic behaviour of parameters of the distributions.

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The evolution of a rarefied gas can be described by the nonlinear Boltzmann equation (BE). Generally, it has a form [1]:

$$D(f) = Q(f,f) \tag{1}$$

$$D(f) = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x}, \tag{2}$$

where f is the distribution function we want to find, and the concrete form of the collision integral $Q(f,f)$ depends on the model of interaction between molecules. Except the special case of Maxwell molecules and some its generalizations [2-5], the only exact solution of the BE, which is known up to now, is Maxwellian, global (i.e. independent on the time t and the position $x \in R^3$) or local. For physically significant models of hard and rough [6] spheres the collision integrals are of the form accordingly:

$$Q_H(f,f) = \frac{d^2}{2}\int\limits_{R^3} dv_1 \int\limits_{\Sigma} d\alpha \left|(v_1 - v, \alpha)\right| \cdot \left[f(t,v_1',x)f(t,v',x) - f(t,v_1,x)f(t,v,x)\right] \tag{3}$$

$$Q_R(f,f) = d^2 \int\limits_{R^3} dv_1 \int\limits_{R^3} d\omega_1 \int\limits_{\Sigma} d\alpha\, h\left((v_1 - v,\alpha)\right) \cdot \left[f(t,v_1^*,\omega_1^*,x)f(t,v^*,\omega^*,x) - f(t,v_1,\omega_1,x)f(t,v,\omega,x)\right] \tag{4}$$

$$v' = v - \alpha(v - v_1,\alpha),\ v_1' = v_1 + \alpha(v - v_1,\alpha), \tag{5}$$

$$v^* = v - \frac{1}{b+1}\left\{b(v - v_1) + \alpha(v - v_1,\alpha) + \frac{1}{2}bd[\alpha \cdot (\omega + \omega_1)]\right\},$$

$$v_1^* = v_1 + v - v^*,$$

$$\omega^* = \omega + \frac{2}{d(b+1)}\left\{[\alpha \cdot (v - v_1)] + \frac{1}{2}d[\alpha(\alpha,\omega + \omega_1) - \omega - \omega_1]\right\},$$

$$\omega_1^* = \omega_1 + \omega^* = \omega. \tag{6}$$

Here d is the diameter of a molecule; Σ is the unit sphere in R^3, the function h has a form:

$$h(u) = 1/2(u + |u|), \tag{7}$$

$v, v_1, v', v_1', v^*, v_1^*$ are the linear velocities of molecules; $\omega, \omega_1, \omega^*, \omega_1^*$ are their angular velocities; the parameter $b \in [0; 2/3]$ is connected with a moment of inertia I of the molecule by the relation:

$$I = bd^2/4. \tag{8}$$

Let us call the equation (1), (2), (4) as the Bryan-Pidduck equation (BPE), because Bryan was the first, who had taken into consideration (in 1894) the model of rough spheres [6], and some later Pidduck carried out its investigations.

The form of the global Maxvellians for these models are well-known, and the general form of the local ones is very complicated. So, let us consider only the special cases of the local Maxwellians, which correspond to equilibrium, stationary, inhomogeneous states of a gas –

the so-called spiral-type Maxwellians (in short – spirals):

$$M(x,v) = \rho e^{\beta \bar{\omega}^2 r^2} \left(\frac{\beta}{\pi} \right)^{3/2} e^{-\beta (v - \bar{v} - [\bar{\omega} \times x])^2} \quad (9)$$

for hard spheres, and

$$M(x,v,\omega) = \rho e^{\beta [\bar{\omega} \times (x - x_0)]^2} I^{3/2} \left(\frac{\beta}{\pi} \right)^3 \cdot \cdot e^{-\beta [(v - \bar{v})^2 + I(\omega - \bar{\omega})^2]} \quad (10)$$

for rough spheres (the parameter $\beta = 1/2T$ is the inverse temperature of a gas).

These distributions describes the rotation of a gas in whole as a rigid body with the bulk angular velocity $\bar{\omega}$ about the axis which pass through the point x_0, and its translational movement along this axis (the linear velocity is connected with the parameter $\bar{v} \in R^3$), besides

$$x_0 = \frac{1}{\bar{\omega}^2} [\bar{\omega} \times \bar{v}] \bar{\omega}, \quad (11)$$

and

$$r^2 = \frac{1}{\bar{\omega}^2} [\bar{\omega} \times (x - x_0)]^2 \quad (12)$$

is the square of a distance from the axis of rotation. Note, that the densities now depend on x too, and have the constant value ρ only on this axis.

The necessity of the construction of explicit approximate solutions of the BE and BPE was compelled by the absence of adequate description of the process of interaction between two equilibrium flows in a gas (i.e. absence of the exact solution which corresponds to the non-equilibrium state). So, in [7-10] such the solutions were built as the bimodal distributions with modes, which have the form of global Maxwellians.

The analogous results, but for more complicated case of spirals, was obtained in [11,12] for BE, and in the present paper-for BPE. The statement of the problem is as follows.

Let us seek an explicit approximate solution of BE (1), (2), (3) or BPE (1), (2), (4) in the form of the bimodal distribution:

$$f = \sum_{i=1}^{2} \varphi_i(t,x) M_i, \quad (13)$$

where $M_i \; (i = 1,2)$ are from (9) or (10) respectively with arbitrary values of parameters $\rho_i, \beta_i, \bar{v}_1, \bar{\omega}_1, x_{0i} \; (i = 1,2)$ and $\varphi_i \; (i = 1,2)$ are some non-negative, smooth coefficient functions. It is required to find $\varphi_i \; (i = 1,2)$ such, that the remainders Δ_{1H}, Δ_R or Δ_{1R} tend to zero with corresponding asymptotic behaviour of all parameters of the distribution. Here:

$$\Delta_H = \sup_{t,x} H(D,Q), \quad (14)$$

$$\Delta_{1H} = \int_{R^1} dt \int_{R^3} dx\, H(D,Q), \quad (15)$$

$$\Delta_R = \sup_{t,x} R(D,Q), \quad (16)$$

$$\Delta_{1R} = \int_{R^1} dt \int_{R^3} dx\, R(D,Q), \quad (17)$$

where

$$H(D,Q) = \int_{R^3} dv \left| D(f) - Q_H(f,f) \right|, \quad (18)$$

$$R(D,Q) = \int_{R^3} dv \int_{R^3} d\omega \left| D(f) - Q_R(f,f) \right|. \quad (19)$$

Let us now enumerate several main situations which, under some restrictions on the analytic properties of the functions $\varphi_i \; (i = 1,2)$, ensure the infinitesimality of the values $\Delta_H, \Delta_{1H}, \Delta_R$ or Δ_{1R} (some rigorous proofs one can find in [11,12]). First of all, put:

$$\bar{\omega}_i = \bar{\omega}_{0i} s_i \beta_i^{-m_i} \; (i = 1,2), \quad (20)$$

and

$$\varphi_i(t,x) = \psi_i(t,x) \exp\left(-\beta_i \bar{\omega}_i^2 r_i^2 \right) \; (i = 1,2), \quad (21)$$

where $\bar{\omega}_{0i}$ are arbitrary fixed vectors, $s_i > 0$ are constants, and the functions ψ_i, $i = 1,2$ are independent on β_i, $i = 1,2$. Then the remainder Δ_H becomes vanishingly small with $\beta_i \to +\infty$, $i = 1,2$ under anyone of the suppositions:

$$m_i = 1, \; i = 1,2, \quad (22)$$

$$\bar{v}_1 = \bar{v}_2 = 0, \; \psi_i = \psi_i(x), \; i = 1,2, \quad (23)$$

or

$$\bar{v}_1 = \bar{v}_2 \neq 0, \quad (24)$$

$$[\bar{\omega}_{0i} \times \bar{v}_i] = 0 \; (i = 1,2), \quad (25)$$

$$\psi_i = C_i(x - \bar{v}_i t) \; (i = 1,2), \quad (26)$$

where $C_i \geq 0$ are any smooth finite or fast decreasing functions on R^3. In particular, instead of (26) can be

$$\psi_i = C_i [x \times \bar{v}_i] \; (i = 1,2). \quad (27)$$

$$m_i = 1/2, \; i = 1,2, \quad (28)$$

and one of the assumptions (23) or (24) with (26) are true, and, in addition to that, the restriction (25) is fulfilled, or

$$s_i \to 0, \; i = 1,2. \quad (29)$$

$$m_i = 1/4, \; i = 1,2, \quad (30)$$

and (25), (29) are valid, together with (23), or (24) and Eq. (26).

The analogous statement for the remainder Δ_{1H} is also true, but under some other suppositions, because, for example, the functions (26) or (27) do not ensure the existence of the value (15) at all. Namely, the value Δ_{1H} tends to zero with $\beta_i \to +\infty$, i = 1,2, if (20), (21) are fulfilled, and at least one of the requirements is satisfied:

1) (22) is fulfilled, and (23) is true, or the functions $\psi_i \ (i = 1,2)$ are of the form of finite "plateaus" [9, 10,12], such that the measure of projections of their supports on the hyperplane $t = 0$ in R^4 tends to zero, and

$$\bar{v}_i^{-k} \operatorname{mes}(\sup \psi_i)_k \to 0 \ (i = 1,2; k = 1,2,3). \qquad (31)$$

(here the index k denotes that the accordant projection corresponds to the hyperplane $x^k = 0$), and the supposition (24) is satisfied, or

$$\sup \psi_1 \cap \sup \psi_2 = \varnothing. \qquad (32)$$

2) One of the conditions of the point 1) is fulfilled, and (28) takes place together with (25) or (29), or (30), (25), (29) are valid simultaneously.

In respect to BPE (1), (2), (4), the same results on the behaviour of the values Δ_R, Δ_{1R} can be obtained, as the mentioned ones for Δ_H and Δ_{1H}, but some technical difficulties arise because of more complicated mathematical structure of (4), (6) and (19) than (3), (5) and (18) respectively. Nevertheless, the methods, investigated in [10-12], give the possibility to extend the results connected with the BE on the case of the BPE. So, the process of interaction between the spiral flows in a gas of rough spheres can be described, in principle, in the same way, as for a gas of hard spheres.

The detailed interpretation of the obtained results one can find in [9-12]. Thus, let us now confine ourselves to the reminder of more important facts only.

All the distributions described above correspond to the cooling down gas ($\beta_i \to +\infty$, $i = 1,2$) with decelerating rotation ($\omega_i \to 0$, $i = 1,2$) of both spirals, but in different degree, in accordance with (20), (22), (28), (29), (30). The densities of the spirals depend on x, but not on their temperatures, i.e. on β_i, $i = 1,2$, because of (21). The condition (25) together with (11) mean, that $x_{0i} = 0$, i.e. the axis of rotation of the *i*-th spiral pass through the origin of coordinates. Next, (31) describes objects (flows) in a gas of an incomplete dimensionality (their classification was carried out in [9]), and (32) corresponds to the stratification of the objects in the space R^4. Finally, (23) means, that the spirals have not move translationally, but only rotate about their axis, and (24) shows, that the spirals asymptotically (when $\beta_i \to +\infty$, $i = 1,2$) fly in parallel to each other.

So, the bimodal distributions (13), constructed in [11,12] and this paper, give the approximate description of the transitional regime (in other words, of the process of interaction) between two equilibrium, stationary, inhomogeneous flows in a gas of hard or rough spheres under some restrictions on the hydrodynamic parameters of these flows and the coefficient functions of the distributions.

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