FORMALISM FOR CHAOTIC BEHAVIOR OF THE BUNCHED BEAM

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Using Cesaro mid of Fourier series the quasi-linear Vlasov's equation is transformed to the integral Fredholm equation. New results on the oscillatory behavior of solution are obtained. An extension to perturbing equation is also included.

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1. INTRODUCTION

The self-consistent Vlasov equation is one of the most frequently used equations for the time dependent description of many-particle systems. Especially in nuclear physics this equation has been employed to describe multifragmentation phenomena and collective oscillations. It is apparently not widely known that there exists an analytical solvable model from which the effects of self-consistency can be studied. Here such a model is presented which shows that self-consistency can lead to self-focused and acceleration of bunched beam.

The kinetic equation for the beam distribution function f has the form

$$\frac{\partial f}{\partial t} + v\partial_r f + F_l \partial_v f = 0\,, \tag{1}$$

where $r = (x_1, x_2, x_3)$ is a three-dimensional vector, x_i, $i = 1,2,3$ are Cartesian coordinates; $v = (v_1, v_2, v_3)$ is their velocity. In this solution the Lorentz force $F_l = q(E + \frac{1}{c}[v\times H])$ acting on a nonrelativistic driving beam. Here q is the particle charge and E is the electric field: $E = E_1 + E_2$ where E_1 is given field and E_2 is generated by a charged bunch, H is the magnetic field and $H = H_1 + H_2$ too. The fields should satisfy the Maxwell system

$$< \dot v > = \frac{\int\limits_{(\infty)} \dot v f(r,v,\dot v,t)\, d\dot v}{f(r,v,t)} = \frac{e}{m}(E +$$

$$+ \frac{1}{c}[vH]),\ rot\, H - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c} q \int\limits_{(\infty)} v f\, dv,$$

$$div\, E = 4\pi\, q \int\limits_{(\infty)} f\, dv \tag{2}$$

$$(rot\, E + \frac{1}{c}\frac{\partial H}{\partial t} = 0,\ div\, H = 0),$$

where $\rho\,(t,r) = q\int f(t,r,v)\, dv$, $j(t,r) = q\int v f(t,r,v)\, dv$ are a charge and a current of the beam, c is the speed of light.

If we formally let $c = \infty$, $H = 0$ and replace qE by E and ρ/q by ρ, we get the Vlasov-Poisson system:

$$\partial_t f + v\partial_x f + E(t,r)\partial_v f = 0 \tag{3}$$

$$\Delta U(t,r) = -4\pi\rho\,(t,r) \tag{4}$$

$$\rho = \int f(t,r,v)\, dv\,.$$

This system was considered by A.A. Vlasov in his treatise on many-particle theory and plasma physics [1]. To determine the focusing and accelerating fields we use the following auxiliary postulate.

The postulate of the existence electric and magnetic fields realizing any motion of the bunch beam: it is shown [2] that for any field of the velocity of charged particle exist electric & magnetic fields that yields same velocity field satisfying Maxwell's equations. This postulate makes it feasible to construct the optimal fields using the optimal control theory [3].

2. APPROXIMATE SOLUTION OF VLASOV'S EQUATION

Letting $f(t,r,v) = f_0(r,v)\,c^{-i\omega t}$ into (1) yields

$$L f_0 \equiv v\partial_r f_0 + F_l \partial_v f_0 = i\omega f_0\,. \tag{5}$$

Suppose the solution Eq. (5) can be represented in the form

$$f_0 = \sum_{k=-\infty}^{\infty} c_k e^{ikx}\,, \tag{6}$$

where the vector $k = (k_1, k_2, \ldots, k_6)$, k_i is an integer, $i = 1,2,\ldots,6$; the vector $x = r + v$, i.e. it's sum of the vector of position and vector of velocity of the particle orbit, $kx = \sum_1^6 k_i x_i$,

$x_i = r_i$, $i = 1,2,3$; $x_i = v_i$, $i = 4,5,6$.

Let us assume that the the vector r falls into a domain Δ_1, the vector $r\in\Delta_2$, and

$\Delta=\Delta_1\times\Delta_2=\Delta_{11}\times\Delta_{12}\times\Delta_{13}\times\Delta_{21}\times\Delta_{22}\times\Delta_{23}$,

where Δ_{ij} is some line segment, a sign x is the right multiplication sign. By c_k we denote Fourier's coefficients

$$c_k\equiv c_k(f_0)=\frac{1}{(2\pi)^6}\int_\Delta f_0(x)e^{-ikx}dx.$$

Thus the formula (6) is a expansion of the function f_0 in the Fourier series.

Summing Eq. (6) by the method of Cesaro for any $N\in[1,2,\dots,\infty)$ we get

$$\sigma_N f_0(x)=\frac{1}{\pi^6}\int_\Delta \Phi_N(x-y)f_0(y)\,dy, \quad (7)$$

here $\Phi_N(u)$ is the Cesaro's kernel

$$\Phi_N(u)=\prod_{j=1}^{N}F_N(u_j),$$

$$F_N(u_j)=\frac{1}{2(N+1)}\left(\frac{\sin\frac{N+1}{2}u}{\sin\frac{u}{2}}\right)^2$$

It is easy to see

$$\int_\Delta \Phi_N(u)\,dy=1$$

and

$$\lim_{N\to\infty}\left\|\sigma_N f_0-f_0\right\|_{c(\Delta)}\to 0,$$

where $\|\cdot\|$ is a norm in the space of continuous functions on $\Delta : c(\Delta)$.

Let us write down the function f_0 as follows $f_0=\sigma_n f_0+g$, where $\int_\Delta \sigma_N f_0\cdot g\,dx=0$ and for $N\to\infty$ is vanishing.

Now differentiating formula (6) by the Eq. (5), we obtain the following equation

$$\lambda\sigma_N f_0-g_0=\frac{1}{\pi^6}\int_\Delta\tilde{\Phi}_N(x-y)\,f_0(y)\,dy, \quad (8)$$

where $g_0=-\lambda g$, $\tilde{\Phi}=L\Phi$ $x,y\in\Delta$

This reasoning yields Fredholm equation for the function $\sigma_N f_0$ if g_0 is a given function then:

$$\lambda\sigma_N f_0-g_0=\frac{1}{\pi^6}\int_\Delta\tilde{\Phi}(x-y)\sigma_N f_0(y)\,dy \quad (9)$$

Define a matrix $k=[k_{qr}]_1^N$, as follows

$$k_{qr}=\frac{1}{\pi^6}\int_\Delta\tilde{\Phi}(x-y)\,e^{iqx}e^{-iry}dx\,dy$$

It is easy to see that the matrix K is the Toeplitz matrix which generates a vector-function $X=\{v,F_l\}$ [4].

The Eq. (9) is transformed to the linear algebraic equation as follows:

$$\lambda^{-1}c_q=\sum_{r=1}^{N}k_{qr}c_r,\ q=1,\dots,N, \quad (10)$$

on set $\Psi=\left\{e^{ikx}\right\}_1^N$, here $k_i\in[1,\dots,N]$.

The Eq. (9) is an integral equation with degenerated kernel [5].

Corollary 1. *We can always find a sufficiently large N such that there exists $\varepsilon>0$, such that the following relations are true:*

$$\left|f(t,r,v)-\sigma_N f_0 e^{-i\omega t}\right|<\varepsilon,$$

$$\lim_{N\to\infty}\left|f(t,r,v)-\sigma_N f_0 e^{i\omega t}\right|\to 0,$$

where f is continuous at every point (r,v) of the domain Δ. The function $\sigma_N f_0$ is a solution of Eq. (9), and the ω is the eigenvalue of the matrix K, thus it is frequency of a wave motion of the bunched beams.

The number ω, generally, maybe any complex number: $\omega=\alpha+i\beta$.

It can be shown in the usual way that if $\mathrm{Im}\,\omega>0$ then $|f(t,r,v)|\to 0$ for $t\to\infty$, if $\mathrm{Im}\,\omega<0$ then the solution f goes out from the domain Δ. Finally, may be the case such that $\omega=0$. These results are discussed in more details in the next section. Under this condition we have a stationary solution of Eq. (1).

Definition. *The solution $f^0=0$ of Eq. (1) is said to be an asymptotically stable if for any $t_0\ge 0$ and arbitrary $\varepsilon\ge 0$ it is possible to find such $\delta>0$ that implies $\rho\left(f_{00},f^0\right)\le\delta\to\rho\,(f(t,r,v),0)\le\varepsilon$ and $\rho\,(f(t,r,v),0)\to 0$ as t tends ∞. Here* $\rho\,(f,0)=\max_{x\in\Delta}\|f\|$, $\|f\|^2=\int_\Delta|f|^2dx$,

$f_{00}=f(t_0,r_0,v_0)$.

3. CHAOTIC BEHAVIOR OF THE BUNCHED BEAM

The motion of particles of bunched beam is evolving in the space

$\Omega_r\times\Omega_v, \Omega_r=\{r:|r|\le\infty\}$, $\Omega_v=\{v:|v|\le\infty\}$.

It is well known that the variables r,v are governed by the following equation

$$\dot r=v,$$

$$\dot v=\frac{e}{m}\left(E+\frac{1}{c}[vH]\right). \quad (11)$$

Here $\Omega_r\times\Omega_v\subset R^6$.

Thus

$$\sum_1^3 \frac{\partial v_s}{\partial x_s} = \sum_1^3 \frac{\partial [vH]k}{\partial v_k} \equiv 0$$

for this reason (well known Liouville theorem) the measure $\partial \mu = dr \times dv$ is the invariant measure for a group T_t (11), i.e. $T_t \mu_0 = \mu_t$ for all $t \in [-\infty, \infty]$, that is easy to see. Consider an invariant measure on $\Omega_r \cdot \Omega_v$, simplify to solve linear partial differential (4) by eigenvalue method, because we now have the eigenvalue problem with electro-magnetic dependent coefficients and the zero eigenvalue. We claim that the eigenvalues will be points of the continuous spectrum and eigenvector of (5) will be chaotic in the phase space in the present case. It is interesting to know if it is the case and how should one solve this kind of eigenvalue problem when the system (11) is chaotic.

Let us consider the following operator L that is selfadjoint extensions of the operator L_0 in Hilbert space $L^2(\Omega)$. In accordance with the Stone theorem, the operator $L = L^*$ generates a group of transformation $U_t = e^{itL}$, such that

$$iL = \lim_{t \to 0} \frac{U_t \varphi - \varphi}{t}.$$

Let $e_k(\lambda)$ be an eigenfunction of the group U_t then

$$U_t e_k(\lambda) = e^{i\lambda t} e_k(\lambda),$$

$k = 1, \ldots, \dim L_\lambda$, where L_λ is a multiple of the point $\lambda \in \sigma(L)$ and σ is the spectrum of the L.

The element $e_k(\lambda)$ belongs to the space $H_{-1}(\Omega_v) = H_1(\Omega_v)^*$ ([8] p. 387).

It is a direct consequence of the existence of the invariant measure in dynamical system (11).

It is well known that $e_k(\lambda) \in H_{-1}(\Omega_v)$ and $e_k(\lambda) \notin C(\Omega_v)$ if it is the point of the continuous spectrum.

In this case the first integral will be absent for dynamical system (5) and it has become the transitive system. In particular this reasoning yields the first integral destruction. A. Einstein, [7] has given conditions under which the first integral disappears.

Corollary 2. *The electro-magnetic field in (11) can generate the ergodic or chaotic motion. Suppose that ergodic is equivalent to the chaos. This reasoning yields an approach of the problem of deterministic chaos.*

We return back to the Eq. (1) and assume that there is the stationary solution $f_0(g_0, r, v)$ for which:

1° There exists a function $V(g_0, r, v)$ such that $V = 0$ on the solution $f_0(g_0, r, v)$, where $f_0 = s_0$ at $r = r_0$, $v = v_0$,$^\circ$

2° The function V is positive defined and founded on an arbitrary solution $f(s, t, r, v)$ of Eq. (1), here s is an arbitrary function such that

$s = f(t_0, r_0, v_0)$, $\| f(t_0, r_0, v_0) - f_0 \|_c \le \delta$

3° The derivative $\dot{V}$ of which in view of Eq. (1) is negative.

We are going to show that in this case the solution f_0 of Eq. (1) is orbital asymptotically stable.

In fact, for the function $V(t, r, v)$ mentioned above we have an estimate

$$V(t_2, r, v) \le V(t_1, r, v), t$$

if $t_2 < t_1$ and $\lim_{t \to \infty} V = 0$.

Thus the function V is decreasing and $\dot{V}$ is representing its total time derivative, taken under the assumption that r, v are function of t, satisfying differential Eq. (5).

Note that a perturb have initial value, i.e. a perturbation motion appears due to perturb of the function s only. Now introduce into consideration a function

$$V(t, r) = \int_\Delta \eta(r) f(t, r, v)\, dt$$

and will show the one fulfils the conditions $1^\circ - 3^\circ$. A function η is an arbitrary symmetric function such that

$$\int_\Delta \eta(r) f_0(r, v)\, dv = 0.$$

Its derivative has the form

$$\dot{V} = \frac{\partial}{\partial t} \int_\Delta \eta(r) f_0 dv + div_r \int_\Delta \eta(r) f_0 dv +$$

$$+ div_v \int_\Delta \eta(r) f_0 dv = 0.$$

Indeed, in the case under consideration we get

$$\frac{\partial}{\partial t} \int \eta f\, dv = 0,$$

$$div_r \int \eta f_0 dv + div_v \int \eta f_0 dv =$$

$$\int \left[v \frac{\partial \eta}{\partial r} f_0 + \eta \left(v \frac{\partial f_0}{\partial r} + \frac{\partial f_0}{\partial v} F_l \right) \right] dv, \qquad (12)$$

while

$$\int \frac{\partial \eta}{\partial r} v f_0\, dv = 0, \; v \frac{\partial f_0}{\partial r} + \frac{\partial f_0}{\partial v} F_l = 0.$$

The first integral equal to zero under the following condition

$$f_0 \Big|_{v \to \infty} \to 0,$$

i.e. the function f is a quickly decreasing with the increasing velocity v.

Corollary 3. *The function* $\dot{V}$ *be no positive if* $R_e \omega < 0$. *It is easy to verify (see above) that*

$$\int \frac{\partial \eta}{\partial r} vf\, dv = 0.$$

By using this reasoning Eq. (12) yields

$$\dot{V} = \int_{\Delta^*} \eta\ L f(s,t,r,v)\, dv = \int_{\Delta^*} \eta \cdot i\omega f\, dv$$

or $R_e V \geq 0$ and $R_e \dot{V} \leq 0$.

Thus the particles beams under the above condition be orbital asymptotically stable for solution f_0.

Speaking about the condition of the asymptotically stable, we mean that the postulate in respect to the field (E,H) holds.

Thus this consideration proves that in domain $\Delta^* \subset \Delta$ there is (E,H) such that the solution f_0 Eq. (5) is the orbital asymptotically stable. Note that if the velocity $v(v_1,v_2,v_3)$ is such that the following condition $v_1^2 + v_2^2 + v_3^2 = const$ holds, then we have case focusing and acceleration of bunched beam around f_0. It is easy to see that we can choose any unperturbed motion such that one is a motion of bunched beam along arbitrary axis of rotation. This can do always, because always, there exist electric and (or) magnetic fields satisfying the Maxwell equation for a given arbitrary motion, i.e. any (or) magnetic fields which satisfy the Maxwell Eq. (2).

It follows that we can choose optimal fields.

4. CONSTRUCTION OF AN OPTIMAL ELECTRIC FIELD

We can assume without loss of generality that the matrix K is given in the following form

$$\sum k_{ii} = -1, k_{ii} = -\alpha_i,,$$

where α_i is given number, the vector r is one-dimension vector $r \equiv x$, the velocity $v = \dot{x}$ and $-1 \leq v \leq 1$, i.e. it is normalized on c (the speed of light). Then $\Delta = \Delta_1 \times \Delta_2, \Delta_1 = \{x : |x| \leq \pi\}$,

$$\Delta_2 = \{v : |v| \leq 1\}, \left\{\varphi_n(x) = \frac{1}{\sqrt{2\eta}} e^{inx}\right\}_{n=+\infty}^{+\infty},$$

$$\left\{P_n(v) = \sqrt{\frac{2n+1}{2}}\,\hat{P}_n(v)\right\}_{n=0,1,\dots},$$

$\hat{P}_n$ are the polynomials of Legendre.

Next we show how to choose the electrostatic field E for the Vlasov-Poisson system

$$\partial_t f + v\partial_x f + E(t,x)\partial_v f = 0 \qquad (13)$$

$$\Delta U(t,x) = -4\pi\rho\,(t,x),$$

where $E(t,x) = -\partial_x U(t,x)$,

$\rho\,(t,x) = \int_{\Delta} f(t,x,v)\, dv$, and E is such that the beam of particle focused and accelerated along axis x. For this purpose the distribution function $f(t,x,v)$ of the particles in phase space Δ will be sought in the form $f = f_0(x,v)\, e^{-i\omega t}$. The substitution of $f_0 e^{i\omega t}$ for f yields $v\partial_x f_0 + E\partial_v f_0 = i\omega f_0$.

Next, we construct the function $\sigma_N f_0$. Thus we arrive at the following matrix $K = [k_{qr}]_1^N$,

$$k_{qr} = \int_{\Delta_1} e^{i(q-r)x} E(x)dx \sum_{-N}^{N} \int_{\Delta_2} P_n \frac{\partial P_m}{\partial v} dv,$$ but

$$\sum_{-N}^{N} \int_{\Delta_2} \frac{\partial P_m}{\partial v} dv = \int_{-1}^{1} \sum_{-N}^{N} \partial\,(P_n P_m) = \sum_{-N}^{N} P_n P_m \Big|_{-1}^{1} = C_0$$

here $P_n(1) = 1$, $P_n(-1) = (-1)^n$, $|n| = 1,..,N$.

Let the matrix K be given then the problem arises of finding the field E under which the formulas $[k_{qr} = C_0 \int_{-\pi}^{\pi} e^{i(q-r)x} E(x)dx]_1^N$ are fulfilled. Note that in this situation the $N \times N$ number k_{qr} are given and $N \times N$ function $e^{i(q-r)x}$ are given too, it is necessary to find the function $E(t,x)$.

Let exist some number $L > 0$ that $|E(t,x)| \leq L$. Thus we obtain the well known L - problem of moments [3].

Now from Eq. (4) we can find ρ and U.

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