ON THE AVERAGING PROCEDURE OVER THE CANTOR SET

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The procedure of averaging a smooth function over the normalized density of the Cantor set (A. Le Mehaute, R.R. Nigmatullin, L. Nivanen. *Fleches du temps et geometric fractale*. Paris: "Hermes", 1998, Chapter 5) has been shown not to reduce exactly the convolution to the classical fractional integral of Riemann-Liouville type. Although the asymptotic behavior of the self-similar convolution kernel is very close to the product of a power and a log-periodic function, this is not obviously enough to claim the direct relationship between the fractals and the fractional calculus.

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1. INTRODUCTION

The result [1,2] of convolution of a smooth function with the normalized density of the Cantor set in the limit $N \to \infty$ has caused a great interest [3,4] (and references therein). The point is that a clear relationship between fractal geometry and fractional calculus has been long sought by the scientific community. In the paper [1] such a relation has been asserted to be found. The criticism expressed in [3] has stated the approach of [1] under serious doubts and has required the reconsideration of the previous result in [2]. Nevertheless, the detailed analysis of the modification has shown [5] that the procedure of averaging a smooth function on fractal sets does not allow one to obtain the kernel corresponding to the fractional integral.

The goal of this paper is to present the supplementary results verifying the main conclusion of [5].

2. THE CONVOLUTION OVERTHE CANTOR SET

Let a value $J(t)$ be related with a function $f(t)$ by the convolution operation

$$J(t) = K(t) * f(t) = \int_0^t K(t-\tau) f(\tau) d\tau , \qquad (1)$$

where $K(t)$ is the memory function determined on the segment $[0,T]$ so that $\int_0^T K(t)dt = 1$. The binary Cantor sets with fractal dimension $\nu = \ln 2/\ln(1/\xi)$ is built iteratively on the interval $[0,T]$ by deleting, at the first step, the middle part of length $2(1-\xi)T$, where $\xi = 2^{-\nu}$, $0 < \xi \le 1/2$. Each following step repeats the previous one on the all remaining intervals. The height of each Cantor bar on any stage of the construction provides the conservation of normalization. The result $K(t)$ of the procedure on the N-th stage has been given by the recurrence relation [2] with the Laplace image of $K_2^{(N)}(t)$ taking the following form

$$\overline{K}_2^{(N)}(p) = \int_0^\infty \exp(-pt) K_2^{(N)}(t) dt = \frac{1-\exp(-pT\xi^N)}{pT\xi^N} Q_2^{(N)}[pT(1-\xi)] ,$$

where

$$Q_2^{(N)}(z) = \prod_{n=0}^{N-1} \frac{1+\exp(-z\xi^n)}{2} \qquad (2)$$

and $z = pT(1-\xi)$. The product (2) converges due to the numerator tending exponentially to 2 for $n \gg 1$. In the limit $N \to \infty$ the Laplace image of $J_2^{(N)}(t) = K_2^{(N)}(t) * f(t)$ is of the form

$$\overline{J_2}(p) = Q_2[pT(1-\xi)]\overline{f}(p) ,$$

where $Q_2(z)$ is the limit of the product (2). It satisfies the functional equation $Q_2(z) = Q_2(z\xi)\{1+\exp(-z)\}/2$ that reduces to the scaling relation $Q_2(z) \approx Q_2(z\xi)/2$ for $|z| \gg 1$.

In the Cantor sets, having M bars ($M\xi < 1$) in each stage of its construction, the corresponding recurrence relation [2] leads to

$$\overline{K}_M^{(N)}(p) = M^{-N} \prod_{n=0}^{N-1} \frac{1-\exp[-z\xi^n M/(M-1)]}{1-\exp[-z\xi^n/(M-1)]} ,$$

$z = pT(1-\xi)$.

In the framework proposed in [2] the Cantor sets with an exponential damping or random localization of random bars were also considered. From the particular cases it follows that the memory function is written as

$$\overline{K}^{\Sigma}(p) = \lim_{N\to\infty} \overline{K}^{(N)}(z) = G(z) = \prod_{n=0}^{\infty} \hat{g}(z\xi^n), \qquad (3)$$

$z = pT(1-\xi)$,

where $\hat{g}(x)$ is an entire function (without zero), and $\ln \hat{g}(x) = \sum_{k=1}^{\infty} c_k x^k / k!$, where c_k are some constants. Assume that the product (3) converges. The expansion of $\ln \hat{g}(x)$ can be obtained by means of analytic computer calculations. So, in the case of binary Cantor set (2) we have

$$\ln\frac{1+e^{-x}}{2} = -1/2\cdot x + 1/8\cdot x^2 - 1/192\cdot x^4 + 1/2880\cdot x^6 -$$
$$-17/645120\cdot x^8 + 31/14515200\cdot x^{10} -$$
$$-\,691/3832012800\cdot x^{12} + 5461/348713164800\cdot x^{14} -$$
$$-929569/669529276416000\cdot x^{16} + \cdots .$$

The product (3) will be also an entire function, which has no zero in the whole of finite complex plane, and can be represented in the form of $G(z) = \exp[A(z)]$, where $A(z)$ is an entire function [6]. The function $G(z)$ has the essential singular point at infinity and becomes vanishing small for $\operatorname{Re} z \to \infty$. This indicates its non-analytic asymptotic behavior.

3. EXPONENTIAL FORM OF THE MEMORY FUNCTION

By taking logarithm, the product $\overline{K}^{\Sigma}(p)$ can be represented in the form of the convergent series $A(z,\xi) = \sum_{n=0}^{\infty} \ln \hat{g}(z\xi^n) = \sum_{n=0}^{\infty} h(n)$. Here z plays the role of a parameter. By means of the Poisson summation formula

$$\sum_{n=0}^{\infty} h(n) = \tfrac{1}{2}h(0) + \int_0^{\infty} h(x)dx + 2\sum_{m=1}^{\infty}\int_0^{\infty} h(x)\cos(2\pi mx)dx,$$

where $h(x) = \hat{g}[\exp(x\ln\xi)z]$, and the change of variables $y = \exp(x\ln\xi)$, we have

$$A(z,\xi) = \tfrac{1}{2}\hat{g}(z) + \frac{1}{\ln\xi}\sum_{m=-\infty}^{\infty}\int_1^{\infty} y^{-1+2\pi im/\ln\xi}\ln\hat{g}(yz)dy. \quad (4)$$

Since $\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\exp(ims) = \delta(s)$ is the Dirac δ-function, and $\ln\hat{g}(0) = 0$, the expression

$$\frac{1}{\ln\xi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} y^{-1+2\pi im/\ln\xi}\ln\hat{g}(yz)dy$$

also equals 0, too. Then the remaining terms of (4) become

$$A(z,\xi) = \tfrac{1}{2}\hat{g}(z) - \frac{1}{\ln\xi}\sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty}\frac{c_k z^k}{k![k+2\pi im/\ln\xi]}. \quad (5)$$

Using the identity

$$\sum_{m=-\infty}^{\infty}\frac{1}{[k+2\pi im/\ln\xi]} = \tfrac{1}{2}\ln\xi\,\coth[\tfrac{1}{2}\ln\xi\,k],$$

and interchanging the summations in the double series by virtue of its convergence, the expression (5) reduces to the following form

$$A(z,\xi) = \sum_{k=1}^{\infty}\frac{c_k z^k}{k!(1-\xi^k)}. \quad (6)$$

Thus, we obtain $\overline{K}^{\Sigma}(p) = \exp[A(z,\xi)]$. By the direct substitution of the solution into the functional equation $G(\xi z) = G(z)/\hat{g}(z)$ it is easy to check the correctness of the result. It should be observed that the expansion of $\ln\hat{g}(x)$ defines completely, in some sense, the exponential form of $\overline{K}^{\Sigma}(p)$. If $\ln\hat{g}(z)$ is an entire function of degree $\alpha = \lim_{n\to\infty}\sqrt[n]{|c_k|}$, $A(z,\xi)$ will be also an entire function with the same degree.

Since $\lim_{x\to\infty}\hat{g}(x) = \overline{g}(x) = const < 1$ (in particular, for binary Cantor set $\lim_{x\to\infty}[1+\exp(-x)]/2 = 1/2$, and for Cantor set with M bars the limit is $1/M$), the functional equation $G(\xi z) = G(z)/\hat{g}(z)$ reduces to the scaling relation $G(\xi z) \approx G(z)/\overline{g}$ which has the unique non-trivial solution $Bz^{\mu}L(z)$, where B is a constant, $\mu = \ln(1/\overline{g})/\ln(1/\xi)$, $L(\xi z) = L(z)$ is a log-periodic function. A more precise asymptotic representation of $\overline{K}^{\Sigma}(p)$ can be reached with the help of Euler-Maclaurin formula [2]. Although in some cases the asymptotic behaviour (after averaging of the log-periodic term) can be well approximated by the power function with non-integer exponent for $|z| > 1$, the analytic background of $\overline{K}^{\Sigma}(p)$ cannot be fully ignored.

4. CONCLUDING REMARKS

We have shown that the exponential form of $\overline{K}^{\Sigma}(p)$ is a direct consequence of properties of the memory function. In the Laplace-image space the convolution kernel for the averaging procedure over the Cantor set is an analytic (entire transcendental) function, and the kernel of fractional calculus is non-analytic. None of non-analytic functions is impossible to represent as a product of analytic functions on the whole of complex plane. Therefore, the approach of [2] does not give the correct procedure for passing from fractal geometry (Cantor set) to the fractional integral of Riemann-Liouville type. Thus, the treatment (both mathematical and physical) of [2] to the notion of fractional integral in terms of fractals requires the revision.

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