ANALYTICAL TREATMENT OF THE CHAOTIC BEHAVIOUR OF THE DETERMINISTIC PSEUDOLINEAR MAP: DECAY OF CORRELATIONS AND STABILITY OF PERIODICAL SYSTEMS

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It was received the rate of chaotization for pseudolinear mapping. It was shown that the rate of chaotization is proportional to the dimension of the phase space and maximal Lyapunov exponent. It was shown also that the problem of the rate of chaotization is not correct and must be regularized. It was investigated also the two-dimensional dynamical system stability in the case of two and three step periodical standard maps. The stability conditions were obtained. The analytical expressions of the bounders of stability regions were written. It had been shown that the summary region of stability is expanded, when compared to the case of the one-step map, but the number of stable points decreases.

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INTRODUCTION

The problem of chaotization of deterministic mechanical system arose after the kinetic theory was built. This theory aims at a mechanical explanation of thermodynamical processes [1,2]. It was found some decades ago that a dynamical chaos arises also in dynamical systems with a small number of degrees of freedom [3-5]. Soon it turned out that the dynamical chaos is a rule rather than an exception [6].

Since the equations considered are nonlinear and nonintegrable, the analytical results were obtained only in a few cases. The majority investigations of the dynamical chaos rely upon numerical simulation. However, computer calculations are badly suited for the treatment of the most interesting final stage of the chaotization. Besides this when analysing experimental data, evaluating of perspectives and searching of possible ways of optimization of various devices, the numerical calculations considerably lose relative to the analytical ones in clearness.

Therefore many important questions leaved unanswered:

1. What is the law of the decay of correlation? Is it an exponential law $\exp(-\alpha t)$ (only in this case we speak about chaotization) or is not it a more slow law $\exp(-\beta t^{\gamma})$ $(\gamma < 1)$? Maybe this law is ever power $t^{-\delta}$, $(\delta > 0)$?

2. What is the rate of chaotization α? Is α proportional to the maximal Lyapunov exponent or is it proportional to the KS - entropy? Is α proportional to the dimension of unstable subspace or is it proportional to the dimension of all phase space?

3. Does the rate of chaotization α depend on the initial indeterminacy or on the uncertainty of the measuring device? Does α tend to infinity when the indeterminacy of the initial state or the uncertainty of the measuring device tends to zero? If it does so what is the law of this tendency?

4. Does the rate of chaotization depend on the shape of the initial and the final region in the phase space? If it depends what complementary condition must be imposed on the dynamical system in order to the rate of chaotization was independent from the shape of the initial and the final regions?

The aim of this review is to answer above questions in the case of a model of the dynamical system, which can be solved exactly.

DECAY OF CORRELATION'S AND MIXING

We consider the dynamical system, the state of which at the time t is described by a d-dimensional vector $x(t)$

$$x(t) = \left(x_1^{(t)}, x_2^{(t)}, ..., x_d^{(t)}\right). \tag{1}$$

If there is no chaos the state vector $x(t)$ changes with time deterministically. The chaos means that the point in phase space $x(t)$ moves in a very complicated manner. Therefore, deterministic description is incorrect. If the chaos takes place the state of the system must be characterized not by the single state vector $x(t)$, but by the distribution function (the probability density in the phase space) $f(x,t)$. One of the manifestations of the chaos consists in lose of the memory about the initial state. Moreover, if the phase volume is conserved, every distribution function tends as $t \to \infty$ to the sole distribution function $\mu(x)$ (equilibrium distribution).

We see that with the chaos the notions of the necessity and the randomness in some sense interchange their places. Namely, without the chaos the behaviour of the system is random in the sense that it is determined

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by the outer relative to the system initial state $x(0)$). On the contrary, with the chaos every quadratically integrable initial distribution $f(x,0)$ tends (in the metric L^2) as $t \to \infty$ to the same limiting distribution $\mu(x)$.

The measure of the memory about the initial state is the correlation

$$C(t) \equiv \langle f(x,t)g(x,0)\rangle - \langle f(x,t)\rangle\langle g(x,0)\rangle . \qquad (2)$$

Here $f(x)$ and $g(x)$ are any two functions and the angle brackets mean the average over the phase space

$$\langle \varphi(\vec{x},t)\rangle = \frac{1}{V(\Gamma)}\int_\Gamma \varphi(\vec{x},t)d\Gamma , \qquad (3)$$

$$d\Gamma = \mu(x)dx_1 dx_2 ... dx_d , \qquad (4)$$

$$V(\Gamma) = \int_\Gamma \mu(\vec{x})dx_1 dx_2 ... dx_d . \qquad (5)$$

Thus one of the manifestations of the randomness consists in the decay of all correlations:

$$\lim_{t\to\infty} C(t) = 0 . \qquad (6)$$

With randomness there occurs a mixing of the phase space. For the simplicity we limit ourselves to the case when the equilibrium distribution is homogeneous:

$$\mu(x) = const . \qquad (7)$$

In this case the mixing means that after elapse of sufficiently long time t the probability $P(\Gamma_f,t)$ that the state vector x gets in an arbitrary region Γ_f of the phase space is proportional to its volume $V(\Gamma_f)$

$$\lim_{t\to\infty} P(\Gamma_f,t) = \frac{V(\Gamma_f)}{V(\Gamma)} . \qquad (8)$$

The region Γ_f plays a role of a measuring device, which determines the degree of the mixing. An error of the measuring device is characterized by the quantity $V(\Gamma_f)$. This error is the less the smaller is $V(\Gamma_f)$.

The mixing may be defined also as the decay of correlations. Indeed the probability $P(\Gamma_f,t)$, which is contained in the formula (8) equals:

$$P(\Gamma_f,t) = \int f(x,t)\chi(x,\Gamma_f)dx \qquad (9)$$

Here $\chi(x,\Gamma_f)$ is the characteristic function of the region Γ_f:

$$\chi(\vec{x},\Gamma_f) = \begin{cases} 1, & if \quad x \in \Gamma_f \\ 0 & in \quad the \quad opposite \quad case \end{cases} \qquad (10)$$

We note that the integral over the whole phase space of the characteristic function equals to the volume $V(\Gamma_f)$. Besides this the normalizing condition is fulfilled:

$$\int_\Gamma f(\vec{x},t)dx = 1 . \qquad (11)$$

Taking into account for this remarks the condition of mixing (8) may be represented as decay of correlations

$$\lim_{t\to\infty}\{\langle f(x,t)\cdot\chi(x,\Gamma_f)\rangle - \langle f(x,t)\rangle\langle\chi(x,\Gamma_f)\rangle\} = 0 . \qquad (12)$$

If correlations damp by an exponential law

$$C(t) \sim \exp(-\alpha\cdot t) \qquad (t \to \infty) \qquad (13)$$

the mixing is called a chaotization.

SETS OF ZERO MEASURE

We note that the singular (i.e. δ - type) distribution

$$f(x) = \delta(x - x_0) \qquad (14)$$

which corresponds to the exactly determined initial state tends to nothing. In this case correlations do not decay and there is no mixing. But such distribution does not correspond to any physical situation. In practice the initial distribution $f(x,0)$ differs from zero in some initial region Γ_i (initial indeterminacy).

"...in the problem of Cauchy the solution must be unique. It must be fully determined by initial conditions and consequently quite predictable. How an indeterminacy can arise? It turns out that the posing of the Cauchy problem is not legitimate while chaotic movements are investigated. This problem never corresponds to the conditions of an experiment (natural or numerical) because the initial conditions in principle cannot be absolutely exact. Therefore there is a reason to formulate the problem in the statistical language" [7].

In the ergodic theory in order to avoid patalogical situations in statistical considerations one neglects sets of measure zero. It means in particular that isolated points in the phase space are not considered. In other words, one treats almost all sets in the phase space. In formulations of ergodic theory theorems there are words "almost everywhere".

"Mathematicians who do not like the speculations in which the expressions "almost all" and "neglecting sets of zero measure" occur, may be objected that this is the only way to mathematically interpret what "as a rule" takes place in the nature" [8].

"The most important principle of the theory of measure is a neglect of sets of zero measure. In accordance with this principle, spaces with measure and their endomorphisms must be studied only disregarding the sets of zero measure, or, as people say, "module 0" (mod 0) ... often the addition "mod 0" is implied but is not included in the wording obviously" [9].

On the same ground the final region Γ_f cannot be a single point: it must have some positive measure. In other words the uncertainty of the measure device cannot equal to zero.

We may say that the phase space consists not of points but rather of infinitesimal cells. This causes the drastical change of properties of the elements of the phase space: a point has precisely defined place and has no shape. On the contrary, a cell has definite shape. With the lapse of time the shape of the cell changes. According to the Liouville's theorem, the area of the cell remains constant under the natural motion of the system. It becomes more and more complicated. Eventually it resembles a spider or a sponge. Thus the cell has an "age". In other words, in the theory there is the time arrow. It means that the paradoxes of

Loschmidt [10, p. 152] and Zermelo [10, p. 155; 17] disappear.

THE PSEUDOLINEAR MAPPING

As the time elapses the phase point $x(t)$ intersects many times the Poincare section

$$x^{(t+1)} = Tx^{(t)} \qquad (15)$$

where $x^{(t)}$ is the phase point which corresponds to the t-th intersection of the Poincare section and T is some nonlinear operator. Instead of the investigation of $x(t)$ we shall consider the sequence of the phase points $x^{(1)}, x^{(2)}, \dots$. In other words we will pass on to discrete time t.

The dynamical chaos is intimately connected with two effects:
1) the exponential growth of infinitesimal perturbations,
2) the non linear limitation of the perturbation growth when perturbation increases to finite value.
The first effect is described by linearized map

$$x_i^{(t+1)} = \alpha_{ij} x_j(t). \qquad (16)$$

Here α_{ij} are given constants and it is implied the repeated indices are summarized from 1 to d.

In order to consider the second effect we limit ourselves for simplicity to the stochastic acceleration and the stochastic diffusion. In the action-angle variables (I, θ) they are described by the standard map [4]:

$$I^{(t+1)} = I^{(t)} + K \sin\left(2\pi \theta^{(t)}\right) \qquad (17)$$

$$\theta^{(t+1)} = \theta^{(t)} + I^{(t+1)}. \qquad (18)$$

As θ stands under the sign of sinus we can take the fractional part of (18)

$$\theta^{(t+1)} = \left\{\theta^{(t)} + I^{(t+1)}\right\}. \qquad (19)$$

Thus our phase space is the cylinder

$$(I, \theta), \quad \theta \equiv \theta + 1. \qquad (20)$$

Now we replace the equation (17) by the more tractable one

$$I^{(t+1)} = \left\{I^{(t)} + K \sin\left(2\pi \theta^{(t)}\right)\right\} \qquad (21)$$

Doing so, we identify the points $I = 0$ and $I = 1$. In other words, we transform the phase space from the cylinder (20) into the torus

$$(I, \theta), \quad \theta \equiv \theta + 1, \quad I \equiv I + 1. \qquad (22)$$

The solution of the modified system (19), (21) is less stochastic than the solution of the previous system. Indeed, if, for example, the value of I jumps erratically from zero to one, there and back, its fractional part will be always equal to 0.

"It may appear that it is a very special class of dynamical systems. But it is not so: many important dynamical systems turn out to be nonergodic. Their phase space splits into invariant tori" [11, p. 66].

Then we replace $\sin(2\pi\theta)$ by $2\pi\theta$ for simplicity. Thus we obtain the pseudolinear map:

$$I^{(t+1)} = \left\{I^{(t)} + 2\pi K \theta^{(t)}\right\}, \qquad (23)$$

$$\theta^{(t+1)} = \left\{I^{(t)} + (1 + 2\pi K)\theta^{(t)}\right\}, \qquad (24)$$

The pseudo linear map looks as linear. But it is essentially non linear, as it does not admit the transformation

$$I \to cI, \quad \theta \to c\theta. \qquad (25)$$

The pseudo linear map (23), (24) is a very crude approximation but it preserves the simplicity of the linearized map and, at the same time, takes into account the non linear limitation of the perturbation growth. In this approximation one can obtain a series of exact results. In the d-dimensional case the pseudo linear map has the form

$$x_i^{(t+1)} = \left\{\alpha_{ij} x_j^{(t)}\right\}, \quad (i, j = 1, 2, \dots, d). \qquad (26)$$

We further assume the coefficients α_{ij} in these equations to be integers. If α_{ij} in (26) are integers the limiting distribution $f(x, \infty)$ will be homogeneous:

$$f(x, \infty) \equiv \mu(x) = 1. \qquad (27)$$

When

$$\alpha_{ij} = \begin{pmatrix} 1 & k \\ 1 & 1+k \end{pmatrix} \qquad (28)$$

the pseudo linear mapping was investigated in the work [7]. If $k = 1$, the transformation (28) is named the "Arnold's cat". The pseudo linear map arises also in the one-dimensional theory of a crystal [6].

THE GENERAL SOLUTION

Let the initial distribution be

$$\vec{f(x)} = \begin{cases} \prod_{j=1}^{d} \dfrac{1}{\varepsilon_j^i} & when \quad x_j \in \Gamma_i \\ 0 & in \quad opposite \quad case \end{cases} \qquad (29)$$

We assume that the initial region Γ_i is determined by the relations

$$x_j^i - \frac{1}{2}\varepsilon_j^i \le x_j \le x_j^i + \frac{1}{2}\varepsilon_j^i, \quad (j = 1, 2, \dots, d) \qquad (30)$$

and the final region Γ_f is

$$x_j^f - \frac{1}{2}\varepsilon_j^f \le x_j \le x_j^f + \frac{1}{2}\varepsilon_j^f, \quad (j = 1, 2, \dots, d) \qquad (31)$$

Then the correlation equals [12]

$$C(t) = \frac{1}{\pi^d} \sum_{m_1, m_2, \dots, m_d = -\infty}^{\infty} F(m_1, \dots, m_d) \prod_{j=1}^{d} K_j, \qquad (32)$$

$$K_j = \frac{\exp\left(2\pi i \sum_{k=1}^{d} m_k (\alpha^{-t})_{kj} x_j^f\right)}{\sum_{k=1}^{d} m_k (\alpha^{-t})_{kj}} \sin\left(\pi \sum_{k=1}^{d} m_k (\alpha^{-t})_{kj} x_j^f\right).$$

Here $F(m_1, \dots, m_d)$ are the Fourier coefficients of the initial distribution $f(x)$:

$$F(m_1, \dots, m_d) = \frac{1}{\pi^d} \prod_{j=1}^{d} \frac{\sin\left(\pi \cdot m_j \varepsilon_j^i\right)}{m_j \varepsilon_j^i} \exp\left(-2\pi i m_j x_j^i\right), \qquad (33)$$

and the stroke at the sign of summation means that the term $m^1 = m^2 = m^d = 0$ is omitted, $\left(a^{-t}\right)_{ij}$ are the elements of the matrix T^{-t}.

With arbitrary Γ_i and Γ_f the evaluation of the d-fold sum in (32) is embarrassing. Therefore we investigate two particular cases:

1) A crude initial state and a fine measuring device

$$f\left(\vec{x}\right) = \begin{cases} \dfrac{1}{\varepsilon^i}, & if \quad x_1^i - \dfrac{1}{2}\varepsilon^i \le x_1 \le x_1^i + \dfrac{1}{2}\varepsilon^i, \\ & \quad 0 \le x_j \le 1, (j = 2,...,d) \\ 0, & in \; opposite \; case \end{cases} \quad , \quad (34)$$

Γ_f is determined by the relations (31). In this case the expansion of the initial distribution function into a Furier series collapses from the d-dimensional sum into the one-dimensional:

$$f\left(\vec{x}\right) = \sum_{m=-\infty}^{\infty} F(m)\exp(2\pi imx_1) . \quad (35)$$

According to the well-known formula

$$Tf(x) = f\left(T^{-1}x\right) \quad (36)$$

the evolution of correlation is determined by the inverse transformation T^{-1}. So

$$T^t f\left(\vec{x}\right) = \sum_{m=-\infty}^{\infty} F(m)\prod_{j=1}^{d} \exp\left(2\pi im\left(\alpha^{-t}\right)_{1j}x_j\right). \quad (37)$$

Therefore the correlation in this case equals to [26]

$$C_1(t) = \frac{2}{\pi^{d+1}\varepsilon^i\prod_{j=1}^{d}\left(\alpha^{-t}\right)_{1j}}\sum_{m=1}^{\infty} K_m \prod_{l=1}^{d} \sin\left[\pi m\left(\alpha^{-t}\right)_{ll}\varepsilon_l^f\right]$$

$$\quad (38)$$

$$K_m = \frac{1}{m^{d+1}}\cos\left(2\pi m\left[\sum_{j=1}^{d}\left(\alpha^{-t}\right)_{1j}x_j^f - x_1^i\right]\right)\sin\left(\pi m\varepsilon^i\right) \quad .$$

When $t \to \infty$

$$\left|C_1(t)\right| \le const \cdot \exp\left(-t\left|L_{\min}\right|d\right) \quad (39)$$

where $L_{\min}$ is the minimal Lyapunov's exponent of the linearized system (16).

2) A fine initial state and a crude measuring device Γ_i is defined by the formula (30) and Γ_f is defined by the relations:

$$x_1^f - \frac{1}{2}\varepsilon^f \le x_1 \le x_1^f + \frac{1}{2}\varepsilon^f, \; 0 \le x_j \le 1, \; (j = 1,2,...,d) . \quad (40)$$

The calculations analogous to the preceding case lead to the following expression for the correlation [26]:

$$C_2(t) = \frac{2}{\pi^{d+1}\prod_{l=1}^{d}\varepsilon_l^i\left(\alpha^{-t}\right)_{dl}}\sum_{m=1}^{d} K_m \prod_{j=1}^{d} \sin\left[\pi m\left(\alpha^t\right)_{dj}x_j^i\right]$$

$$K_m = \frac{1}{m^{d+1}}\cos\left(2\pi m\left[\sum_{k=1}^{d}\left(\alpha^t\right)_{dk}x_k^i - x_d^f\right]\right)\sin\left(\pi m\varepsilon^f\right) \quad . \; (41)$$

When $t \to \infty$

$$\left|C_2(t)\right| \le const \cdot \exp\left(-tL_{\max}d\right), \quad (42)$$

here $L_{\max}$ is the maximal Lyapunov's exponent.

We see that the rate of chaotization in both cases is proportional to the dimension of the phase space and has nothing to do with the dimension of the unstable subspace. If the initial state is crude and the measuring device is fine, the rate of chaotization is proportional to the $\left|L_{\min}\right|$, whereas on the opposite case the rate of chaotization is proportional to $L_{\max}$. We note that the rate of chaotization is nothing to do with the KC-entropy. The later equals to the sum of all positive Lyapunov's exponents

$$K = \sum_{L_i > 0} L_i . \quad (43)$$

We note further that the rate of chaotization depends on the choice of $f(x,0)$ and $\chi\left(x,\Gamma_f\right)$. This dependence disappears if

$$L_{\max} = \left|L_{\min}\right| . \quad (44)$$

The last condition is fulfilled for a Hamilton system.

TWO-DIMENSIONAL CASE

In the case $d = 2$, $t \to \infty$ the correlation C_1 takes the form [12]

$$C_1 = \frac{sh^2 L}{3\varepsilon^i\left(\alpha_{22} - e^{-L}\right)\alpha_{12}}\exp\left(-2L_{\max}t\right)\sum_{\pm} A_i$$

$$A_i = \left[\{a_u\}\left(1 - 3\{a_u\} + 2\{a_u\}^2\right) - \{a_g\}\left(1 - 3\{a_g\} + 2\{a_g\}^2\right)\right] \quad (45)$$

Here a_g are various sums of the form

$$P \pm \frac{1}{2}\varepsilon^i \pm \frac{1}{2}\left(\alpha^t\right)_{22}\varepsilon_1^f \pm \frac{1}{2}\left(\alpha^t\right)_{12}\varepsilon_2^f \quad (46)$$

with even number of minuses, and a_u - the same sums with odd number of minuses, and

$$P = \left(\alpha^t\right)_{22}x_1^f - \left(\alpha^t\right)_{12}x_2^f - x_1^i. \quad (47)$$

There is analogous expression for $C_2(t)$.

Note that in the two-dimensional case

$$L_{\max} = -L_{\min} \quad (48)$$

Formula (45) shows that the decay of correlation is not exponential, but rather erratic. In contrast to the thermodynamics the correlation (45) always approaches zero nonmonotonically ever after elapse of an arbitrary long interval of time.

However, the majoranta of correlation is exponential

$$Sup\left|C_1(t)\right| = \frac{4sh^2 L_{\max}}{9\sqrt{3}\varepsilon^i\left|\alpha_{22} - e^{-L_{\max}}\right|\alpha_{12}}\exp\left(-2L_{\max}t\right). \quad (49)$$

We note that the multiplier before the exponent in (45) remains finite when $\varepsilon \to 0$, $t \to \infty$.

CONTINUOUS FUNCTIONS

Up to now we assumed the functions $f\left(\overset{\cup}{x},0\right)$ and $\chi\left(\overset{\cup}{x}\right)$ to be piece-wise constant and discontinuous. Let

us now consider more complicated function $\chi\left(\overset{\sqcup}{\vec{x}}\right)$. Namely

$$\chi\left(\vec{x}\right) = \begin{cases} px_1 & if \quad 0 \le x_1 \le \dfrac{1}{6} \quad 0 \le x_2 \le 1 \\ \left(\dfrac{1}{3} - x_1\right) & if \quad \dfrac{1}{6} \le x_1 \le \dfrac{1}{3} \\ 0 & in \quad opposite \quad case \end{cases} \qquad (50)$$

As to the function $f\left(\overset{\sqcup}{\vec{x}},0\right)$ it is defined by the expression

$$f\left(\vec{x},0\right) = \begin{cases} 1, & if \quad 0 \le x_1 \le \dfrac{1}{2}, \quad 0 \le x_2 \le 1 \\ 0 & in \quad opposite \quad case \end{cases} \qquad (51)$$

Then for the Arnold's cat the asymptotic value of the correlation when $t \to \infty$ is

$$\underset{t}{Sup}\left|C(t)\right| = A(p-1)e^{-2Lt} + Be^{-3Lt}, \qquad (52)$$

here $L = \ln \dfrac{3 + \sqrt{5}}{2}$.

We see that when p differs from unity, i.e. when the function $\chi\left(\overset{\sqcup}{\vec{x}}\right)$ is discontinuous, the rate of the decay of correlation is $2L$ as before. On the other hand, if $p = 1$, i.e. if the function $\chi\left(\overset{\sqcup}{\vec{x}}\right)$ is continuous, the decay rate of the correlation equals $3L$. This means that infinitesimal change of the coefficient p provides an essential change of the correlation. We see that the problem of the calculation of the correlation is incorrect.

REGULARIZATION OF THE PROBLEM

A mathematical problem is called to be correct in the sense of Hadamard if following conditions are satisfied:
1. The solution exists.
2. The solution is unique.
3. The solution continuously depends on initial data.

As we have seen in the preceding section the problem of determining of the rate of decay of correlation is incorrect. We conjecture that in this case the correctness must be understood in the sense of Tikhonov [13] rather than in the sense of Hadamard. In other words the problem, which is incorrect in the sense of Hadamard must be regularized. The regularization consists in the reduction of the class of admissible functions. For example, the problem of Cauchy for the Laplace equation becomes correct, if the solution is searched in the class of bounded functions [14].

In the case of the problem of the decay of correlations the regularization is based on the fact that almost all chosen by chance functions are discontinuous (except of the set of zero measure). This reasoning is analogous to the conclusion about incommensurability of frequencies in a conditionally - periodical motion [15].

According to this we divide the phase space into cells and specify the number of particles in each cell [2].

In this case the rate of chaotization for two-dimensional phase space is the doubled maximal Lyapunov exponent L. More detailed specifying the functions $f\left(\overset{\sqcup}{\vec{x}},0\right)$ and $\chi\left(\overset{\sqcup}{\vec{x}}\right)$ influences only on a multiplier before the exponent.

STABILITY OF A PERIODICAL SYSTEM

We will investigate a periodical structure in a period of τ time:
$$T_\tau = T_1 \cdot T_2 \cdot \ldots \cdot T_N \qquad (53)$$

Here N is a number of stages (steps) in that period. As it is well known (see, for example, [16]), the question of periodical systems stability is determined by means of investigating of eigen value signification ρ of the matrix of monodromy $\hat{M}$ that satisfies the equation
$$\rho^2 - \rho \cdot Sp \cdot \hat{M} + 1 = 0, \qquad (54)$$
(with significant regard of the phase volume (3)). Here $\hat{M} -$ is a matrix of monodromy that displaces the solution of equation (54) in a period of time
$$\hat{M}_{ij} \cdot x_j^0 = x_j^\tau . \qquad (55)$$

It is supposed [16] that
$$\rho = e^{i\phi} \qquad (56)$$

and for the phase ϕ we have the equation
$$\cos\phi = \dfrac{1}{2} Sp \cdot \hat{M} . \qquad (57)$$

Real meaning of ϕ corresponds to the conditions of system stability, e.g. the condition of stability is:
$$-2 \le Sp \cdot \hat{M} \le 2 . \qquad (58)$$

For determination of the value $Sp \cdot \hat{M}$ we will find two solutions for the system (16). The first solution $\left(x_j^t\right)^I$ is for such initial conditions is
$$\left(x_1^0\right)^I = 1, \left(x_2^0\right)^I = 0 . \qquad (59)$$

Then, from (55) follows that
$$\left(x_1^\tau\right)^I = \hat{M}_{11} \qquad (60)$$

The second decisión for the system (16) $\left(x_j^t\right)^{II}$ made up for the following initial conditions
$$\left(x_1^0\right)^{II} = 0, \left(x_2^0\right)^{II} = 1 . \qquad (61)$$

Then, from (55) follows that
$$\left(x_2^\tau\right)^{II} = \hat{M}_{22} . \qquad (62)$$

Consequently
$$Sp \cdot \hat{M} = \left(x_1^\tau\right)^I + \left(x_2^\tau\right)^{II} , \qquad (63)$$
and the periodical structure conditions of stability can be written as
$$-2 \le \left(x_1^\tau\right)^I + \left(x_2^\tau\right)^{II} \le 2 \qquad (64)$$

It is recalled that in the case of the standard map [4,6], the conditions of stability have the following aspect
$$-4 \le k \le 0 . \qquad (65)$$

TWO STEP PERIODICITY

We will now investigate the periodic map T with a period of time τ, which consists of two successive standard maps

$$T_\tau = T_1 \cdot T_2 \ , \tag{66}$$

here

$$T_q = \begin{pmatrix} 1 & k_q \\ 1 & \cdots 1 + k_q \end{pmatrix} \qquad (q = 1,2) \ . \tag{67}$$

The conditions of stability in this case are

$$-1 \le 1 + k_1 + k_2 + \frac{1}{2} k_1 k_2 \le 1 \ . \tag{68}$$

We have to note that the inequality (68) is symmetrical in relation to the substitution of the members $k_1 \Leftrightarrow k_2$, e.g. in relation to the order of T_1 and T_2 in (66).
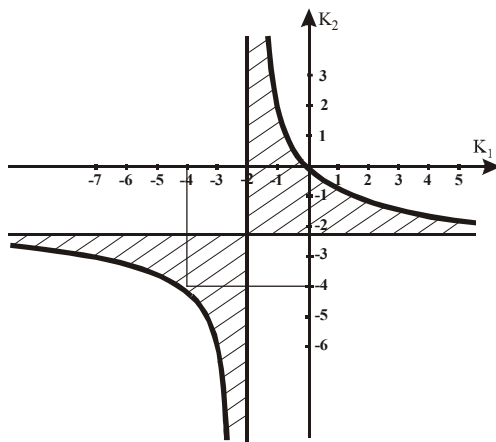

Fig. 1. *The stability regions*

In the plane $(k_1 k_2)$ in Fig. 1, the stability regions of the periodical structure (66), determined by the inequality (68), are pointed out in shaded lines. [The square formed by the points (0,0), (-4,0); (0,-4), (-4,4) represents the stability region of nonperioducal structure].

In our opinion, interesting results were obtained: some stable points standing in the boundary of an nonperiodical structure [for example, the points (-3, -1); (-1, -3) become unstable, but at the same time, some unstable points [for instance, the points (1, -1); (-5, -3); (3-, -5); (-1,1) become stable. In other words, in the case of a periodical map, there's a change in the stability conditions in the standard map.

THREE STEP PERIODICITY MAP

This time, we'll investigate the periodical map T_τ (τ – is any period), which consists of three steps. In order to get obvious results of stability regions on the plane, we'll consider only one variant, when two steps coincide, i.e.

$$T_\tau = T_1 \cdot T_1 \cdot T_2 \tag{69}$$

Following the mentioned scheme of the calculations $Sp \cdot \hat{M}$ we can obtain the stability conditions for the variant (69):

$$-1 \le \frac{k_2}{2}(k_1 + 3)(k_1 + 1) + k_1^2 + 3k_1 + 1 \le 1 \tag{70}$$

The stability regions on the plane (k_2, k_1) are indicated in Fig. 2 as the shaded regions
The boundary curves of the stability regions conformable to the sign of the equality (70) are given by the following equations:

$$k_2 = -2 + \frac{2}{k_1 + 1} \quad \text{(the curves A)}, \tag{71}$$

$$k_2 = -2 + \frac{2}{k_1 + 3} \quad \text{(the curves B)}. \tag{72}$$
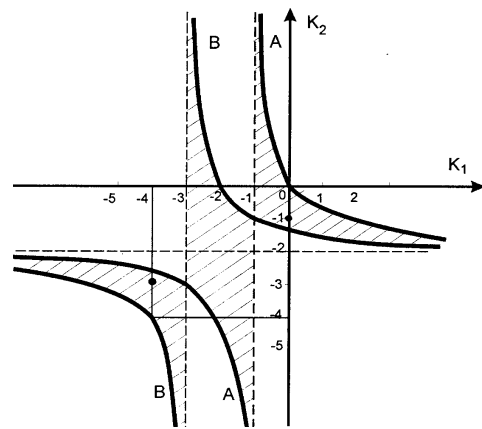

Fig. 2. *The regions of stability*

Therefore, the horizontal rect-line $k_2 = -2$ and two vertical rect-lines $k_1 = -1$ and $k_1 = -3$ represent the boundary lines. Notice that the obtained results do not depend on the step order in (69).

In Fig. 2, we can see that two points, just (-3,-4); and (-1, 0) standing on the boundary of stability square (65) of two inperiodical maps, become stable in the case of periodical maps with the same steps T_1 and T_2. Therefore, the other two points, just (-3, -5) and (-1, 1) standing outside of the stability square of the inperiodical map stability, fall on the bounds of the stability regions of the three step periodical map.

In our opinion, it's interesting the decay of the one stability region (65) in the three stability regions in the case of the three-step periodicity. (We could regard that in the case of two-step periodicity, such regions were only two). The points (-3,3) and (-1,1) of the bounders of the stability regions crossing are interesting too. The investigation of the region around these two points in the case of small disturbances k_q in the T_1 $(l = 1,2)$ shows that if the significance of disturbances falls into in the stability regions, it means that the additions in the right part of (57) decreases the module of its signification, e.g. to convert the equalities (58) into inequalities.

CONCLUSIONS

As this paper is divided into two sections two groups of conclusions are made. The first group of conclusions concerns the decay of correlations. They are:

1. The decay of correlations is going in a complicated non-exponential way.

2. The majoranta of the correlation function is an exponent.

3. The problem of the rate of chaotization is incorrect in the sense of Hadamard: the rate of the decay of correlation depends on the smoothness of the initial and the final functions.

4. We conjecture that the algorithm of regularization of this problem consists in dividing phase space into cells and then specifying a number of particles in each cell.

5. The rate of chaotization is proportional to the dimension of the phase space.

6. 1n general the rate of chaotization essentially depends on the initial and the final functions. This dependence disappears if the system is invariant under the time inversion. In that case the rate of chaotization is proportional to the maximal Lyapunov exponent.

The second group includes the conclusions about stability of periodical mapping. The two-dimensional dynamical system stability was investigated in the case of two- and three-step periodical maps. The conditions of stability were obtained. The influence of periodicity of standard maps on the stability of chaotic dynamical systems was investigated. In the case of a two step periodical mapping there's a change in the stability conditions in the standard map: some stable points standing in the boundary of an inperiodical structure become unstable, but at the same time, some unstable points become stable. In the case of the three-step periodicity it's interesting the decay of the one stability region (65) in to the three stability regions.

However, the most interesting, to our opinion, is: internal points from the squire of stability (65) of one-step mapping, there are the point (-2,-2) in Fig. 1 and the point (-1,-1) and (-3,-3) in Fig. 2 do not correspond to absolute stability. But these point correspond to one step mapping with k=1,-2 and -3 there are, for the stable value of k. Therefore the stability of these points depends from the meaning of perturbations (which appear in any moment). So we can make such conclusion: mapping (28) with integer k has no points of absolute stability.

REFERENCES

1. N.N. Bogoliubov. *Problemy dinamicheskoi teorii v statisticheskoi fizike*. Gostekhizdat, Moscow, 1946, 314 p. (in Russian).

2. N.S. Krylov. Works on the foundations of Statistical Physics. Princeton Univ. Press, Princeton, New-Jersey, 1979.

3. J. Mozer. Stable and random motions in dynamical systems. Princeton Univ. Press, Princeton, New-Jersey, 1973.

4. B.V. Chirikov. A universal instability of many-dimentional oscillator systems // *Phys. Rep.* 1979, v. 52, №5, p. 263-379.

5. G.M. Zaslavsky. *Chaos in dynamical systems*. 1985, Harvard, New-York.

6. G.M. Zaslavsky, R.Z. Sagdeev. *Vvedenie v nelineynuyu fiziku*. 1988, Moscow: "Fizmatlit", 368 p. (in Russian).

7. Yu.I. Neumark, P.S. Landa. *Stochastic and chaotic oscillations*. 1987, Moscow, 424 p. (in Russian).

8. E. Hopf. *Ergodentheorie Ergebnisse der Mathematik und ihrer Grenzgebiete. V. 2*. 1937, Berlin: "Springer".

9. V.A. Rokhlin. Liktsii po entropijnoj teorii preobrazovanij invariantnoj mery // *Uspekhi Matemat. Nauk.* 1967, v. 22, №5, p. 3-10 (in Russian).

10. R.C. Tolman. *The Principles of Statistical Mechanics.* 1948, Oxford: "Oxford University Press", .796 p.

11. J.P. Kornfel'd, Ya.G. Sinai, S.V. Fomin. *Ergodicheskaya teoriya.* 1980, Moscow: "Nauka", p. 190 (in Russian).

12. V.P. Demutskiy. R.V. Polovin. Model of dynamic chaos with analitical solution // *Journal of Technical Physics.* 1991, v. 61, p. 1-6.

13. A.N. Tikhonov. Ob ystoychivosti obratnykh zadach // *DAN SSSR.* 1943 v. 39, p. 195-198 (in Russian).

14. S.G. Krein. O klassakh korrektnosti dliy nekotorykh granichnykh zadach // *DAN SSSR.* 1987, v. 114, p. 1162-1165 (in Russian).

15. V.I. Arnol'd. Dokazatelstvo teoremy A.S. Kolmogorova o sokhranenii uslovno-periodicheskogo dvizeniay pri malom izmenenii funkstii Gamiltona *Uspekhi Matematicheskikh Nauk //* 1963, v. 18, p. 13-40 (in Russian).

16. D. Mishkis. *Mathematic.* 1971. Moscow: "Nauka", 632 p. (in Russian).