ROUTES TO CHAOS THROUGH THE INTERACTION OF HIGH- AND LOW-FREQUENCY OSCILLATIONS

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Roads to chaos for systems with the interaction of high- and low-frequency oscillations are considered. Results are presented for quisi-periodically forced Duffing and Van-der-Pole oscillators and a two-mode system driven by a harmonic force. The focus is made on the conditions for chaos under weak nonlinearity of the system.

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1. INTRODUCTION

It is well known that interaction of two or more oscillations or modes may lead to chaotic oscillations. Up to now, the most extensively studied systems were ones with the resonant interaction of oscillations, when the following conditions were met [1-4]:

$$\omega_1/\omega_2 \approx n/m \text{, or } \omega = \omega_1 \pm \omega_2, \qquad (1)$$

where ω_1 and ω_2 are external frequencies or the natural frequencies of interacting modes, n and m are comparatively small integers, and ω stands for the frequency of an external force or the third interacting mode. The most recent results [5] show that even the interaction of oscillations with substantially different natural frequencies, i.e., when the following condition is met:

$$\omega_1 >> \omega_2, \qquad (2)$$

may considerably change the system dynamics.

In this paper we present results of recent investigations of different dynamical systems, like quisiperiodically forced Duffing and Van-der-Pole oscillators and a two-mode system driven by a harmonic force. We show that interaction of low- and high-frequency oscillations in such systems leads to the chaos onset even in the limit of weak nonlinearity. Both, numerical and analytical methods are used for the study of chaotic states.

2. CHAOS IN DUFFING OSCILLATOR WITH HIGH- AND LOW-FREQUENCY EXTERNAL FORCING

Let us consider a two-frequency forced Duffing oscillator:

$$\ddot{x} + \mu'\dot{x} + \omega_0^2 x - \beta'' x^2 + \gamma'' x^3 = F_0' \cos k\Omega t + F' \cos\omega t, \qquad (3)$$

under the following condition: $\omega \approx \omega_0 >> \Omega$, where $k = 1,2$, ω and Ω are the frequencies of external excitation, and ω_0 is the natural frequency of oscillator. This equation has been studied already for the case when the following condition is met: $n\omega = m\omega_0$, where $n, m = 1,2,\ldots$, or $\omega_0 = \omega_1 + \omega_2$.

Our study has demonstrated that the oscillator (3) demonstrates chaotic behavior as a result of interaction of high- and low-frequency oscillations even in the weakly nonlinear limit. For this case the equation (3) can be simplified by applying the standard averaging technique. This leads to the following averaged equations:

$$\begin{aligned} \dot{u} &= -\mu u - \left[\Delta + \gamma\left(u^2 + v^2\right) + \beta B \cos\varepsilon\tau + 2\gamma B^2 \cos 2\varepsilon\tau\right] v, \\ \dot{v} &= -\mu v + \left[\Delta + \gamma\left(u^2 + v^2\right) + \beta B \cos\varepsilon\tau + 2\gamma B^2 \cos 2\varepsilon\tau\right] u - P. \end{aligned} \qquad (4)$$

for k=1, and

$$\begin{aligned} \dot{u} &= -\mu u - \left[\Delta - P(\beta - 2\gamma P) + \gamma\left(u^2 + v^2\right) + 2B(\beta - 2\gamma P)\cos\varepsilon\tau + 4\gamma B^2 \cos^2 \varepsilon\tau\right] v, \\ \dot{v} &= -\mu v + \left[\Delta + P(\beta + 2\gamma P) + \gamma\left(u^2 + v^2\right) + 2B(\beta + 2\gamma P)\cos\varepsilon\tau + 4\gamma B^2 \cos^2 \varepsilon\tau\right] u. \end{aligned} \qquad (5)$$

for k=2.

Here the point means derivative with respect to a slow time τ, $\varepsilon << 1$ is a normalized frequency of low-frequency external forcing, P is a normalized amplitude of high-frequency external forcing, B is amplitude of the low-frequency external forcing, β and γ are nonlinearity coefficients, and Δ is frequency detuning parameter.

We have studied this system by using two methods: Melnikov technique and the method of second averaging. The first one allows us to find conditions for the chaos onset in the system, and the second one allows finding conditions for the period-doubling bifurcation and for the tangential bifurcation. The results obtained by using the both techniques show good correspondence with the results of numerical experiments (see Fig. 1). The criterion obtained in accordance with the Melnikov technique, is shown by solid line. Crosses show regions of chaos, obtained in numerical experiment. The border of the first period-doubling bifurcation, obtained by the

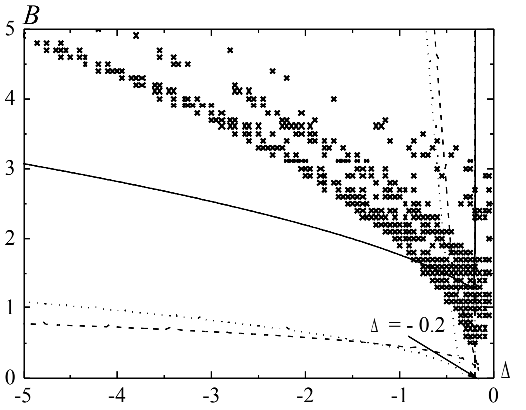

Fig. 1. *Chaos regions on the* (Δ, B) *parameters plane for k=1,* $\beta = 0.1$, $\gamma = 0.1$, $P = 0.1$, $\varepsilon = 0.08$, *and* $\mu = 0.01$

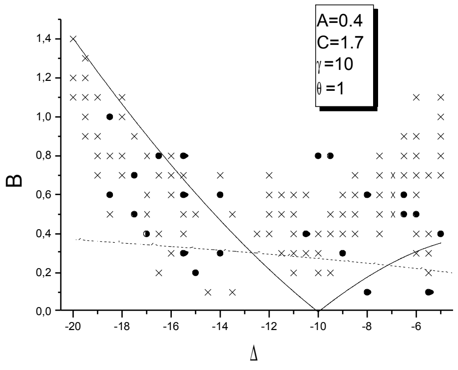

Fig. 2. *Bifurcation diagram on the* (B, Δ) *plane for given values of parameters*

second averaging method, is shown by dashed line, and the border of the tangential bifurcation is shown by dotted line. With respect to the initial equation (3) the transition to chaos is realised via destruction of two- dimensional tori.

3. CHAOS IN THE VAN-DER-POLE OSCILLATOR WITH LOW-FREQUENCY ANODE VOLTAGE MODULATION

The next system under consideration is the Van-der-Pole oscillator with a low- and high-frequency external forcing. This system is described by the following equation:

$$\ddot{q} - \left(2\mu_0 - \mu_1 q^2\right)\dot{q} + q - \beta q^2 + \gamma q^3 = - A' \sin(\theta \tau) + B' \cos(\nu \tau) \quad (6)$$

Here μ_0 and μ_1 are the damping coefficients, β and γ are coefficients of nonlinearity, A' and θ are the amplitude and phase of low-frequency modulation, correspondingly, B' and ν are the amplitude and phase of the synchronizing force ($\nu \approx 1$, i.e., ν is close to the natural frequency of the oscillator).

After the application of the averaging technique to (6) we obtain the following equations:

$$\frac{da}{d\tau} = \left(1 - a^2\right)a - B\sin\varphi - a A \sin^2\theta\tau,$$
$$\frac{d\varphi}{d\tau} = \Delta + \gamma a^2 - \frac{B}{a}\cos\varphi - C\sin\theta\tau + \gamma A \sin^2\theta\tau. \quad (7)$$

Here C is related to A and β .

Conditions for the stability of the synchronous mode are as following:

$$a^2 > \frac{1}{2}, \quad (8)$$

$$\left(1 - a^2\right)\left(1 - 3a^2\right) + \left(\Delta + \gamma a^2\right)\left(\Delta + 3\gamma a^2\right) > 0. \quad (9)$$

The application of method of the second averaging has also allowed us to obtain conditions for the period-doubling bifurcation in this system.

Results of numerical experiments as well as conditions for the period-doubling bifurcation and for the stability of the synchronous mode are shown in Fig. 2. Crosses here represent chaotic regions obtained from numeric simulations; points are period-2 oscillations; the border of synchronous oscillations is shown by solid line, and the boundary for the period-doubling bifurcation obtained analytically is shown by dashed line.

4. CHAOS IN A NONLINEAR TWO-MODE HARMONICALLY FORCED SYSTEM

In this section we review results of the investigation of a harmonically forced system of two coupled passive oscillators, which natural frequencies differ essentially. In the general case such system can be described by the following equations:

$$\frac{d^2 x_{HF}}{d\tau^2} + x_{HF} = - 2\varepsilon\mu_1 \frac{dx_{HF}}{d\tau} - \varepsilon(2\gamma x_{HF} x_{LF} - S\cos\nu\tau),$$
$$\frac{d^2 x_{LF}}{d\tau^2} + \varepsilon^2 x_{LF} = -2\varepsilon\mu_2 \frac{dx_{LF}}{d\tau} - \varepsilon^2 \gamma x_{HF}^2. \quad (10)$$

Here x_{HF} and x_{LF} are variables describing high- and low-frequency oscillators, correspondingly, μ_1 and μ_2 represent damping in high- and low-frequency oscillators, correspondingly; γ is the coefficient of nonlinearity; τ is a slow time; S is the amplitude of the external forcing; ν is relevant to the natural frequency of high-frequency oscillator.

A possible physical realization of the above system is shown in Fig. 3. Here L_1, R_1, and C_1 represent resonantly driven by an external harmonic force high-frequency circuit I. L_2, R_2, and C_2 represent low-frequency circuit II. It is generally believed that if the conditions $L_1 \approx \varepsilon L_2, C_1 \approx \varepsilon C_2$, are met, the influence of the low-frequency circuit on the dynamics of the whole system can be neglected. Our studies [6] have shown that such interaction can exert a considerable influence on the system dynamics.

The first equation of (10) can be considered as the motion equation of a quasilinear oscillator. So one can apply an averaging technique to it. After performing corresponding transformations one can come to the following system of averaged equations:

$$\begin{aligned}\dot{a} &= -\mu_1 a - S\sin\varphi ,\\ \dot{\varphi} &= -\Delta + \gamma u - \frac{S}{a}\cos\varphi ,\\ \dot{u} &= v,\\ \dot{v} &= -2\mu_2 v - u - \frac{1}{2}\gamma a^2 .\end{aligned} \qquad (11)$$

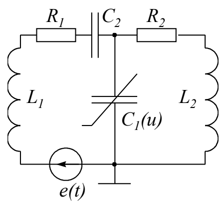

Fig. 3. *Two-mode externally forced system*

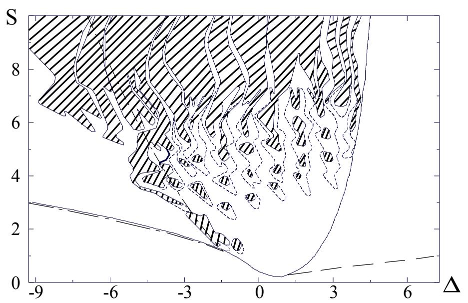

Fig. 4. *Obtained from numerical simulations bifurcation diagram on the parameter plane (S, Δ) at γ=1.0; μ1=0.7 and μ_2=0.01*

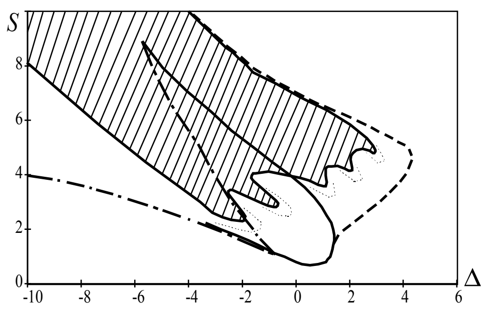

Fig. 5. *Experimental bifurcation diagram on the parameter plane (S, Δ) with the same notations as in Fig. 4. The fine structure of the chaos region is not indicated*

The overdot here denotes differentiation with respect to the slow time $\varepsilon\tau$, $v \equiv \dot{x}_{LF}$, $u \equiv x_{LF}$ are independent variables which define the state of the low-frequency oscillator, and $\Delta = \left(\nu^2 - 1\right)/\left(2\varepsilon\nu\right)$ is the parameter of the frequency mismatch.

Bifurcation diagram of the system (1) obtained numerically is shown in Fig. 4. It should be noted that the system demonstrates chaotic behaviour in a wide range of variation of control parameters, and that the threshold for chaos onset is going down with a decrease of the damping coefficients. Results of experimental investigations of the circuit in Fig. 3 are shown in Fig. 5 on the same parameter plane. A good qualitative agreement between these results should be mentioned.

CONCLUSION

The results of the presented investigations allow us to make the following conclusions:

(i) periodically excited systems with the interaction of high- and low-frequency oscillations are susceptible to chaotic instabilities to a great extent,

(ii) the chaotic oscillations can arise under weakly nonlinear conditions of excitation,

(iii) chaotic instabilities due to the interaction of low- and high-frequency oscillations can exert a strong influence on the dynamics of many practical systems.

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