DETERMINATION OF MINIMUM DISTANCE BETWEEN ORBITS OF CELESTIAL BODIES

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A method is suggested to determine of minimum distance between orbits of two celestial bodies. The method is based on finding of the singular points of one variable function. The orbits of celestial bodies can be elliptic, parabolic or hyperbolic. The orbits of celestial bodies can also be described universal variable. On the basis of the given method the program is developed, using which the accounts of minimal distance between orbits of comets and orbits of planets are executed. The given method can also be used for finding of the minimal distances both between orbits of asteroids and orbits of planets, and between asteroids orbits.

INTRODUCTION

The minimal distances between orbits of celestial bodies are heavy weighed. If they are small enough that probably close approach celestial bodies. It is especially important in case of comet, when the small value of this minimal distance can serve indication that the close approach of the comet and planet could take place, as a result of which there could be an essential change of elements of an orbit of a comet. There is one more case, when it is important to know this minimal distance is a case of orbits asteroids and orbit of the Earth, when small distance between their orbits can be indication on an opportunity of collision this asteroid with the Earth.

DETERMINATION OF MINIMUM DISTANCE BETWEEN ORBITS

In [2] the case of elliptic orbits of two bodies is considered and is shown that the given task can be reduce to finding of zero function by one variable. By the author [1] is shown that the finding of minimal distance between an elliptic orbit both hyperbolic and parabolic orbits can also be reduce to a finding of zero function by one variable. In connection with this it represents the great interest, including practical, whether a finding of minimal distance between orbits of any type, including in a case, when for the description of an orbit are used universal variable, is possible, to reduce a finding of zero function by one variable, to that the present work is devoted.

Let us consider a task of a finding of minimal distance between orbits of two celestial bodies in the most general case.

The system of coordinates, where the XY plane coincides with a celestial body orbital plane, and the X axis is directed in an orbit perihelion may be written as

$$\left\{ \begin{array}{l} x = r_x(a, e, \phi), \\ y = r_y(a, e, \phi), \end{array} \right. \tag{1}$$

where r_x and r_y are also functions dependent on major axis, eccentricity and variable ϕ, which will be concretized below. It is clear that the radius-vector $r = \sqrt{x^2 + y^2}$ will be function of the same variable.

As it is known, in ecliptic system of coordinates the radius-vector r can be expressed by r_x^p, r_y^p, and also by argument perihelion ω, longitude of the ascending node Ω and inclination of an orbit i:

$$\mathbf{r} = \mathbf{P}\mathbf{r_x} + \mathbf{Q}\mathbf{r_y}, \tag{2}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are the orbital vectors of the constants, which are expressed by ω, Ω, and i:

$$\begin{array}{ll} P_x = \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i, & Q_x = -\sin\omega\cos\Omega + \cos\omega\sin\Omega\cos i, \\ P_y = \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i, & Q_y = -\sin\omega\sin\Omega - \cos\omega\cos\Omega\cos i, \\ P_z = \sin\omega\sin i, & Q_z = \cos\omega\sin i. \end{array} \tag{3}$$

Marking everything, concerns to the first body by 1, and to the second one by 2 it is possible to write the following formula for a square of distance between bodies ρ:

$$\rho^2 = (\mathbf{r_2} - \mathbf{r_1}, \mathbf{r_2} - \mathbf{r_1}) = \mathbf{r_2^2} - \mathbf{2}(\mathbf{r_2}, \mathbf{r_1}) - \mathbf{r_1^2}, \tag{4}$$

where ρ is a function of two variables ϕ_1 and ϕ_2. It will depend on parameters of orbits of two celestial bodies. As it is known, for a finding of its minimum it is necessary to equate with zero its derivative of variable and to solve concerning them the turned out equations system. If such decisions will be a little bit that to find distances for all values of the decisions and minimal of the found distances will be required minimal distance between orbits of these two celestial bodies. From above-stated it is received the following equations system:

$$\begin{cases} \dfrac{\partial \rho^2}{\partial \phi_1} = 2r_1 \dfrac{dr_1}{d\phi_1} - 2\left(\mathbf{r_2}, \dfrac{\mathbf{dr_1}}{\mathbf{d\phi_1}}\right) = 0, \\ \dfrac{\partial \rho^2}{\partial \phi_2} = 2r_2 \dfrac{dr_2}{d\phi_2} - 2\left(\dfrac{d\mathbf{r_2}}{d\phi_2}, \mathbf{r_1}\right) = 0. \end{cases}$$

The obtained system can also be rewrote as follows:

$$\begin{cases} r_1 \dfrac{dr_1}{d\phi_1} - \left(\mathbf{r_2}, \dfrac{\mathbf{dr_1}}{\mathbf{d\phi_1}}\right) = 0, \\ r_2 \dfrac{dr_2}{d\phi_2} - \left(\dfrac{d\mathbf{r_2}}{d\phi_2}, \mathbf{r_1}\right) = 0. \end{cases} \tag{5}$$

With (1) and (2) the equations system (5) is possible to rewrite in the following kind:

$$\begin{cases} r_1 \dfrac{dr_1}{d\phi_1} - S_1 \dfrac{r_{1x}}{d\phi_1} - S_3 \dfrac{r_{1y}}{d\phi_1} = 0, \\ r_2 \dfrac{dr_2}{d\phi_2} - S_2 r_{1x} - S_4 r_{1y} = 0, \end{cases} \tag{6}$$

where $S_1 = P_1P_2r_{2x} + P_1Q_2r_{2y}$, $S_2 = P_1P_2\dfrac{dr_{2x}}{d\phi_2} + P_1Q_2\dfrac{dr_{2y}}{d\phi_2}$, $S_3 = Q_1P_2r_{2x} + Q_1Q_2r_{2y}$, $S_4 = Q_1P_2\dfrac{dr_{2x}}{d\phi_2} + Q_1Q_2\dfrac{dr_{2y}}{d\phi_2}$, and where by P_1P_2, Q_1P_2, P_1Q_2, and Q_1Q_2 defined scalar products $(\mathbf{P_1}, \mathbf{P_2})$, $(\mathbf{Q_1}, \mathbf{P_2})$, $(\mathbf{P_1}, \mathbf{Q_2})$, $(\mathbf{Q_1}, \mathbf{Q_2})$.

The further consideration we shall take at once for all types of orbits, including for a case of use for the description of an orbital universal variable. In Table 1 for all these cases the values of r_x, r_y, r, $dr_x/d\phi$, $dr_y/d\phi$, $dr/d\phi$ are given without their expressions in order to save place for our paper. For an elliptic orbit the eccentric anomaly E is chosen as independent variable, for a parabolic orbit $\sigma = q\tan\dfrac{f}{2}$ is chosen, where q is the perihelion distance, f is the true anomaly, for a hyperbolic orbit H is chosen, which is as follows expressed by true anomaly: $\tanh\dfrac{H}{2} = \sqrt{\dfrac{e+1}{e-1}}\tan\dfrac{f}{2}$, for a case of use for the description of an orbital universal variable, *i.e.*, generalized anomaly s, determined by expression $\mu\dfrac{dt}{ds} = r$, where μ is the gravitational constant, p is an orbital parameter. The functions $C_n(z)$ are defined by the formulas:

$$C_n(z) = \sum_{k=0}^{k=\infty} (-1)^k \frac{z^k}{(2k+n)!}; \qquad n = 1, 2, 3, \dots. \tag{7}$$

Let us note that there are 10 variants of the equations system (6): $E-E$, $E-P$, $E-H$, $E-U$, $P-P$, $P-H$, $P-U$, $H-H$, $H-U$, $U-U$, where for orbits types the following designations are entered: E is elliptic, P is parabolic, H is hyperbolic, U is used for the description of an orbital universal variable. We shall consider only four cases, when the first orbit is consistently chosen elliptic, parabolic, hyperbolic and written down in universal constant while for the second orbit we shall keep recording in a general view.

Elliptic orbit. The equations system (6) with the accounting Table 1 accepts the following form:

$$\begin{cases} M\sin E_1 + N\cos E_1 = K\sin E_1\cos E_1, \\ A\sin E_1 + B\cos E_1 = 0, \end{cases} \tag{8}$$

Table 1. Expression for r_x, r_y, r, $dr_x/d\phi$, $dr_y/d\phi$, $dr/d\phi$ for various types of orbits

Expression	Elliptic orbit	Parabolic orbit	Hyperbolic orbit	Universal variable
r_x	$a(\cos E - e)$	$q(1-\sigma^2)$	$\mid a \mid (e - \cosh H)$	$q - s^2 C_2(\alpha s^2)$
r_x	$a\sqrt{1-e^2}\sin E$	$2q\,\sigma$	$\mid a \mid \sqrt{e^2-1}\sinh H)$	$\sqrt{q} s C_1(\alpha s^2)$
r	$a(1 - e\cos E)$	$q(1+\sigma)$	$\mid a \mid (e\cosh H + 1)$	$q\,C_0(\alpha\, s^2) s^2 C_2(\alpha s^2)$
$\dfrac{dr_x}{d\phi}$	$-a\sin E$	$-2q\sigma$	$- \mid a \mid \sinh H$	$sC_1(\alpha s^2)\dfrac{\sqrt{\mu}}{r}$
$\dfrac{dr_y}{d\phi}$	$a\sqrt{1-e^2}\cos E$	$2q$	$\mid a \mid \sqrt{e^2-1}\cosh H$	$C_0(\alpha s^2)\dfrac{\sqrt{p\,\mu}}{r}$
$\dfrac{dr}{d\phi}$	$ae\sin E$	$2q\,\sigma$	$\mid a \mid e\sinh H$	$(1-\alpha q)sC_1(\alpha s^2)$

where $M = S_1 - a_1 e_1,\ N = -\sqrt{1-e_1^2}S_3,\ K = -a_1 e_1,\ A = a_1\sqrt{1-e_1^2}S_4,\ B = a_1 S_2,\ C = r_2\dfrac{dr_2}{d\phi_2}$. It follows from (8) that A, B, C, M, N, and K depend only on ϕ_2 and do not depend on E_1. Thus, to reduce the decision of the given equations system to a finding of zero function by one variable it is necessary to find $\sin E_1$ and $\cos E_1$ from the second equation and to substitute the received expressions in the first equation.

By solving the second equation from the equations system (8) we obtain the following expressions for $\sin E_1$ and $\cos E_1$:

$$\begin{aligned} \sin E_1 &= \frac{AC \pm B\sqrt{A^2+B^2-C^2}}{A^2+B^2}, \\ \cos E_1 &= \frac{BC \mp A\sqrt{A^2+B^2-C^2}}{A^2+B^2}. \end{aligned} \tag{9}$$

As follows from (9) the decision of the equations system (8) only exists, when the condition: $A^2+B^2-C \geq 0$ is satisfied.

Substituting the obtained expressions for $\sin E_1$ and $\cos E_1$ in the first equation of the system (8) after algebraic transformations one can get:

$$F = [C\tilde{E}(MA+NB) + KAB(G-C^2)] - G[KC(B^2-A^2) - E(MB-NA)]^2 = 0, \tag{10}$$

where $\tilde{E} = A^2 + B^2$, $G = \tilde{E} - C^2$.

Parabolic orbit. The equations system (6) with the accounting Table 1 accepts the following form:

$$\left\{ \begin{array}{l} M\sigma_1^3 + N\sigma_1 = K, \\ A\sigma_1^2 + 2B\sigma_1 = C, \end{array} \right. \tag{11}$$

where $M = q_1$, $N = q_1 - S_1$, $K = S_3$, $A = q_1 S_2$, $B = q_1 S_4$, $C = r_2\dfrac{dr_2}{d\phi_2} - q_1 S_2$. Solving the second equation of the system (11) the following equation for σ may be written:

$$\sigma_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{A}. \tag{12}$$

As follows from (12) the decision of the equations system (11) only exists, when the condition: $B^2 - AC \geq 0$ is satisfied.

In this case substituting the obtained expression for $\sigma_{1,2}$ in the second equation of the system (11) similarly previous one the following function F by one variable is obtained:

$$F = (3MB^2 + B^2 - AC + A^2N)(B^2 - AC) - (A^3K + MB^3 + 3B^2 - 3BAC + A^2NB) = 0. \tag{13}$$

Thus, and in this case the searching for a minimal distance between a parabolic orbit and other orbit form was reduced to a finding of zero function by one variable.

Hiperbolic orbit. The equations system (6) with the accounting Table 1 may be written by the equations:

$$\left\{ \begin{array}{l} M\sinh H_1 + N\cosh H_1 = K\sinh H_1\cosh H_1, \\ A\sinh H_1 + B\cosh H_1 = 0, \end{array} \right. \tag{14}$$

where $M = \mid a \mid - S_1$, $N = -\sqrt{e_1^2 - 1} S_3$, $K = \mid a_1 \mid e_1$, $A = \mid a_1 \mid \sqrt{e_1^2 - 1} S_4$, $B = - \mid a_1 \mid S_2$, $C = r_2 \dfrac{dr_2}{d\phi_2} S_2$.
From the second equation in (14) we find:

$$\begin{aligned} \sinh H_1 &= \frac{-AC \pm B\sqrt{A^2 + C^2 - B^2}}{B^2 - C^2}, \\ \cosh H_1 &= \frac{BC \mp A\sqrt{A^2 + C^2 - B^2}}{B^2 - C^2}. \end{aligned} \tag{15}$$

As follows from (15) the decision of the equations system (14) only exists, when the condition: $A^2 + C^2 - B^2 \geq 0$ is satisfied. The values $\sinh H_1$, $\cosh H_1$ in the first equation of the system (14) were substituted for these values from (15) and the function F was obtained:

$$F = [C\tilde{E}(AM - NB) + KAB(C^2 + G)]^2 - G[\tilde{E}(MB - NA) + KC(A^2 + B^2)]^2 = 0, \tag{16}$$

where $\tilde{E} = A^2 - B^2$, $G = \tilde{E} + C^2$.

The case of use for the description of an orbital universal variable. We can write the equations system (6) with the accounting Table 1 as follows:

$$\begin{cases} Ms_1C_1(\alpha_1 s_1^2) + Ns_1^2 c_2(\alpha_1 s_1^2) - L = Ks_1C_1(\alpha_1 s_1^2) s_1^2 C_2(\alpha_1 s_1^2), \\ A(s_1C_1(\alpha_1 s_1^2))^2 + 2Bs_1C_1(\alpha_1 s_1^2) + G = 0, \end{cases} \tag{17}$$

where $M = \dfrac{\mu S_1}{r_1} - q_1(1 - \alpha_1 e_1)$, $N = -\dfrac{\alpha_1\sqrt{\mu p_1}}{r_1}$, $L = \sqrt{\mu p_1} S_3$, $K = 1 - \alpha_1 q_1$, $A = \alpha_1 + (\alpha_1 q_1 S_4)^2$, $B = \sqrt{q_1} S_4 \left(\alpha_1 r_2 \dfrac{dr_{2x}}{d\phi_2} - (1 - \alpha_1 q_1) S_2\right)$, $C = \left(\alpha_1 r_2 \dfrac{dr_{2x}}{d\phi_2} - (1 - \alpha_1 q_1) S_2\right)^2 - 1$.

Let us note that at reception of the second equation of the system (17) the following expression is used:

$$\alpha s^2 C_2(\alpha s^2) = 1 - \sqrt{1 - \alpha(sC_1(\alpha s^2))^2}.$$

A solution of the second equation may be presented as:

$$(s\, C_1(\alpha s^2))_{1,2} = \frac{-B \pm \sqrt{B^2 - AG}}{A}. \tag{18}$$

As follows from (18) the decision of the equations system (17) only exists, when the condition: $B^2 - AQ \geq 0$ is satisfied. The following expression for function F is derived after substitution of the obtained decision in the first equation of the system (17) taking into account (18):

$$F = U^2 - \frac{E^2 K^2 W}{A^2}, \tag{19}$$

where $U = (N - L)^2 - 2(N - L)\left(\frac{M-K}{A}\right)B + \left(\frac{M-K}{A}\right)B^2 + W\left(\frac{M-K}{A}\right)^2 - \frac{R}{A^2}\left(V^2 + \frac{WK^2}{A}\right) - \frac{4BWV}{A^2}$, $E = 2B\left(\frac{M-K}{A}\right)^2 - \frac{2RV}{A^2} - \frac{2B}{A^2}\left(V^2 + \frac{K^2 W}{A}\right) - \frac{2R}{A^2}\left(\frac{KB}{A}\right)$, $V = \frac{KB}{A} + N$, $W = B^2 - AG$, $R = A^2 + B^2 - AG - \alpha_1 B^2$.

Thus, one see in this case the searching for a minimal distance between orbits of two bodies was reduced to a finding of zero function by one variable as well as in the three above-mentioned cases too.

It is necessary to note that our research was carried out without a concrete definition of the second orbit. However, from stated above clearly that the transition to the concrete kind of the second orbit is reduced to replacement in the received formulas of general expressions by the concrete expressions according to Table 1.

In summary we shall note that from our consideration is followed that the definition of minimal distance between orbits of two celestial bodies can be executed as follows. To find a dependence on types of orbits or form of their description it is necessary to find zero function by one variable. Then, for these values variable the second variable is founded, and the distances between celestial bodies orbits are founded for the obtained pairs of variables. Minimum of these distances will be minimal distance between the given orbits of celestial bodies.

[1] *Babenko Yu. G.* // Astrometry and Astrophys.–1983.–**49**.–P. 22–26.

[2] *Vasiljev N. N.* // Bull. Inst. Theor. Astron.–1978.–**14**.–P. 266–268.