

# BIHAMILTONIAN APPROACH TO THE GAUGE MODELS: CLOSED STRING MODEL

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The closed string model in the background gravity field and the antisymmetric B-field is considered as the bihamiltonian system in assumption that string model is the integrable model for particular kind of the background fields. It is shown that bihamiltonicity is origin of two types of the T-duality of the closed string models. The dual nonlocal Poisson brackets, depending of the background fields and of their derivatives, are obtained. The integrability condition is formulated as the compatibility of the bihamiltonicity condition and the Jacobi identity of the dual Poisson bracket. It is shown, that the dual brackets and dual hamiltonians can be obtained from the canonical (PB) and from the initial hamiltonian by imposing of the second kind constraints on the initial dynamical system, on the closed string model in the constant background fields, as example. The closed string model in the constant background fields is considered without constraints, with the second kind constraints and with first kind constraints as the B-chiral string. The two particles discrete closed string model is considered as two relativistic particle system to show the difference between the Gupta-Bleuler method of the quantization with the first kind constraints and the quantization of the Dirac bracket with the second kind constraints.

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## 1. INTRODUCTION

The bihamiltonian approach [1-3] to the integrable systems was initiated by Magri [4] for the investigation of the integrability of the KdV equation. A finite dimensional dynamical system with  $2N$  degrees of freedom  $x^a$ ,  $a=1...2N$  is integrable, if it is described by the set of the  $n$  integrals of motion  $F_1, \dots, F_n$  in involution under some Poisson bracket (PB)

$$\{F_i, F_k\}_{PB} = 0. \quad (1)$$

The dynamical system is completely solvable, if  $n=N$ . Any of the integral of motion (or any linear combination of them) can be considered as the hamiltonian  $H_k = F_k$ . The bihamiltonicity condition has following form:

$$\dot{x}^a = \frac{dx^a}{dt} = \{x^a, H_1\}_1 = \dots = \{x^a, H_N\}_N. \quad (2)$$

The hierarchy of new (PB) is arosed in this connection:

$$\{, \}_1, \{, \}_2, \dots, \{, \}_N. \quad (3)$$

The hierarchy of new dynamical systems is arosed under the new time coordinates  $t_k$ .

$$\frac{dx^a}{dt_{n+k}} = \{x^a, H_n\}_{k+1}. \quad (4)$$

The new equations of motion are described the new dynamical systems, which are dual to the original system, with the dual set of the integrals of motion. The dual set of the integrals of motion can be obtained from original one by the mirror transformations and by the contraction of the integrals of motion algebra. The contraction of the integral of motion algebra means, that the dynamical system is belong to the orbits of corresponding generators and is describe the invariant subspace. The set of the commuting integrals of motion belongs to Cartan subalgebra of this algebra. Consequently, duality is property of the integrable models. KdV equation is one of the most interesting

examples of the infinite dimensional integrable mechanical systems with soliton solutions.

We are considered the dynamical systems with constraints. In this case, first kind constraints are the generators of the gauge transformations and they are integrals of motion. First kind constraints  $F_k(x^a) \approx 0$ ,  $k=1,2...n$  form the algebra of constraints under some (PB).

$$\{F_i, F_k\}_{PB} = C_{ik}^l F_l \approx 0. \quad (5)$$

The structure functions  $C_{ik}^l$  may be functions of the phase space coordinates in general case. The second kind constraints  $f_k(x^a) \approx 0$  are the representations of the first kind constraints algebra. The second kind constraints is defined by the condition

$$\{f_i, f_k\} = C_{ik} \neq 0. \quad (6)$$

The reversible matrix  $C_{ik}$  is not constraint and it is function of phase space coordinates also. The second kind constraints take part in deformation of the  $\{, \}_{PB}$  to the Dirac bracket  $\{, \}_D$ . As rule, such deformation leads to the nonlinear and to the nonlocal brackets. First kind constraints are imposed upon the vector states under the quantization:  $F_k|\Psi\rangle = 0$ . The same spectrum of the excitations and the wave functions are obtained under the Gupta-Bleuler method of the quantization. One-half of the second kind constraints can be considered as first kind constraints and they must be imposed upon vector states in Gupta-Bleuler method of quantization. The (PB) is not deformed to the Dirac bracket in this connection. The bihamiltonicity condition leads to the dual (PB), which are nonlinear and nonlocal brackets as rule.

We suppose, that the dual brackets can be obtained from the initial canonical bracket under the imposition of the second kind constraints. We make this conclusion from the consideration of two dynamical models, closed string model in the constant background fields and two particles discrete closed string model, as examples. The Gupta-Bleuler method of the quantization may be more preferable in some case, if the dual bracket is nonlocal.

We have applied [5,6] bihamiltonity approach to the investigation of the integrability of the closed string model in the arbitrary background gravity field and antisymmetric B-field. The bihamiltonity condition and the Jacobi identities for the dual brackets have considered as the integrability condition for a closed string model. They led to some restrictions on the background fields. The local dual (PB) of the similar type have considered [7] in the application to the hamiltonian hydrodynamical models. The (PB) of the hydrodynamical type for the phase coordinate functions  $u^i(x,t)$  is defined by formula

$$\{u^i(x), u^k(y)\} = g^{ik}(u(x)) \partial_x \delta(x-y) + b_i^{jk}(u(x)) u_x^j \delta(x-y). \quad (7)$$

There  $g^{ik}(u)$ ,  $b^{ik}(u)$  are the arbitrary functions of the phase space coordinates and  $u_x = \partial_x u$ . The Jacobi identity is satisfied under the following conditions:

1. Tensor  $g^{ik}$  is symmetric tensor and it is define some metric on the phase space.

2.  $b^{ik}(u) = -g^{ij} \Gamma_{ij}^k$  and connection  $\Gamma_{ij}^k$  is consistent to metric  $g^{ik}$  and it has zero curvature and zero torsion. Therefore, there are such local coordinates, that  $g^{ik} = \text{const}$ ,  $b^{ik} = 0$ . This (PB) was used for description of the hamiltonian system of the hydrodynamical type. That is systems with functionals of the hydrodynamical type. The density of this functionals does not depend of the derivatives  $u_x^k$ ,  $u_{xx}^k, \dots$  and hamiltonian is functional of the hydrodynamical type also. In opposite this models, the functionals of the closed string model is depended of the derivatives of the string coordinates. As result, we need to introduce additional nonlocal term with the step function  $\varepsilon(x-y) = 2\partial_x^{-1} \delta(x-y)$ , which is the origin of the difficulty of the Jacobi identity proof.

The plan of the paper is following. In the second section we are considered closed string model in the arbitrary background gravity field and antisymmetric B-field as the bihamiltonian system. We suppose, that this model is integrable model for some configurations of the background fields. The bihamiltonity condition and the Jacobi identities for the dual (PB) must be result to the integrability condition, which is restricted the possible configurations of the background fields. The well-known examples of the integrable gravity models with the gravity metric tensor, which is depended of one or of two variables only.

In this paper we are assumed the metric dependence of the arbitrary number of the variables for generality and we did not analyzed the particular cases of the metric dependence. In the third section we are considered three examples of the closed string model in the constant background fields: without constraints, with the second kind constraints and the B-chiral string with the first kind constraints. In the four section we are considered two particles discrete closed string model to show the difference between the (PB) structure under the Gupta-Bleuler method of the quantization and under the quantization of the system with the second kind constraints.

## 2. CLOSED STRING IN THE BACKGROUND FIELDS

The closed string in the background gravity field and the antisymmetric B-field is described by first kind constraints

$$h_1 = \frac{1}{2} g^{ab}(x) [p_a - \alpha B_{ac}(x) x'^c] [p_b - \alpha B_{bd}(x) x'^d] + \frac{1}{2} g_{ab}(x) x'^a x'^b \approx 0, h_2 = p_a x'^a \approx 0, \quad (8)$$

where  $a, b = 0, 1, \dots, D-1$ ,  $x^a(\sigma)$ ,  $p_a(\sigma)$  are the periodical functions on  $\sigma$  with the period on  $\pi$ ,  $\alpha$  -arbitrary parameter. The original (PB) are the symplectic (PB)

$$\{x^a(\sigma), p_b(\sigma')\}_1 = \delta_b^a \delta(\sigma - \sigma') \quad (9)$$

$$\{x^a, x^b\}_1 = \{p_a, p_b\}_1 = 0.$$

The hamiltonian equations of motion of the closed string, in the arbitrary background gravity field and antisymmetric B-field under the hamiltonian

$$H_1 = \int_0^\pi h_1 d\sigma \quad \text{and (PB) } \{, \}_1, \text{ are}$$

$$\begin{aligned} \dot{x}^a &= g^{ab} [p_b - \alpha B_{bc} x'^c], \\ \dot{p}_a &= \alpha B_{ab} g^{bc} p'_c + [g_{ab} - \alpha^2 B_{ac} g^{cd} B_{db}] x''^b \\ &\quad - \frac{1}{2} \frac{\partial g^{bc}}{\partial x^a} p_b p_c - \alpha \frac{\partial}{\partial x^a} (B_{bd} g^{dc}) x'^b p_c + \\ &\quad + \alpha \frac{\partial}{\partial x^b} (B_{ad} g^{dc}) x'^b p_c - \\ &\quad - \frac{1}{2} \frac{\partial}{\partial x^a} [g_{bc} - \alpha^2 B_{bd} g^{de} B_{eb}] x'^b x'^c + \\ &\quad + \frac{\partial}{\partial x_b} [g_{ac} - \alpha^2 B_{ad} g^{de} B_{ec}] x'^b x'^c. \end{aligned} \quad (10)$$

The dual (PB) are obtained from the bihamiltonity condition

$$\begin{aligned} \dot{x}^a &= \{x^a, \int_0^\pi h_1 d\sigma'\}_1 = \{x^a, \int_0^\pi h_2 d\sigma'\}_2 \\ \dot{p}_a &= \{p_a, \int_0^\pi h_1 d\sigma'\}_1 = \{p_a, \int_0^\pi h_2 d\sigma'\}_2, \end{aligned} \quad (11)$$

and they have the following form:

$$\begin{aligned} \{A(\sigma), B(\sigma')\}_2 &= \\ \frac{\partial A}{\partial x^a} \frac{\partial B}{\partial x^b} &[[\omega^{ab}(\sigma) + \omega^{ab}(\sigma')] \varepsilon(\sigma' - \sigma) + \\ &[\Phi^{ab}(\sigma) + \Phi^{ab}(\sigma')] \frac{\partial}{\partial \sigma'} (\sigma' - \sigma) + \\ &[\Omega^{ab}(\sigma) + \Omega^{ab}(\sigma')] \delta(\sigma' - \sigma)] + \\ \frac{\partial A}{\partial p_a} \frac{\partial B}{\partial p_b} &[[\omega_{ab}(\sigma) + \omega_{ab}(\sigma')] \varepsilon(\sigma' - \sigma) + \\ &[\Phi_{ab}(\sigma) + \Phi_{ab}(\sigma')] \frac{\partial}{\partial \sigma'} (\sigma' - \sigma) + \\ &[\Omega_{ab}(\sigma) + \Omega_{ab}(\sigma')] \delta(\sigma' - \sigma)] + \end{aligned} \quad (12)$$

$$\begin{aligned} & \frac{\partial A}{\partial x^a} \frac{\partial B}{\partial p_b} + \frac{\partial A}{\partial p_b} \frac{\partial B}{\partial x^a} [[\omega_b^a(\sigma) + \omega_b^a(\sigma')] \varepsilon(\sigma' - \sigma) + \\ & [\Phi_b^a(\sigma) + \Phi_b^a(\sigma')] \frac{\partial}{\partial \sigma'} (\sigma' - \sigma)] + \\ & [\frac{\partial A}{\partial x^a} \frac{\partial B}{\partial p_b} - \frac{\partial A}{\partial p_b} \frac{\partial B}{\partial x^a}] [\Omega_b^a(\sigma) + \Omega_b^a(\sigma')] \delta(\sigma' - \sigma). \end{aligned}$$

The arbitrary functions A, B,  $\omega$ ,  $\Phi$ ,  $\Omega$  are the functions of the  $x^a(\sigma)$ ,  $p_a(\sigma)$ . The functions  $\omega^{ab}$ ,  $\omega_{ab}$ ,  $\Phi^{ab}$ ,  $\Phi_{ab}$  are the symmetric functions on a, b and  $\Omega^{ab}$ ,  $\Omega_{ab}$  are the antisymmetric functions to satisfy the condition  $\{A, B\}_2 = -\{B, A\}_2$ . The equations of motion under the hamiltonian  $H_2$  and (PB)  $\{, \}_2$  are

$$\begin{aligned} \dot{x}^a &= -2\omega_b^a x^b + 4\omega^{ab} p_b + 2\Phi^{ab} p_b'' - \\ & 2\Phi_b^a x^{''b} + 2\Omega_b^a x'^b - 2\Omega^{ab} p_b' + \int_0^\pi d\sigma' [\omega_b^a x'^b + \\ & \frac{\partial \omega^{ac}}{\partial x^b} x'^c p_c + \frac{\partial \omega^{ac}}{\partial p_b} p_b' p_c] \varepsilon(\sigma' - \sigma) \end{aligned} \quad (13)$$

$$\begin{aligned} & + (\frac{\partial \Phi^{ac}}{\partial x^b} x'^b + \frac{\partial \Phi^{ac}}{\partial p_b} p_b') p_c' - \\ & - (\frac{\partial \Phi_c^a}{\partial x^b} x'^b + \frac{\partial \Phi_c^a}{\partial p_b} p_b') x'^c, \\ \dot{p}_a &= -2\omega_{ab} x^b - 2\Phi_{ab} x^{''b} + 2\Omega_{ab} x'^b + \\ & + 4\omega_a^b p_b + 2\Phi_a^b p_b'' + 2\Omega_a^b p_b' + \int_0^\pi d\sigma' (\omega_{ab} x'^b - \\ & - \frac{\partial^2 \omega^{cd}}{\partial x^a \partial x^b} x'^b p_c p_d - \frac{\partial \omega^{bc}}{\partial x^a} p_b p_c') \varepsilon(\sigma' - \sigma) - \\ & - (\frac{\partial \Phi_{ac}}{\partial x^b} x'^b + \frac{\partial \Phi_{ac}}{\partial p_b} p_b') x'^c + \\ & + (\frac{\partial \Phi_a^c}{\partial x^b} x'^b + \frac{\partial \Phi_a^c}{\partial p_b} p_b') p_c'. \end{aligned} \quad (14)$$

The bihamiltonity condition (11) is led to the two constraints

$$\begin{aligned} & -2\omega_b^a x^b + 4\omega^{ab} p_b + 2\Phi^{ab} p_b'' - \\ & 2\Phi_b^a x^{''b} + 2\Omega_b^a x'^b - 2\Omega^{ab} p_b' + \int_0^\pi d\sigma' [\omega_b^a x'^b + \\ & \frac{\partial \omega^{ac}}{\partial x^b} x'^c p_c + \frac{\partial \omega^{ac}}{\partial p_b} p_b' p_c] \varepsilon(\sigma' - \sigma) \end{aligned} \quad (15)$$

$$\begin{aligned} & + (\frac{\partial \Phi^{ac}}{\partial x^b} x'^b + \frac{\partial \Phi^{ac}}{\partial p_b} p_b') p_c' - \\ & - (\frac{\partial \Phi_c^a}{\partial x^b} x'^b + \frac{\partial \Phi_c^a}{\partial p_b} p_b') x'^c = \\ & = g^{ab} p_b - \alpha g^{ab} B_{bc} x'^c, \\ & -2\omega_{ab} x^b - 2\Phi_{ab} x^{''b} + 2\Omega_{ab} x'^b + \\ & + 4\omega_a^b p_b + 2\Phi_a^b p_b'' + 2\Omega_a^b p_b' + \int_0^\pi d\sigma' (\omega_{ab} x'^b - \\ & - \frac{\partial^2 \omega^{cd}}{\partial x^a \partial x^b} x'^b p_c p_d - \frac{\partial \omega^{bc}}{\partial x^a} p_b p_c') \varepsilon(\sigma' - \sigma) - \\ & - (\frac{\partial \Phi_{ac}}{\partial x^b} x'^b + \frac{\partial \Phi_{ac}}{\partial p_b} p_b') x'^c + \\ & + (\frac{\partial \Phi_a^c}{\partial x^b} x'^b + \frac{\partial \Phi_a^c}{\partial p_b} p_b') p_c' = \end{aligned} \quad (16)$$

$$\begin{aligned} & = \alpha B_{ab} g^{bc} p_c' + [g_{ab} - \alpha^2 B_{ac} g^{cd} B_{db}] x^{''b} - \\ & - \frac{1}{2} \frac{\partial g^{bc}}{\partial x^a} p_b p_c - \alpha \frac{\partial}{\partial x^a} (B_{bd} g^{dc}) x'^b p_c + \\ & + \alpha \frac{\partial}{\partial x^b} (B_{ad} g^{dc}) x'^b p_c - \\ & - \frac{1}{2} \frac{\partial}{\partial x^a} [g_{bc} - \alpha^2 B_{bd} g^{de} B_{eb}] x'^b x'^c + \\ & + \frac{\partial}{\partial x_b} [g_{ac} - \alpha^2 B_{ad} g^{de} B_{ec}] x'^b x'^c. \end{aligned}$$

In really, there is the list of the constraints depending on the possible choice of the unknown functions  $\omega$ ,  $\Phi$ ,  $\Omega$ . In the general case, there are as the first kind constraints as the second kind constraints too. Also, it is possible to solve the constraints equations as the equations for the definition of the functions  $\omega$ ,  $\Phi$ ,  $\Omega$ . We are considered last possibility and we obtained the following consistent solution of the bihamiltonity condition:

$$\begin{aligned} \Phi^{ab} &= 0, \Omega^{ab} = 0, \Phi_b^a = 0, \omega^{ab} = C g^{ab} \\ \omega_{ab} &= \frac{C}{2} \frac{\partial^2 g^{cd}}{\partial x^a \partial x^b} p_c p_d, \omega_b^a = -C \frac{\partial g^{ac}}{\partial x^b} p_c \\ \Phi_{ab} &= -\frac{1}{2} [g_{ab} - \alpha^2 B_{ac} g^{cd} B_{db}], \\ \Omega_{ab} &= \frac{1}{2} (\frac{\partial \Phi_{bc}}{\partial x^a} - \frac{\partial \Phi_{ac}}{\partial x^b}) x'^c - \\ & - (\frac{\partial \Omega_b^c}{\partial x^a} - \frac{\partial \Omega_a^c}{\partial x^b}) p_c, C = \frac{1}{2(n+2)} \\ 2\Omega_b^a &= -\alpha g^{ac} B_{cb}, \frac{\partial g^{ab}}{\partial x^c} x'^c = n g^{ab}, \frac{\partial \omega^{ab}}{\partial p_c} = 0. \end{aligned} \quad (17)$$

The metric tensor  $g^{ab}(x)$  is the homogeneous function of  $x^a$  order n and C is arbitrary constant. In the difference of the (PB) of the hydrodynamical type, we are needed to introduce the separate (PB) for the coordinates of the Minkowski space and for the momenta because, the gravity field is not depend of the momenta. Although, this difference is vanished under the such constraint as  $f(x^a, p_a) \approx 0$ . One can see, that the main term in the (PB) with the metric tensor is the term with the step function  $\varepsilon(\sigma - \sigma')$ . The functions  $\omega_b^a$ ,  $\Omega_{ab}$  are proportional to the connection and to the torsion. The function  $\omega_{ab}$  is proportional to the curvature and the product of the connections. The dual (PB) for the phase space coordinates are:

$$\begin{aligned} \{x^a(\sigma), x^b(\sigma')\}_2 &= [\omega^{ab}(\sigma) + \omega^{ab}(\sigma')] \varepsilon(\sigma' - \sigma), \\ \{p_a(\sigma), p_b(\sigma')\}_2 &= [\omega_{ab}(\sigma) + \omega_{ab}(\sigma')] \varepsilon(\sigma' - \sigma) + \\ & [\Phi_{ab}(\sigma) + \Phi_{ab}(\sigma')] \frac{\partial}{\partial \sigma'} (\sigma' - \sigma) + \\ & [\Omega_{ab}(\sigma) + \Omega_{ab}(\sigma')] \delta(\sigma' - \sigma), \\ \{x^a(\sigma), p_b(\sigma')\}_2 &= [\omega_b^a(\sigma) + \omega_b^a(\sigma')] \varepsilon(\sigma' - \sigma) + \\ & + [\Omega_b^a(\sigma) + \Omega_b^a(\sigma')] \delta(\sigma' - \sigma). \end{aligned} \quad (18)$$

The functions  $\omega^{ab}$ ,  $\omega_{ab}$ ,  $\Phi_{ab}$ ,  $\Omega_{ab}$ ,  $\omega_a^b$ ,  $\Omega_a^b$  are defined in (17). It is rather easy to prove the Jacobi identities for the local part of the dual (PB)  $\{, \}_2$ . It does not understand, how to prove the Jacobi identities for the

nonlocal part of it. The principal term of the Jacobi identities with the step functions only is the term with the structure function  $\omega^{ab}(x)$ :

$$\begin{aligned} & \left[ \frac{\partial g^{ab}(\sigma)}{\partial x^d} [g^{dc}(\sigma) + g^{dc}(\sigma'')] - \frac{\partial g^{ac}(\sigma)}{\partial x^d} [g^{db}(\sigma) \right. \\ & \left. + g^{db}(\sigma')] \right] \varepsilon(\sigma' - \sigma) \varepsilon(\sigma'' - \sigma) + \\ & \left[ \frac{\partial g^{cb}(\sigma')}{\partial x^d} [g^{da}(\sigma') + g^{dc}(\sigma)] - \frac{\partial g^{ab}(\sigma')}{\partial x^d} [g^{dc}(\sigma' \right. \\ & \left. + g^{dc}(\sigma'')] \right] \varepsilon(\sigma - \sigma') \varepsilon(\sigma'' - \sigma') + \\ & \left[ \frac{\partial g^{ac}(\sigma'')}{\partial x^d} [g^{db}(\sigma'') + g^{db}(\sigma')] - \frac{\partial g^{cb}(\sigma'')}{\partial x^d} [g^{da}(\sigma'') \right. \\ & \left. + g^{da}(\sigma)] \right] \varepsilon(\sigma - \sigma'') \varepsilon(\sigma' - \sigma'') = 0. \end{aligned} \quad (19)$$

It is possible to reduce this condition to the unique equation

$$\begin{aligned} & \left[ \frac{\partial g^{ab}(\sigma)}{\partial x^d} [g^{dc}(\sigma) + g^{dc}(\sigma'')] - \frac{\partial g^{ac}(\sigma)}{\partial x^d} [g^{db}(\sigma) \right. \\ & \left. + g^{db}(\sigma')] \right] \varepsilon(\sigma' - \sigma) \varepsilon(\sigma'' - \sigma) = 0. \end{aligned} \quad (20)$$

One of the possible ways of the solution of this problem is the consideration of the metric tensor on the phase space to count the contribution of the structure functions  $\Phi$  and  $\Omega$  to this expression. The second possible way is the consideration of the second kind constraints from the list of the constraints (15,16), instead of the solution (17). It is necessary to introduce this constraints to the initial model with the hamiltonian  $H_1$ , the (PB)  $\{, \}_1$  and to obtain the bihamiltonity condition in this case. As we will see later on the some examples, the second kind constraints of the  $f(x^a, p_a) \approx 0$  type can lead to the nonlocal Dirac bracket with the step function from the one side, and they can introduce the dependence of the metric tensor of the momenta on the solutions of this constraints from the other side. At present, these problems are under the consideration.

### 3. CONSTANT BACKGROUND FIELDS

The bihamiltonity condition (15,16) is reduced to the following constraints on the phase space:

$$\begin{aligned} & -2\omega^a_b x^b + 4\omega^{ab} p_b + 2\Phi^{ab} p'_b - 2\Phi^a_b x''^b + \\ & 2\Omega^a_b x'^b = g^{ab} p_b - \alpha g^{ab} B_{bc} x'^c, \Omega^{ab} = 0, \\ & -4\omega_{ab} x^b - 2\Phi_{ab} x''^b + 2\Omega_{ab} x'^b + 4\omega^b_a p_b + \\ & 2\Phi^b_a p'_b + 2\Omega^b_a p'_b = \alpha B_{ab} g^{bc} p'_c + \\ & + [g_{ab} - \alpha^2 B_{ac} g^{cd} B_{db}] x''^b. \end{aligned} \quad (21)$$

There is the unique solution without constraints

$$\begin{aligned} \omega^{ab} &= \frac{1}{4} g^{ab}, 2\Omega^a_b = -\alpha g^{ac} B_{cb} = \alpha B_{bc} g^{ca}, \\ -2\Phi_{ab} &= g_{ab} - \alpha^2 B_{ac} g^{cd} B_{db}. \end{aligned} \quad (22)$$

The rest structure functions are equal zero. In this section we are supplemented the bihamiltonity condition (11) by the mirror transformations of the integrals of motion.

$$\dot{x}^a = \{x^a, H_1\}_1 = \{x^a, \pm H_2\}_{\pm 2}. \quad (23)$$

The dual (PB) are:

$$\begin{aligned} \{x^a(\sigma), x^b(\sigma')\}_{\pm 2} &= \pm \frac{1}{2} g^{ab} \varepsilon(\sigma' - \sigma), \\ \{x^a(\sigma), p_b(\sigma')\}_{\pm 2} &= \mp \alpha g^{ac} B_{cb} \delta(\sigma' - \sigma), \\ \{p_a(\sigma), p_b(\sigma')\}_{\pm 2} &= \\ &= \mp [g_{ab} - \alpha^2 B_{ac} g^{cd} B_{db}] \frac{\partial}{\partial \sigma'} \delta(\sigma' - \sigma). \end{aligned} \quad (24)$$

The dual dynamical system

$$\dot{x}^a = \{x^a, \pm H_2\}_1 = \{x^a, H_1\}_{\pm 2} \quad (25)$$

is the left(right) chiral string :

$$\dot{x}^a = \pm x'^a, \dot{p}_a = p'_a. \quad (26)$$

In the terms of the Virasoro operators

$$\begin{aligned} L_k &= \frac{1}{4\pi} \int_0^\pi (h_1 + h_2) e^{ik\sigma} d\sigma, \\ \bar{L}_k &= \frac{1}{4\pi} \int_0^\pi (h_1 - h_2) e^{ik\sigma} d\sigma \end{aligned} \quad (27)$$

the first kind constraints form the  $\text{Vir} \oplus \text{Vir}$  algebra under the (PB)  $\{, \}_1$ :

$$\begin{aligned} \{L_n, L_m\}_1 &= -i(n-m)L_{n+m}, \\ \{\bar{L}_n, \bar{L}_m\}_1 &= -i(n-m)\bar{L}_{n+m}, \\ \{L_n, \bar{L}_m\}_1 &= 0. \end{aligned} \quad (28)$$

The dual set of the integrals of motion is obtained from initial it by the mirror transformations

$$H_1 \rightarrow H_2, L_0 \rightarrow \pm L_0, \bar{L}_0 \rightarrow \bar{L}_0, \tau \rightarrow \sigma \quad (29)$$

and by the contraction of the first kind constraints algebra  $L_n=0$ , or,  $\bar{L}_n = 0, n \neq 0$ .

#### 3.1. SECOND KIND CONSTRAINTS

Another way to obtain the dual brackets is the imposition of the second kind constraints on the initial dynamical system, by such manner, that  $F_i = F_k$  for  $i \neq k, i, k = 1, 2, \dots$  on the constraints surface  $f(x^a, p_a) = 0$ .

Let us consider the closed string model with  $B_{ab}=0$  for simplicity:

$$\begin{aligned} h_1 &= \frac{1}{2} g^{ab} p_a p_b + \frac{1}{2} g_{ab} x'^a x'^b, \\ h_2 &= p_a x'^a. \end{aligned} \quad (30)$$

The constraints  $f_a^{(-)}(x, p) = p_a - g_{ab} x'^b \approx 0$ , or  $f_a^{(+)}(x, p) = p_a + g_{ab} x'^b \approx 0$  (do not simultaneously) are the second kind constraints:

$$\begin{aligned} \{f_a^{(\pm)}(\sigma), f_b^{(\pm)}(\sigma')\}_1 &= C_{ab}^{(\pm)}(\sigma - \sigma') = \\ &= \pm 2g_{ab} \frac{\partial}{\partial \sigma'} \delta(\sigma' - \sigma). \end{aligned} \quad (31)$$

The inverse matrix  $(C_{ab}^{(\pm)})^{-1}$  has following form:

$$C^{(\pm)ab}(\sigma - \sigma') = \pm \frac{1}{4} g^{ab} \varepsilon(\sigma' - \sigma). \quad (32)$$

There is only one set of the constraints, because consistency conditions

$$\begin{aligned} \{f^{(\pm)}(\sigma), H_1\}_1 &= f'^{(\pm)}(\sigma) \approx 0, \dots \\ \{f^{(n)(\pm)}(\sigma), H_1\}_1 &= f^{(n+1)(\pm)}(\sigma) \approx 0 \end{aligned} \quad (33)$$

are not produce the new sets of constraints. By using the standard definition of the Dirac bracket, we are obtained following Dirac brackets for the phase space coordinates:

$$\begin{aligned} \{x^a(\sigma), x^b(\sigma')\}_D &= \pm \frac{1}{4} g^{ab} \varepsilon(\sigma' - \sigma), \\ \{p_a(\sigma), p_b(\sigma')\}_D &= \mp \frac{1}{2} g_{ab} \frac{\partial}{\partial \sigma'} \delta(\sigma' - \sigma), \\ \{x^a(\sigma), p_b(\sigma')\}_D &= \frac{1}{2} \delta_b^a \delta(\sigma' - \sigma). \end{aligned} \quad (34)$$

The equations of motion under the hamiltonians  $H_1 = h_1$ ,  $H_2 = h_2$  and Dirac bracket

$$\begin{aligned} \dot{x}^a &= \{x^a, H_1\}_D = \{x^a, H_2\}_D = g^{ab} p_b = \pm x'^a, \\ \dot{p}_a &= \{p_a, H_1\}_D = \{p_a, H_2\}_D = g_{ab} x''^b = \pm p'_a \end{aligned} \quad (35)$$

are coincide on the constraints surface. The dual brackets  $\{, \}_{(\pm)}$  are coincide with the Dirac brackets also. The contraction of the algebra of the first kind constraints means that the integrals of motion  $H_1 = H_2$  are coincide on the constraints surface too.

### 3.2 B-CHIRAL STRING

Let us consider the following constraint from the list (24)

$$\varphi_a = p_a + \beta B_{ab} x'^b \approx 0 \quad (36)$$

The consistency condition

$$\begin{aligned} \{\varphi_a, H_1\}_1 &= (\alpha + \beta) B_{ab} g^{bc} \varphi'_c + \\ &+ [g_{ab} - (\alpha + \beta)^2 B_{ac} g^{cd} B_{db}] x''^b \end{aligned} \quad (37)$$

shows that under the additional condition on the B-field

$$g_{ab} = (\alpha + \beta)^2 B_{ac} g^{cd} B_{db}, \quad (38)$$

the constraints  $\varphi_a \approx 0$  are first kind constraints. The motion equations are:

$$\begin{aligned} \dot{x}^a &= -(\alpha + \beta) g^{ab} B_{bc} x'^c, \\ \dot{p}_a &= -(\alpha + \beta) B_{ab} g_{bc} p'_c, \ddot{x}^a = x''^a. \end{aligned} \quad (39)$$

This model is the bihamiltonian model under (PB) (26) also. The B-chiral string model is dual to the chiral model also.

## 4. TWO PARTICLES DISCRETE STRING

In this section we are considered two particles discrete closed string model, as two relativistic particles model, to show the difference between the Dirac brackets quantization and the Gupta-Bleuer quantization methods.

Two and three pieces discrete string in Gupta-Bleuer method of quantization was considered in the paper [8] and it is described by the following constraints:

$$h = \frac{1}{4}(p^2 + q^2) + \omega_0^2 r^2, \quad (40)$$

$$f_1 = pq \approx 0, f_3 = qr \approx 0,$$

$$f_2 = pr \approx 0, f_4 = \frac{1}{4}q^2 - \omega_0^2 r^2.$$

This model is the two relativistic particles system with the oscillator interaction. The constraints (40) are two particles discrete analog of the Virasoro constraints and p, q, r are the collective variables  $p^a = p_2^a + p_1^a$ ,  $q^a = p_2^a - p_1^a$ ,  $r^a = x_2^a - x_1^a$ .

Under the hamiltonian  $H=h$  and the canonical (PB)  $\{r^a, q^b\} = \eta^{ab}$ , the constraints  $f_i$  are the second kind constraints:

$$\{f_1, f_2\} = 2p^2 \neq 0, \quad (41)$$

$$\{f_3, f_4\} = q^2 + 4\omega_0^2 r^2 \neq 0.$$

The string coordinates are satisfied to following Dirac brackets:

$$\{r_a, r_b\}_D = \frac{r_b q_a - r_a q_b}{q^2 + \omega_0^2 r^2}, \{q_a, q_b\}_D = 4\omega_0 \{r_a, r_b\},$$

$$\{r_a, q_b\}_D = 2(\eta_{ab} - \frac{p_a p_b}{p^2} - \frac{q_a q_b + 4\omega_0^2 r_a r_b}{q^2 + 4\omega_0^2 r^2}). \quad (42)$$

In the terms of the amplitudes  $a_a^{(+)}, a_a$  of the equations of motion solutions

$$\begin{aligned} r_a &= a_a e^{i\omega_0 \tau} + a_a^{(+)} e^{-i\omega_0 \tau}, \\ q_a &= 2i\omega_0 (a_a e^{i\omega_0 \tau} - a_a^{(+)} e^{-i\omega_0 \tau}) \end{aligned} \quad (43)$$

they have the form:

$$\begin{aligned} \{r_a, r_b\}_D &= \frac{i(a_b^{(+)} a_a - a_a^{(+)} a_b)}{2\omega_0 a_k^{(+)} a_k}, \\ \{q_a, q_b\}_D &= 2\omega_0 \{r_a, r_b\}_D, \end{aligned} \quad (44)$$

$$\{r_a, q_b\}_D = 2(\eta_{ab} - \frac{p_a p_b}{p^2} - \frac{a_a^{(+)} a_b + a_b^{(+)} a_a}{2a_k^{(+)} a_k})$$

This Dirac brackets is possible to solve in the terms of the variables a,  $a_a^{(+)}$ .

$$\{a_a, a_b^{(+)}\}_D = \frac{i}{2\omega_0} (\eta_{ab} - \frac{p_a p_b}{p^2} - \frac{i a_a^{(+)} a_b}{2\omega_0 a_k^{(+)} a_k}). \quad (45)$$

The hamiltonian and the linear combinations of the constraints  $f_1, f_4$  have following form:

$$H = \frac{1}{4} p^2 + 4\omega_0^2 a_k^{(+)} a_k, \quad (46)$$

$$f_1 = p_k a_k, f_2 = p_k a_k^{(+)}, f_3 = a_k a_k, f_4 = a_k^{(+)} a_k^{(+)}. \quad (46)$$

Under the quantization  $[, ] \rightarrow \{, \}_D$ ,  $a_k, a_k^{(+)} \rightarrow$  operators  $a_k, a_k^{(+)}$  the commutation relation is

$$[a_k, a_l^{(+)}] = \frac{-i}{2\omega_0} (\eta_{kl} - \frac{p_k p_l}{p^2}) + \frac{1}{2\omega_0} (a_i^{(+)} a_i)^{-1} a_k^{(+)} a_l. \quad (47)$$

Last term of the commutation relation has transposed Lorentz indices k,l. The wave function

$$|\Psi_n(p)\rangle = a_{k_1}^{(+)} a_{k_2}^{(+)} \dots a_{k_n}^{(+)} \Psi_{k_1 k_2 \dots k_n}(p) |0\rangle \quad (48)$$

of the physical states must be the own function of the hamiltonian H on the constraints surface. Let us consider the two particles excited state, for example.

$$H |\Psi_2(p)\rangle = (\frac{1}{4} p^2 - 4\omega_0^2) \Psi_{k_1 k_2}(p) a_{k_1}^{(+)} a_{k_2}^{(+)} |0\rangle +$$

terms, which are proportional to the expressions:

$$p_{k_1} \Psi_{k_1 k_2}(p), \eta_{k_1 k_2} \Psi_{k_1 k_2}(p), a_l^{(+)} a_l^{(+)} \Psi_{k_1 k_2}(p). \quad (49)$$

Consequently, we must to impose additional conditions on the wave function  $p_k \Psi_{kl}, \eta_{kl} \Psi_{kl}=0$  to satisfy the request about the own function. Last term  $a_i^{(+)} a_i^{(+)} \Psi_{kl}$  is vanished on the constraints surface. In contrast to the nonlinear and to the nonlocal Dirac bracket (45), we have the canonical (PB) and two first kind constraints  $p_k a_k \approx 0, a_k a_k \approx 0$  in the Gupta-Bleuler method of the quantization. The first kind constraints H,  $f_1, f_3$  are imposed on the vector states and they are led to the following equations on the wave function:

$$(p - 16\omega_0^2) \Psi_{k_1 k_2 \dots k_n}(p) = 0, \quad (50)$$

$$p_k \Psi_{k k_2 \dots k_n}(p) = 0, \eta_{kl} \Psi_{kl \dots k_n}(p) = 0.$$

## 5. RELATIVISTIC PARTICLE IN THE CONSTANT ELECTROMAGNETIC FIELD

The relativistic particle in the constant background electromagnetic field is described by the Hamiltonian

$$H = \frac{1}{2} [(p_a + i\beta B_{ab} x_b)^2 + m^2]. \quad (51)$$

The electromagnetic field is  $A_a(x) = -2F_{ab} x_b = -B_{ab} x_b$ . The simplest constraint from (21) is

$$\phi_a = p_a + i\alpha B_{ab} x_b \approx 0. \quad (52)$$

The consistency condition

$$\{\phi_a, H\} = i\alpha(\alpha + \beta) B_{ab} \phi_b \quad (53)$$

shows that there is a unique set of constraints if  $\alpha + \beta = 0$ . They are the second class constraints:

$$\{\phi_a, \phi_b\} = 2i\alpha B_{ab}.$$

There is the following algebra of the phase space coordinates under the Dirac bracket

$$\{x_a, x_b\}_D = \frac{-i}{2\alpha} (B^{-1})_{ab}, \{x_a, p_b\}_D = \frac{1}{2} \eta_{ab}, \quad (54)$$

$$\{p_a, p_b\}_D = \frac{-i\alpha}{2} B_{ab}.$$

The motion equations under the Dirac bracket

$$\dot{x}_a + 2i\alpha B_{ab} x_b = 0, \dot{p}_a + 2\alpha B_{ab} x_b = 0 \quad (55)$$

have the solution  $x_a(\tau) = (e^{-2i\alpha B\tau})_{ab} x_b(0)$ .

The quantization of the Dirac bracket results in the following commutation relations:

$$[x_a, x_b] = \frac{1}{2\alpha} (B^{-1})_{ab}, [p_a, p_b] = \frac{\alpha}{2} B_{ab}, \quad (56)$$

$$[x_a, p_b] = \frac{i}{2} \eta_{ab}.$$

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