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BEAM INSTABILITY CAUSED BY STOCHASTIC PLASMA DENSITY FLUCTUATIONS

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Beam instability in a plasma with the density varying stochastically with respect to space or time is investigated. In case of spatial variations, the system of equations in moments of arbitrary order is obtained and analyzed. It is shown that each moment in the sequence grows faster than the preceding one. As a result an intermittent character of development of the instability arises, and the appearance of some critical length within which the amplification of a regular signal is yet possible. In case of temporal variations, the maximum time interval of signal destitution due to fluctuation interference is estimated.

Introduction

The theory of interaction of a beam with plasma is developed (see, for example, [1]). The interaction between a beam and plasma gives rise to instabilities that could lead to microwave amplification. At the time, fluctuations ever present in plasma also rise. Using plasma-beam instability for amplifying signals or heating plasma, it is necessary to know a ratio between the growth rate of a regular signal component and that of fluctuations. Beguiashvily *et al*. [2] and Virchenko *et al*. [3] studied plasma-beam interaction in a plasma with the number density varying stochastically in the x direction. Their analysis was restricted by the correlation theory of instability, i.e. the dynamics of variations of the first and second moments.

This study is dedicated to the investigation of a beam instability in a plasma with the number density varying stochastically with respect to space or time. In case of spatial variations, as distinct from [3], the system of equations in moments of arbitrary order is obtained and analyzed. It is shown that each moment in the sequence grows faster than the preceding one. As a result of such a dependence of the growth rate upon the number of a moment, an intermittent character of development of the instability arises [4], as well as the appearance of some critical length of interaction space within which the amplification of a regular signal is yet possible. The latter conclusion is due to the fact that the growth rate of the second moments is more than two times the growth rate of a regular component of the signal. In case of stochastic temporal variations, the equations describing the dynamics of first and second moments are obtained, their growth rates are found, and the maximum time interval of signal destitution due to fluctuation interference is estimated.

Basic equations

Let an electron beam with a neutralizing ion background have a number density n_b and velocity υ_b in the *x* direction in the plasma. Perturbations in the beam number density $\tilde{n}_b$ and velocity $\tilde{\upsilon}_b$ satisfy the equation of motion and the continuity equation

$$\frac{\partial}{\partial t}\tilde{\upsilon}_b+\upsilon_b\frac{\partial}{\partial x}\tilde{\upsilon}_b=-\frac{e}{m}E(x,t), \qquad (1)$$

$$\tilde{n}_b+\frac{\partial}{\partial x}\left(n_b\tilde{\upsilon}_b+\tilde{n}_b\upsilon_b\right)=0 \qquad (2)$$

where $E(x,t)$ is the electric field intensity.

Perturbations in the ambient plasma number density $\tilde{n}_p$ and velocity $\tilde{\upsilon}_p$ satisfy the equations

$$\frac{\partial}{\partial t}\tilde{n}_p+\frac{\partial}{\partial x}\left(n_p\upsilon_p\right)=0, \qquad (3)$$

$$\frac{\partial}{\partial t}\tilde{\upsilon}_p=-\frac{e}{m}E(x,t) \qquad (4)$$

where n_p is an unperturbed plasma number density which is a stochastic function of time or space coordinates, and its probability properties are known. The electric field intensity and variations in beam and ambient plasma number densities satisfy Maxwell's equation

$$\frac{\partial}{\partial x}E(x,t)=-4\pi e\left(\tilde{n}_p+\tilde{n}_b\right). \qquad (5)$$

Spatially inhomogeneous plasma

Let all variables have the functional form $\exp(-i\omega t)$, and the plasma be spatially inhomogeneous, $n_p=n_p(x)$. On rearranging and choosing boundary conditions in such a way that the constant of integration turns zero, from (1) to (5), we obtain the equations in the Fourier transformed electric field:

$$\left(i\omega+\upsilon_b\frac{\partial}{\partial x}\right)^2\varepsilon(x)E_\omega+\omega_b^2E_\omega=0, \qquad (6)$$

$$\varepsilon(x)=1-\frac{\omega_p^2}{\omega^2},\ \omega_p^2=\frac{4\pi e^2n_p(x)}{m},\ \omega_b^2=\frac{4\pi e^2n_b}{m}.$$

Substituting $E_\omega(x)$ into (6) from $D(\omega,x)\equiv\varepsilon(\omega,x)\mathrm{E}_\omega(x)\exp\left(i\omega_x/\upsilon_b\right)$, we obtain the

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equation similar to (3) for a spatial oscillator with a stochastically changing frequency $\omega_b^2 / \upsilon_b^2 \varepsilon(x)$

$$\frac{d^2 D}{d x^2} + \frac{\omega_b^2}{\upsilon_b^2 \varepsilon(x)} D = 0. \quad (7)$$

Let us consider, as an example, a case of $\varepsilon \langle 0$. Suppose plasma number density fluctuations are $n_p = n_{p0}(1+z_1(x))$, and z_1 is a stationary Gaussian process with a zero mean; then, considering that the amplitude of fluctuation is small, substituting $z \equiv z_1(x)\omega_{p0}^2 / (\omega^2 \varepsilon_p) \langle\langle 1$ into (7), and changing the variables $\varepsilon_{p0} = 1 - \omega_{p0}^2 / \omega^2$, $\omega_{p0}^2 \equiv \langle \omega_p^2 \rangle$, and $\tau \equiv \omega_b x / \upsilon_b \sqrt{|\varepsilon_{p0}|}$, we obtain the set of equations of the first order

$$\dot{D} = -u, \qquad \dot{u} = -(1+z(\tau))D. \quad (8)$$

From Equation (8), a set of equations in moments of any order m could be obtained. To achieve this, multiplying both sides of the first equation in (8) by $u^{m-n}D^{n-1}$, and the second equation by $D^n u^{m-n-1}$, adding these equations and performing the ensemble averaging, we obtain the set of equations of the order n in moments

$$\langle u^{m-n} D^n \rangle'_\tau = -n \langle u^{m-n+1} D^{n-1} \rangle - (m-n)\langle (1+z) D^{n+1} u^{m-n-1} \rangle \quad (9)$$

To split the correlations $\langle z D^{n+1} u^{m-n-1} \rangle$, we use the method of variational derivatives [5] and the relation derived by this method

$$\langle z(t) R[z(t)] \rangle = \int_t \langle z(t) z(\tau) \rangle \left\rangle \frac{\delta R[z(\tau)]}{\delta z(\tau)} \right\langle d\tau \quad (10)$$

where $R[z]$ is an arbitrary functional on z, and z is a Gaussian process with a zero mean. Substituting (10) in (9), and calculating respective variational derivatives, we find

$$\langle u^{m-n} D^n \rangle'_\tau = -n \langle u^{m-n+1} D^{n+1} \rangle - (m-n)(1+z) D^{n+1} u^{m-n-1} \rangle + \frac{B}{2}(m-n-1)(m-n)\langle u^{m-n-2} D^{n+2} \rangle \quad (11)$$

where

$$B\langle u^{m-n-2} D^{n+2} \rangle \equiv \int_t d\tau \langle z(t) z(\tau) \rangle \langle u^{m-n-2} D^{n+2} \rangle \approx \approx \sigma_0^2 r_0 \langle u^{m-n-2} D^{n+2} \rangle,$$

σ_0^2 is the variance, r_0 is the dimensionless radius of correlation of the stochastic process $z(\tau)$.

To analyze the stability of (11), we take out all the moments proportional to $e^{\lambda \tau}$. The determinant of the resulting set of equations, calculated applying Rauss's algorithm, yields the following recursion relation in the coefficients of the characteristic equation

$$Det_n(\lambda) \equiv A_n^m(\lambda) = 0, \; A_0^m = \lambda, A_1^m = \lambda^2 - m,$$

$$A_2^m = \frac{\lambda}{2}(\lambda^2 - m) - \lambda(m-1) - m(m-1)\frac{B}{2},$$

$$A_n^m \neq 0, \forall n < m,$$

$$A_n^m = \frac{\lambda}{n} A_{n-1}^m - \left(\frac{m}{n-1} - 1\right) A_{n-2}^m - \left(\frac{m}{n-2} - 1\right)(m-n+1)\frac{B}{2} A_{n-3}^m, \quad (12)$$

In particular, if $m=1$ (first moments), the characteristic equation takes the form

$$\lambda^2 - 1 = 0, \quad (13)$$

and if $m = 2$ (second moments)

$$\lambda^3 - 4\lambda - 2B = 0. \quad (14)$$

Obviously, the second moment growth rate is more than two times the first moment growth rate. Similarly to [3], we define the variance in the dimensionless form

$$\Delta = \left(\langle D^2 \rangle - \langle D \rangle^2 \right) / \langle D \rangle^2 . \quad (15)$$

In order to amplify a signal, the magnitude of this quantity should be much less than the unity ($\Delta\langle\langle 1$), otherwise the signal will be destructed by fluctuation interference. Substituting the first and second moments from (13) and (14) into (15), and returning to dimensional variables, from the condition $\Delta\langle\langle 1$, we obtain the following expression for the critical length of interaction space within which the amplification of a regular signal is yet possible:

$$x \langle\langle x_m = 4\omega^4 \upsilon_b^2 / \left(\omega_{p0}^2 \omega_b^2 |\varepsilon_{p0}| \sigma_1^2 r_1 \right) \quad (16)$$

where σ_1^2 is the variance, r_1 is the radius of correlation of the stochastic process $z(x)$.

If the magnitude of the amplitude of fluctuations equals zero (*B=0*), using the recursion relation for the coefficients, it can be shown that both $Det_m(m) = 0$, and $\lambda = m$ is the maximum root of Equation (12). Hence, even if the amplitude of fluctuations can be neglected, the difference between growth rates of two successive moments equals the unity.

Of each two successive moments, the next grows faster than the former, but the dimensional variance does not. The nonzero amplitude of fluctuations leads to an additional growth in a difference between the growth rates and, therefore, to an increase in the variance. As Molchanov *et al.* [4] have shown, such a peculiarity of growth of the moments indicates an intermittent character of oscillator motion. Note that such a growth of higher harmonics is characteristic of systems described by Langevin's equations of the first order [6], and, therefore, in accordance with [4], intermittent motion should also exist in such systems.

Plasma with temporal fluctuations in the density

Let us consider a case of the plasma with the number density varying stochastically with respect to time $n_p = n_{p0}(1+z(t))$ where $z(t)$ is a stationary Gaussian process with a zero mean. Because the expressions obtained are cumbersome, we restrict our investigation to a correlation approximation. Assuming all unknowns to vary harmonically with the x coordinate e^{-ikx}, introducing the dimensionless variables $\tau \equiv k\upsilon_b t$, $\alpha \equiv \omega_{p0}/k\upsilon_b$, $\beta \equiv \omega_b / k\upsilon_b$, and eliminating the variable E by using its value from (5), from (1) to (5) we obtain the following set of equations in spatial and temporal Fourier components of perturbations of the number density and velocity of a beam and plasma

$$\frac{\partial \tilde{n}_b}{\partial \tau} + i\,\tilde{n}_b + i\tilde{\upsilon}_b = 0\,, \quad (17)$$

$$\frac{\partial \tilde{\upsilon}_b}{\partial \tau} + i\tilde{\upsilon}_b + i\beta\tilde{n}_b + i\alpha^2\tilde{n}_p = 0\,, \quad (18)$$

$$\frac{\partial \tilde{n}_p}{\partial \tau} + i\tilde{\upsilon}_p(1+z(\tau)) = 0\,, \quad (19)$$

$$\frac{\partial \tilde{\upsilon}_p}{\partial \tau} + i\alpha^2\tilde{n}_p + i\beta^2\tilde{n}_b = 0\,. \quad (20)$$

Performing the ensemble averaging of (17) to (20), splitting the correlation $\langle z\tilde{\upsilon}_p\rangle$ with (10), and taking out all the variables proportional to the $e^{-i\lambda\tau}$ for analysis, we obtain the set of algebraic equations of the fourth order in first moments of number density and velocity perturbations, the determinant of which yields the classical dispersion relation for a beam-plasma system

$$\Delta_p\Delta_b = \alpha^2\beta^2, \Delta_p \equiv \lambda^2 - \alpha^2, \Delta_b \equiv (\lambda-1)^2 - \beta^2 \quad (21)$$

which has an unstable solution at points of intersection of plasma (Δ_p = 0) and beam ($\Delta_b = 0$) resonances of oscillators ($\lambda = 1+\delta, \alpha = 1$) with the maximum growth rate of $\gamma_1 = \left(\frac{1}{2}\beta^2\right)^{1/3}$. (Under the natural assumption of $\omega_{p0} \rangle\rangle \omega_b$.)

Having obtained a set of equations in second moments from (17) to (20), and having performed all the operations mentioned above, we obtain two matrix equations of the third order

$$\begin{pmatrix} -\frac{\lambda}{2}+1 & 1 & 0 \\ \beta^2 & -\lambda+2 & 1 \\ 0 & \beta^2 & -\frac{\lambda}{2}+1 \end{pmatrix} \begin{pmatrix} \langle n_b^2\rangle \\ \langle n_b\upsilon_b\rangle \\ \langle \upsilon_b^2\rangle \end{pmatrix} = \alpha^2 \begin{pmatrix} 0 \\ \langle n_p n_b\rangle \\ \langle n_p\upsilon_b\rangle \end{pmatrix} \quad (22)$$

$$\begin{pmatrix} -\frac{\lambda}{2} & 1 & B \\ \alpha^2 & -\lambda & 1 \\ 0 & \alpha^2 & -\frac{\lambda}{2} \end{pmatrix} \begin{pmatrix} \langle n_p^2\rangle \\ \langle n_p\upsilon_p\rangle \\ \langle \upsilon_p^2\rangle \end{pmatrix} = -\beta^2 \begin{pmatrix} 0 \\ \langle n_p n_b\rangle \\ \langle n_b\upsilon_p\rangle \end{pmatrix} \quad (23)$$

in second moments of number density and velocity perturbations of beam and plasma oscillators ($B \equiv -ik\upsilon_b\sigma_0^2\tau_R/2$; σ_0^2, τ_R are the variance and the time of correlation of the stochastic process $z(\tau)$), and one matrix equation of the fourth order

$$\begin{pmatrix} -\lambda+1 & 1 & 1 & 0 \\ \beta^2 & -\lambda+1 & 0 & 1 \\ \alpha^2 & 0 & -\lambda+1 & 1 \\ 0 & \alpha^2 & \beta^2 & -\lambda+1 \end{pmatrix} \begin{pmatrix} \langle n_p n_b\rangle \\ \langle n_p\upsilon_b\rangle \\ \langle n_b\upsilon_p\rangle \\ \langle \upsilon_p\upsilon_b\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^2\langle n_p^2\rangle \\ \beta^2\langle n_b^2\rangle \\ \alpha^2\langle n_p\upsilon_p\rangle + \beta^2\langle n_b\upsilon_b\rangle \end{pmatrix} \quad (24)$$

in their cross-correlation coefficients. Corresponding to Sets (22) to (24), the characteristic equation of the order ten is similarly to Set (12).

To its analyze, we choose the structure of field under which the instability of the first moments occurs ($k\upsilon_b = \omega_p$). In this case, the maximum growth rate of the second moments is localized within the region near $\lambda = 2$. Expanding the characteristic equation in terms of small parameters δ ($\lambda = 2+\delta$) and β, we obtain the expression in the second moment growth rate

$$\gamma_2 = 2^{\frac{2}{3}}\beta^{\frac{2}{3}}\left(1+\beta^{\frac{2}{3}}B\Big/2^{\frac{2}{3}}9\right) \quad (25)$$

i.e. the second moment growth rate is two times the first moment growth rate. Using (15), it is easy to estimate a maximum time interval over which fluctuations are unable to destroy a regular signal

$$t\langle t_m \equiv 9(2\beta)^{-1/3}\Big/\left(\omega_b k\upsilon_b\sigma_0^2\tau_R\right) \quad (26)$$

where σ_0^2 is the variance, τ_R is the correlation time of the stochastic process $z(\tau)$.

Conclusion

Two kinds of the systems with fluctuations are investigated by the moment method. The first system has spatial and the second one – temporal instabilities that lead to amplification of an initial regular signal. The presence of fluctuations changes the situation principally. The fluctuations that always exist in unstable systems lead to destruction of the regular signal amplification. The critical length or time determines the possibility for regular signal amplification. The method of the variational derivatives allows us to obtain the solution in absolutely different physical situations and for quite general model of the fluctuations, as it is Gaussian random signal. It is naturally to suppose that the result obtained here has general field of application and have to be taken into consideration if a system with distributed interaction is analyzed.

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