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FOUR-DIMENSIONAL INTEGRAL EQUATIONS FOR THE MHD DIFFRACTION WAVES IN PLASMA

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Introduction

Let consider low frequency oscillations those can be excited and propagate in plasma. Therefore we shall limit ourselves sufficiently slow development of macroscopic processes. Such a supposition is necessary for the possible application in hydrodynamic description that together with electromagnetic one for the field is expressed by equations of magnetic hydrodynamics [1] relatively to medium velocity $\mathbf{u}(\mathbf{r},t)$, intensity of a magnetic field $\mathbf{b}(\mathbf{r},t)$ and a density of media $\rho(\mathbf{r},t)$.

In linear magnetic hydrodynamics a wave packet contains seven types of characteristics: two rapid magnetoacoustic waves, two slow magnetoacoustic waves, two Alfven waves and entropy wave. The state vector Ψ describing the considered wave packet at each spatial point $M(x,y,z)$ and at any time moment t is determined for initial and boundary conditions of the above waves.

Under definite physical conditions, for example, at the meeting of the solar wind with the Earth magnetic field, or at sudden inclusion of an electromagnetic field, or during collisions of two gas masses and so on arise strong discontinuities under which not only derivatives of the MHD-values are discontinuous along spatial-temporary coordinates, but these values themselves are also discontinuous ones. Jumps of MHD-values on a surface of discontinuities are determined according to integral laws of conservation or integral balance equations. As to differential equations of magnetohydrodynamics the solutions of those are inaccessible at differential of discontinuous values on the surface of discontinuity they can be represented in the integral form [2] completely equivalent to differential equations (induction and Navier-Stores equations) and also initial and boundary conditions above mentioned.

The questions of evolution in magnetohydrodynamics

Here it should be noted two factors connecting to the problem of evolution in magnetic hydrodynamics occur. The first factor may already refereed to the classical one and it has been considered sufficiently well in literature [3] and we named it conditionally evolution of discontinuities in space. As it seemed setting of boundary conditions for discontinuities is not sufficient to determine discontinuity moving of the MHD-medium by the only one method. One needs to take into account an increase of entropy and also wave stability in reference to splitting it into several discontinuous or automodel waves. Such waves in magnetic hydrodynamics are called evolutionary ones. For them infinitely small disturbances of MHD values evolves with time remaining small. The nonevolutionary wave is instantly splitted (in case of an ideal medium).

The problem of evolution of initial disturbance has here a unique solution if a number of expanding waves (a number of unknown disturbances) is equal to a number of independent boundary conditions. In this case the initial discontinuity is evolutionary one. Otherwise the problem has either innumerable quantity of solutions or the solution of this problem is inaccessible generally, i.e. discontinuity is non-evolutionary and splitted.

The evolution conditions of shock wave easily to find, analyzing the linear boundary conditions, written down in laboratory system of coordinates. In brief these evolutionary conditions can be formulated as follows. Relatively the Alfven disturbances exist two domains of evolution $u_{z1} > V_1^a$, $u_{z2} > V_2^a$ (over Alfven) and $u_{z1} < V_1^a$, $u_{z2} < V_2^a$ (up to Alfven). Here index 1 means domain ahead of a shock wave, and index 2 behind of it. $V_{1,2}^a$ − the phase Alfven wave velocities accordingly in medium 1,2.Two evolution domains of shock waves the relation to magnetoacoustic and entropy disturbances: the fast shock wave $V_1^+ < u_{z1}$, $V_2^- < u_{z2} < V_2^+$ and the slow shock wave $u_{z2} < V_2^-$, $V_1^- < u_{z2} < V_1^+$. Here $V_{1,2}^{\pm}$ − the phase velocities of the fast and slow magnetoacoustic waves. As, $V^- \le V^a \le V^+$, then fast shock waves are over alfven, and slow shock waves - up to Alfven .It is necessary to note, that Alfven, tangential and contact types of breaks are always evolutionary.

The second factor leading to the question of wave evolution in magnetic hydrodynamics is connected with taking into account initial conditions. It should be noted that interaction of MHD waves with moving boundary of two media is sufficiently dependent on moving boundary formation (in other terminology - on the surface of MHD-values discontinuity). It can be the interface of motionless and moving media. In this case motion itself is due to external sources. The velocity of boundary movement can be relative arbitrarily to phase velocities of waves in immovable medium. However, as the boundary and medium move along one site from non-uniformity the difficulties to determine the amount of divergent waves do not arise, can be used the above described principle of evolution of MHD waves.

The discontinuity surface itself can be the boundary formed as a result of propagation of rumpling wave changing medium properties but not actuating the latter for movement. In this case the movement of the medium in both sides is absent, the phase velocity of waves do not depend on the velocity of the boundary movement and the characteristics of a transmitted wave are determined only by parameters of the medium.

Considered above two variants of forming boundary movement reduce to varied non-stationary boundary-value problems resulting in wave scattering patterns of sufficiently different nature. There is a very important non-stationary problem with taking into account the evolution which may be according to above mentioned description is called as the temporary MHD evolution. In this connection the initial moment of the non-stationarity is very important, that is the statement of the initial conditions, which make sense to consider at final temporary moment. Usually at research of interaction of a packet of MHD waves with moving nonuniformities assume, that the process if inclusion of a movement occurs adiabatic on infinity. It for some processes can be strong idealism. This idealism can set in loss qualitatively of the new phenomena, connected directly to occurrence of the of this motion. Any real phenomenon begins in a final moment of time, the initial conditions can be considered in a zero moment of time.

Integral formulation of boundary-value problem

In this case a differential formulation of the solution of the boundary-value problem can leads to the difficulties, connected, on the one hand, with the indefinite discrepancy of the secondary waves to the number of boundary conditions, that has coincided fortunately for considering the spatial evolution. On the other hand, it is due to purely mathematical difficulties of mixed boundary-value problems solution.

Therefore it is meaningful to apply an integral formulation of the problem, automatically including the boundary and initial condition. This idea is confirmed by N.A. Khizhnyak and Nerukh A.G. [4] successful application of the non-stationary integral equations by consideration similar of boundary-value problems in electrodynamics. There chain of the integral Volterra equations describe interaction of electromagnetic waves with non-stationarity medium.

For analysis of non-stationarity boundary-value problems in linear magnetic hydrodynamics let us use the integral equations of non-stationarity magnetic hydrodynamics in terms of the constant magnetic field perturbation $\mathbf{b}(\mathbf{r},t)$, medium perturbations velocity $\mathbf{u}(\mathbf{r},t)$ and perturbations density $\rho(\mathbf{r},t)$ [5]:

$$\mathbf{u}(\mathbf{r},t)=\mathbf{u}_0(\mathbf{r},t)+\hat{\mathbf{K}}^{u}_{r,t}\mathbf{u}(\mathbf{r},t)+\hat{\mathbf{K}}^{b}_{r,t}\mathbf{b}(\mathbf{r},t)+\hat{\mathbf{K}}^{\rho}_{\mathbf{r},\mathrm{t}}\rho(\mathbf{r},t)+\hat{\mathbf{K}}_{s}(\mathbf{u},\mathbf{b}), \qquad (1)$$

where $\mathbf{u}_0(\mathbf{r},t)$ is an incident field and $\hat{\mathbf{K}}^{u}_{r,t},\hat{\mathbf{K}}^{b}_{r,t},\hat{\mathbf{K}}^{\rho}_{r,t}$ are the differential-integral operators of distribution of the velocity, magnetic field and density. Each of these operators can be represented as a sum of suboperators, conditionally describing different properties of MHD media, namely

$$\hat{\mathbf{K}}^{u}_{r,t}=\hat{\Gamma}^{u}_{S}+\hat{\Gamma}^{u}_{A}+\hat{\Gamma}^{u}_{v};\ \hat{\mathbf{K}}^{\rho}_{r,t}=\hat{\Gamma}^{\rho}_{U};\hat{\mathbf{K}}^{u}_{r,t}=\hat{\Gamma}^{u}_{S}+\hat{\Gamma}^{u}_{A}+\hat{\Gamma}^{u}_{v}\ \text{and}\ \hat{\mathbf{K}}^{b}_{r,t}=\hat{\Gamma}^{b}_{A}+\hat{\Gamma}^{b}_{U}.$$

Here $\hat{\Gamma}^{u}_{S}$ is responsible for pressure of conducting media and operates as follows $\hat{\Gamma}^{u}_{S}\mathbf{u}=\left(V_{s1}^2-V_{S2}^2\right)\hat{G}\,\mathrm{graddiv}\,\Pi^{\mathrm{u}}$; the operators $\hat{\Gamma}^{u}_{A},\hat{\Gamma}^{b}_{A}$ are responsible for a magnetic tension of MHD media and look like

$$\hat{\Gamma}^{u}_{A}\mathbf{u}=\mathbf{V}^{\mathbf{2}}_{A1}\hat{G}\left[\mathbf{s}_1,\mathrm{rotrot}\left[\mathbf{s}_1-\frac{B_2}{B_1}\mathbf{s}_2,\Pi^{u}\right]\right],\hat{\Gamma}^{b}_{A}\mathbf{b}=-\hat{G}\left[\frac{V_{A1}^2}{B_1}\mathbf{s}_1-\frac{V_{A2}^2}{B_2}\mathbf{s}_2,\mathrm{rot}\frac{\partial}{\partial\ t}\Pi^{b}\right];$$

the operator $\hat{\Gamma}^{u}_{v}\mathbf{u}=\hat{G}\nu_m\dfrac{V_{A1}^2}{B_1}\left[\mathbf{s}_1,\mathrm{rot}\,\Delta\Pi^{\mathrm{u}}\right]$ account presence of magnetic viscosity of media; the operators $\hat{\Gamma}^{b}_{U},\hat{\Gamma}^{\rho}_{U}$ are connected with macromovement of the discontinuity:

$$\hat{\Gamma}^{b}_{U}\mathbf{b}=-\hat{G}\left[\frac{\mathbf{B}_1}{4\pi\ \rho},\mathrm{rotrot}\left[\mathbf{U}_0,\Pi^{b}\right]\right],\hat{\Gamma}^{\rho}_{U}\rho=-\hat{G}\frac{V_{S2}^2}{\rho_1}\mathrm{grad}\left(\mathbf{U}_0,\mathrm{grad}\,\Pi^{\rho}\right).$$

Here, MHD-potentials of velocity, magnetic field and density are defined by analogy to the Hertz potentials in electrodynamics

$$\begin{Bmatrix}\Pi^{u}(\mathbf{r},t)\\ \Pi^{b}(\mathbf{r},t)\\ \Pi^{\rho}(\mathbf{r},t)\end{Bmatrix}=\int_{-\infty}^{\infty}dt'\int_{V(t')}I\left(\mathbf{r}-\mathbf{r}',t-t'\right)\begin{Bmatrix}\mathbf{u}(\mathbf{r}',t')\\ \mathbf{b}(\mathbf{r}',t')\\ \rho(\mathbf{r}',t')\end{Bmatrix}d\mathbf{r}' \qquad (2)$$

And, at last, the operator $\hat{\mathbf{K}}_S(\mathbf{u},\mathbf{b})$ is defined by presence of surface currents.

The kernel of the integral equation (1) is the Green's function $\hat{G}(\mathbf{r}-\mathbf{r}',t-t')$ for a free space, determined by parameters $\{\mathbf{B_1},V_{A1},V_{S1},\rho_1\}$. Such a representation is typical's for the integral equations of scattering problems. The Green's function of MHD linear media is written down in a basis, connected with an undisturbed magnetic field as follows: $\langle\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\rangle,\mathbf{e}_2=\mathbf{s}_1=\mathbf{B_1}/B_1$.The complete its description has been given in [2].

Equation (1) is fully equivalent to the corresponding differential equations of magnetohydrodynamics and the initial and boundary conditions at the surface of inhomogeneous, in laboratory system of coordinates.

Formally, the equation (1) can be considered as a linearised equation of magnetic hydrodynamics with by nonlocal boundary conditions, written in laboratory coordinate system. At the boundary-value problem solution in differential formulation the local boundary conditions can either be satisfied for waves of the same style, or their satisfaction needs usage of several modes. For integral formulation this difficult question is solved automatically, which is caused by physics of the phenomena.

The integral equations (1) contain whole information on scattered waves for the following problem formulation. Assume, that some inhomogeneous, which is characterized by parameters: $\mathbf{B_2}$, V_{A2},V_{S2}, ρ_2,ν_m, occupies a volume $V(t)$ with time-dependent in a general case boundary. Assume, that considered inhomogeneous is placed and move uniformly with velocity $\mathbf{U}_0$ in unbounded MHD medium, described by parameters $\mathbf{B_1},V_{A1},V_{S1},\rho_1(\nu_m=0)$ before its excitation by the incident field $\mathbf{u}_0(\mathbf{r},t),\mathbf{b}_0(\mathbf{r},t)$ correspondingly.

Let carry out an analysis of ratio (1) in frame of a non-stationary problem of diffraction on the inhomogeneous of the volume $V(t)$ using notations and terminology assumed by A.G. Nerukh [4]. So, general ideology is as follows.

Assume, that the object of diffraction before a moment $t=0$ is described by parameters $\mathbf{B_1},V_{A1},V_{S1}$ and ρ_1. At temporary moment $t=0$ the state of an object is changed for $\mathbf{B}_2,V_{A2},V_{S2},\rho_2,\nu_m$ and $\mathbf{U}_0$. Change of the state results in that, (1) disintegrates in the chain of evolutionarily connected expressions.

There are three temporary intervals for the internal field. At $t<0$ the four-dimensional interval of integration of equation (1) for temporary- spatial coordinates is unlimited as the integration is made for the crossing of the region of a transmitted light cone with the top at a point $(t,\mathbf{r})$ and into region given by the characteristic function $\chi(t,\mathbf{r})=\begin{cases}1,\mathbf{r}\in V(t)\\0,\mathbf{r}\notin V(t)\end{cases}$.

After a zero temporary moment the part of the region of integration limited by hyperplane appears $t'=0$. How the object is in a new state. And two regions of integration appear. One of them is completely in four-dimensional spatial region of events corresponding to the object of diffraction. There is no effect of boundaries of a diffraction object and one is taking into account only change of medium properties in pure appearance. If the equation (1) is used for all the four-dimensional space intervals then it will describe the field in unlimited medium with similar properties as the medium within the object.

Starting from the moment $t=\frac{1}{u}d(\mathbf{r})$, where $d(\mathbf{r})$ is a minimal distance from a point of $\mathbf{r}$ to the boundary of volume $V(0)$, a transmitted light cone will already cross the boundary of four-dimensional domain $\chi(t,\mathbf{r})=1$ and a boundary of $V(t)$ will effect on formation of the internal field.

As a result we obtain the following evolution of process of the field interaction with a object of diffraction.

A point of observation is within the region $V(t)$. 1) Before the zero temporary moment the incident field $\mathbf{u}_0(\mathbf{r},t),\mathbf{b}_0(\mathbf{r},t)$ generates the internal fields $\mathbf{u}_1(\mathbf{r},t),\mathbf{b}_1(\mathbf{r},t)$. 2) After changing of state of an object at zero temporary there is no effect of the object boundary of an object in the region and incident field does not obviously take part in creation of the internal field, that is the field $\mathbf{u}_2(\mathbf{r},t),\mathbf{b}_2(\mathbf{r},t)$ is directly generated by the field of $\mathbf{u}_1(\mathbf{r},t),\mathbf{b}_1(\mathbf{r},t)$, formed within the object before changing of the state and conditioned by the field of $\mathbf{u}_0(\mathbf{r},t),\mathbf{b}_0(\mathbf{r},t)$ that is the memory of medium. 3) To form the field $\mathbf{u}_3(\mathbf{r},t),\mathbf{b}_3(\mathbf{r},t)$ together with the field $\mathbf{u}_1(\mathbf{r},t),\mathbf{b}_1(\mathbf{r},t)$ the incident field $\mathbf{u}_0(\mathbf{r},t),\mathbf{b}_0(\mathbf{r},t)$ crossing the boundary and field $\mathbf{u}_2(\mathbf{r},t),\mathbf{b}_2(\mathbf{r},t)$ take part in the process above mentioned.

A point of observation is beyond the region of $V(t)$. Then from the expression (1) we receive the square formula. In this case two temporary intervals are allocated. 1). At $t<\frac{1}{u}l(\mathbf{r})$, where $l(\mathbf{r})-$ distance from a observation point at moment t=0 up to the nearest point of domain $V(t)$, the external field defined by nondisturbance condition of diffraction object by founding field $\mathbf{u}_1(\mathbf{r},t),\mathbf{b}_1(\mathbf{r},t)$. 2). At $t\geq\frac{1}{u}l(\mathbf{r})$, an external field, on which already influence and the new condition, will be defined in view of a field $\mathbf{u}_3(\mathbf{r},t),\mathbf{b}_3(\mathbf{r},t)$.

In this connection the following algorithm to solve the problem is considered. Firstly the solution of non-disturbance problem that can be considered as steadystate one with corresponding choice of point parameters is sought. Secondary is sought the solution of the disturbance problem, but without the effect of boundaries of a diffraction object, that is only the disturbance of the medium itself is present. At the third stage the problem is solved with taking into account the boundary-value effects.

If the non-stationarity dissipation problem of MHD waves on the elementary plane boundary it is possible to consider theoretically stage by stage investigated, in sense mathematical realization there is the set of difficulties. First of all transformation of waves an each other on the boundary of interface a much complicates problem. It is characteristic just of magnetic hydrodynamics. The transformation is displayed already by consideration of a three-dimensional stationary diffraction problem even on the elementary inhomogeneous is half-space. In this case the expression (1) is not divided on the separate equations, describing only Alfven and only magnetoacoustic waves. Here it is possible to look after complete transformation of MHD waves, as was made. That is wave as though «interplace». More simple two-dimensional diffraction problem allows to consider separately of a boundary-value problem for Alfven and in common for fast and slow magnetoacoustic waves. Here it is possible to look after transformation of magnetoacoustic waves. And that, at last, completely to exclude transformation of waves, it is meaningful to consider a spatially single-dimensional problem separately for an each type of waves.

In short a problem of scattering of Alfven wave by plasma half-space (MHD-inhomogeneity) after a initiating its movement is considered.

The model in which one type of MHD waves is coupling with another through a plane boundary between two media is an important first approximation for the description of the propagation of small perturbations in strongly inhomogeneous MHD media.

Let the plane Alfven wave

$$\begin{Bmatrix} \mathbf{u}_0(\mathbf{r},t) \\ \mathbf{b}_0(\mathbf{r},t \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{b}_0 \end{Bmatrix} \exp\left(ik_0^A z - i\omega_0^A t\right)$$

falls onto a plane boundary (z=0) of two nondissipation medium having parameters $\mathbf{B}_i, V_{Ai}, V_{Si}, \rho_i, i = 1,2.$ Let internal medium begins a uniform moment with velocity U_0 to perpendicularly its boundary in moment t=0. Prior to the beginning a movement an inhomogeneous occupied domain (z>0). Then the law of boundary movement will be set by the formula $z(t) = U_0 \theta\,(t)$. Here $\theta\,(t)$ is the Heaviside function.

For such a setting a diffraction problem in case of consideration only the Alfven wave is of scalar form. Then integral relationship (1) to find the Alfven field incident on the MHD inhomogeneous and reflected from the latter is as follows:

$$u_x(\mathbf{r},t) = u_{0x}(\mathbf{r},t) + \frac{1}{B_1}\left(V_{A1}^2 - V_{A2}^2\right)\frac{\partial^2}{\partial z \partial t}\int_{-\infty}^{\infty} dt' \int_{V(t')} b_x(\mathbf{r}',t') G^A(\mathbf{r}-\mathbf{r}', t-t') d\mathbf{r}' -$$

$$-V_{A1}^2\left(1 - \frac{B_2}{B_1}\right)\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} dt' \int_{V(t')} u_x(\mathbf{r}',t') G^A(\mathbf{r}-\mathbf{r}', t-t') d\mathbf{r}',$$

where G^A is the Green function of Alfven component given by

$$G_A = \frac{1}{2V_{A1}}\delta\,(x_1 - x_1')\delta\,(x_2 - x_2')\theta\left(t - t' - \frac{|x_3 - x_3'|}{V_{A1}}\right),$$

here $\theta\,(t)-$ the Heaviside function, $\delta\,(x)-$ the Dirac function.

Three-dimension integral on spatial variable easily to reduce to single-dimension integral. For internal medium we have

$$\int_{V(t')} u_x(\mathbf{r}',t') G^A(\mathbf{r}-\mathbf{r}', t-t') d\mathbf{r}' = \int_{\mathbf{z}(\mathbf{t}')}^{\mathbf{z}} u_x(z',t')\theta\left(t - \frac{z}{V_{A1}} - t' + \frac{z'}{V_{A1}}\right)dz' + \int_{z}^{\infty} u_x(z',t')\theta\left(t + \frac{z}{V_{A1}} - t' - \frac{z'}{V_{A1}}\right)dz'. \qquad (4)$$

And for external medium we have

$$\int_{V(t')} u_x(\mathbf{r}',t') G^A(\mathbf{r}-\mathbf{r}', t-t') d\mathbf{r}' = \int_{z(t')}^{\infty} u_x(z',t')\theta\left(t + \frac{z}{V_{A1}} - t' - \frac{z'}{V_{A1}}\right)dz'. \qquad (5)$$

On structure (4,5) practically coincides with the appropriate integral equation in a dissipation problem of electromagnetic waves by plasma half-space. Omitting the intermediate mathematical calculatious one can note the sufficiently new results of magnetic hydrodynamics arising in the solution of the non-stationary boundary-value problem for the Alfven waves in case of taking into account the initial temporary moment of origin of spatial boundary movement starting under the condition of $U_0 < V_{A1}$.

For the analysis of the internal field after initial movement $(t > 0)$ it follows that the field in the region of $V_{A1}t < z$ does not have either frequency and wave number and the amplitude existed before starting of movement of the Alfven wave is changed and the constant component due to jump of velocity of medium appears. It is connected with that boundary-value effects for the moving boundary do not influence on this field. In the region of $U_0 t \le z \le V_{A1}t$ it is necessary to consider separately cases of «run-away» $U_0 > 0$ and "encouter" $U_0 < 0$ movement with for each case we obtain the Volterra two-dimensional equation of the second type with the Fredholm kernel for that was developed sufficiently well the approach fort solution of problem as uniformly convergent Neiman series.

The reflected field is easily restored by means of the quadrature formulas (5) for the already known internal field. Thus it should be noted the sufficient difference of a spectrum of the scattering field in case of the finite temporary moment from the spectrum of the scattered field for the adiabatic inclusion at infinity. The spectrum of this field consists of waves with different frequencies propagating as from the medium boundary as towards boundary itself then as in case of inclusion at infinity there is only the wave reflected from the boundary in transmitted field.

Conclusions

The superficial analysis of the boundary-value non-stationary problem for Alfven wave has shown the principal possibility of using the method of evolutionary integral equations of non-stationary macroscopic electrodynamical in a case of MHD description of waves in plasma. With the importance of strict mathematical solutions obtained for simple model problems that is the diffraction of one separately taken Alfven wave is that it can be the basis for construction of the approximate solutions of more complex boundary-value problems.

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