COMPACT FORM OF WEAKLY RELATIVISTIC DIELECTRIC TENSOR FOR UNIFORM MAXWELLIAN PLASMA

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A form of weakly relativistic dielectric tensor valid for oblique propagation has been developed in previous works. Nevertheless this form was somehow cumbersome to deal with. In present work, authors start from those results to show that the weakly relativistic dielectric tensor can be written in a compact form that is consistent with the more elegant form of the non-relativistic dielectric tensor.
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1. INTRODUCTION

It is known that theoretical investigation of electron cyclotron wave propagation and absorption in magneto-active plasma requires a relativistic treatment for the temperatures that are usually achieved in the magnetic confinement devices [1]. This treatment is essential in the case of quasi-perpendicular wave propagation, when relativistic effects are dominant. In this regime the non-relativistic treatment can even give wrong results, producing zero absorption in some cases in which it is non-zero.

There exists a vast literature on this topic. In the original paper of Trubnikov [2] there was introduced relativistic treatment but derived form of dielectric tensor was too hard tractable. Dnestrovskii, Kostomarov and Skrydlov [3] derived the tractable form of tensor for perpendicular waves propagation on the base weakly relativistic treatment, in terms of so called Dnestrovsky dispersion functions. Shkarofsky [4] generalized the weakly relativistic treatment in the case of quasi-perpendicular propagation and introduced more general Shkarofsky dispersion functions, that nevertheless, were difficult to estimate. The method to derive the weakly relativistic tensor for arbitrary waves propagation in numerically tractable form was suggested by Airroldi and Orefice [5]. This method allows one, in principle, to obtain dielectric tensor as an expansion in the relativistic parameter $1/\mu$ ($\mu=m_0c^2/T_e$, m_0 is the electron rest mass) and the finite Larmor radius parameter $\lambda=(k_\perp\rho_e)^2$ ($k_\perp$ is the perpendicular wave number and ρ_e is the electron Larmor radius). However, this tensor is somewhat cumbersome to deal with. Krvenski and Orefice [6], using this method, received dielectric tensor in the lowest significant approximation in the parameter $1/\mu$, as an expansion in the parameter λ. This form of dielectric tensor is far from having the compact form that non-relativistic tensor expansions have, because authors did not pay attention in obtaining a final expressions as close as possible to the non-relativistic ones.

In the present work authors, starting from results of the work [6], show that the lowest significant approximation in the parameter $1/\mu$ of the weakly relativistic dielectric tensor may be presented in the compact form that is consistent with one of the possible forms for non-relativistic one. On this way there were received two next approximations in the parameter $1/\mu$.

2. MUTUALLY CONSISTENT FORM OF NON-RELATIVISTIC AND WEAKLY RELATIVISTIC DIELECTRIC TENSOR

It is known that using analytical properties of functions that appear in the non-relativistic dielectric tensor (modified Bessel functions and non-relativistic plasma dispersion function) it is possible to write it in different forms. The choice of any concrete form is defined either by tendency to the extremely possible simplicity of final expression for tensor [7] or by some other considerations [8]. It is easy to verify that this tensor can be presented also in the following form:

$$\hat{\varepsilon} = \hat{1} + Z_0\left(\frac{\omega_p}{\omega}\right)^2 \sum_{n=-\infty}^{+\infty}\left[\hat{P}_n Z(Z_n) + \hat{R}_n Z'(Z_n) + \hat{S}_n Z''(Z_n)\right] \qquad (1)$$

where ω is incident wave frequency and ω_p is plasma frequency, $Z(Z_n)$ is the plasma dispersion function of the argument $Z_n = (\omega - \omega_c)/(\sqrt{2}k_{//}V_t)$, ω_c and V_t are electron cyclotron frequency and thermal velocity and

$$\hat{P}_n = \begin{pmatrix} n^2 A_n/\lambda & -inA'_n & 0 \\ iA'_n & n^2 A_n/\lambda - 2\lambda A'_n & 0 \\ 0 & 0 & A_n \end{pmatrix}$$

$$\hat{R}_n = \begin{pmatrix} 0 & 0 & nA_n/\sqrt{2\lambda} \\ 0 & 0 & -iA'_n\sqrt{\lambda/2} \\ nA_n/\sqrt{2\lambda} & iA'_n\sqrt{\lambda/2} & 0 \end{pmatrix}$$

$$S_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_n/2 \end{pmatrix}$$

$A_n = \exp(-\lambda)I_n(\lambda)$, $\lambda = (k_\perp\rho)^2$, $k_{//}$ and $k_\perp$ are parallel and perpendicular wave numbers.

The main peculiarity of such presentation is total separation of perpendicular and longitudinal dispersion. The perpendicular dispersion is described by tensors $\hat{P}_n$, $\hat{R}_n$, $\hat{S}_n$ and the longitudinal one. Is described by plasma dispersion function, Z, and its first and second derivatives in the parameter Z_n. It is easy to see that this

additional condition totally defines single form of dielectric tensor from all possible ones. Now, let us show that this form of non-relativistic tensor obeys to a deeper sense, from the point of view of consistency with the weakly relativistic tensor. In the work [6] the weakly relativistic tensor was derived in the lowest significant approximation in the parameter 1/μ in the form:

$$\hat{\varepsilon} = \hat{1} - \mu\left(\frac{\omega_p}{\omega}\right)^2 \hat{D} \qquad (2)$$

$$D_{11} = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} s^2(s+k)!2^{s+k}a(s,k)\lambda^{s+k-1}Q^{+}_{s,k}(h=0),$$

$$D_{12} = -D_{21} = -i\sum_{s,k} s(s+k)(s+k)!2^{s+k}a(s,k)\lambda^{s+k-1}Q^{-}_{s,k}(h=0),$$

$$D_{22} = \sum_{s,k}(s+k)!2^{s+k}b(s,k)\lambda^{s+k-1}Q^{+}_{s,k}(h=0),$$

$$D_{13} = D_{31} = \sqrt{\mu\lambda}\sum_{s,k} s(s+k)!2^{s+k}a(s,k)\lambda^{s+k-1}Q^{-}_{s,k}(h=1),$$

$$D_{23} = -D_{32} = i\sqrt{\mu\lambda}\sum_{s,k}(s+k)(s+k)!2^{s+k}a(s,k)\lambda^{s+k-1}Q^{+}_{s,k}(h=1)$$

$$D_{33} = \mu\lambda\sum_{s,k}(s+k)!2^{s+k}a(s,k)\lambda^{s+k-1}Q^{+}_{s,k}(h=2).$$

Above were used next denotings

$$a(s,k) = \frac{(-1)^k[2(|s|+k)]!\left(\frac{1}{2}\right)^{2(|s|+k)}}{[(|s|+k)!]^2(2|s|+k)!k!}, \text{ (here } 1/(-n)!=0\text{)}$$

$$b(s,k) = a(1,k-2), \quad \text{(for s=0)}$$

$$b(s,k) = \frac{1}{4}\left[a(s-1,k)+a(s+1,k-2)-2\frac{|s|+k-1}{|s|+k}a(s,k-1)\right],$$

(for s≠0)

$$\lambda = (N_\perp\omega/\Omega_0)^2/\mu, \quad \Omega_0 = eB_0/m_0c,$$

$$Q^{\pm}_{s,k}(h) = Q_{s,k}(h) \pm Q_{-s,k}(h), \quad (s\neq 0)$$

$$Q^{+}_{0,k} = Q_{0,k}(h), \qquad (s=0).$$

The tensor $\hat{D}$ is development in the parameter λ. In this development Shkarofsky functions of half integer index appear, $F_q(z,a)$:

$$Q_{s,k}(h=0) = F_q(z,a) = -i\int_0^{\infty}\frac{dt}{(1-it)^q}\exp\left(izt - \frac{at^2}{(1-it)}\right),$$

where $z = \mu(1 - s\omega_{c0}/\omega)$, $a = \mu N_{//}^2/2$, $q = |s| + k + 3/2$ (k=0,1,…).

$$Q_{s,k}(h=1) = N_{//}\{F_q - F_{q+1}\},$$

$$Q_{s,k}(h=2) = \mu^{-1}\{F_{q+1} + \mu N_{//}[F_{q+2} + F_q - 2F_{q+1}]\}.$$

Now, let us change argument z in Shkarofsky functions into $Z_n = z/(2\sqrt{a})$. It is homotheticaly correct, since Shkarofsky functions depend mono-valuably on this parameter for any finite value of longitudinal refractive index $N_{//}$ and have derivatives for any real value of this parameter. We have in mind, of course, only those functions $F_q(z,a)$ that appear in dielectric tensor, i.e. we consider $q \geq 5/2$. Obviously $Z_n = (\omega - \omega_c)/(\sqrt{2}k_{//}V_t)$. The advantage of such a change consists of the fact that Shkarofsky functions depend now on the same argument that the non-relativistic plasma dispersion function. It is not difficult to verify term by term that weakly relativistic tensor (2) can be written in the form:

$$\hat{\varepsilon} = \hat{1} - \mu\left(\frac{\omega_p}{\omega}\right)^2 \sum_{n=-\infty}^{+\infty}\left[\hat{P}_n * \vec{F}(Z_n) + \hat{R}_n * \vec{F'}(Z_n) + \hat{S}_n * \vec{F''}(Z_n)\right] \qquad (3)$$

where $F(Z_n)$ is an infinite vector containing all Shkarofsky functions with gradually increasing index starting from $q = 5/2$:

$$F(Z_n) = [F_{5/2}(Z_n), F_{7/2}(Z_n), F_{9/2}(Z_n), \ldots],$$

the sign $*$ denotes the scalar product of the infinite vector $F(Z_n)$ with every left-side matrix element considered as a infinite vector resulting from finite Larmor radius (FLR) – expansion beginning from the term of zeroth order. So, for instance,

$$\left[\hat{P}_n * F(Z_n)\right]_{11} = n^2 A_n/\lambda * F(Z_n),$$

where $\vec{A}_n/\lambda = \left[A_n^0/\lambda, A_n^1/\lambda, A_n^2/\lambda, \ldots\right]$. Here the superscript in A_n^k/λ denotes the order in FLR – expansion of A_n/λ. The order in FLR-expansion in tensor $\hat{R}_n$, that is connected with elements of tensor ε_{13} and ε_{23}, must be considered after dividing these elements by $\sqrt{\lambda}$.

Thus one can see that the form (1) of non-relativistic tensor is very close to the form (3) of weakly relativistic tensor. The perpendicular dispersions defined by tensors $\hat{P}_n$, $\hat{R}_n$, $\hat{S}_n$ are the same in both cases and the longitudinal dispersions defined by analytical properties of non-relativistic plasma dispersion function and the whole family of Shkarofsky functions are very close, cause analytical properties of plasma dispersion function and every Shkarofsky function are very close, especially for oblique propagation.

The comparison of nonrelativistic (1) and weakly relativistic forms (3) of dielectric tensor shows directly that it is enough to take into account the dependence of relativistic electron mass on speed to obtain a split of plasma dispersion function into the family of Shkarofsky functions of half-integer order $q \geq 5/2$. In the weakly relativistic case the role of plasma dispersion function is played by the whole family of Shkarofsky functions and a single Shkarofsky function is directly connected with only one corresponding order in FLR-expansion of perpendicular dispersion for the tensor elements ε_{11}, ε_{12}, ε_{22}, with the corresponding and previous one for the elements ε_{13}, ε_{23} trough its first derivative, and with the corresponding and two previous ones for elements ε_{33} trough its second derivative.

3. WEAKLY RELATIVISTIC DIELECTRIC TENSOR IN FURTHER ORDER APPROXIMATIONS IN PARAMETER μ.

Since the weakly relativistic tensor can be presented in a very close form to the non-relativistic one in the lowest significant order of the expansion in the parameter

μ , one would expect that the next approximations can be written in a more or less compact form using the same argument for Shkarofsky functions and tending to the form of non-relativistic tensor (1). These next approximations are necessary for numerics of EC waves in burning plasmas, when the condition $\mu >> \lambda$ is fulfilled.

Really, using the method suggested by Airoldi and Orefice [5] and described in detail in a recent book by Brambila [7], the next to lowest approximation in the parameter μ can be presented in the form:

$$\hat{\varepsilon}_1 = -\frac{\mu e^{-\mu}}{2K_2(\mu)}\left(\frac{\omega_p}{\omega}\right)^2 \sum_{n=-\infty}^{+\infty}\left[\hat{P}_n * \vec{F}_1(Z_n) + \hat{R}_n * \vec{F}_1'(Z_n) + \hat{S}_n * \vec{F}_1''(Z_n)\right] \quad (4),$$

where $K_2(\mu)$ is McDonald function and subscript 1 stands for first order in μ -approximation of dielectric tensor. The vector F_1 is easily obtained from the vector of Shkarofsky functions F, used in Equation (3), by multiplication by matrix $\hat{B}$, that has comparatively simple structure:

$$\hat{B} = \begin{pmatrix} 0 & b_1 & 0 & \dots & \dots \\ 0 & 0 & b_2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & b_n & 0 \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad b_n = 3/8 + n(n+2)/2 ,$$

$$F_1 = \hat{B}\ F .$$

Coefficients b_n are defined by the expansion coefficients appearing in Airoldi and Orifice's method. In order to derive this approximation, we have neglected the even order derivatives of Shkarofsky functions, $F_q^{(2k)}$ (k≥1), that are small in comparison with the function itself, in the tensor elements $\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22}, \varepsilon_{33}$ and the odd order derivatives, $F_q^{(2k+1)}$ (k≥1), in comparison with first derivatives, $F_q^{(1)}$, in the elements ε_{13} and ε_{23}. These approximations are right for any value of longitudinal refractive index $N_{//}$ in ECR frequency range and it is easily verified numerically. It is worthwhile to note that coefficients b_1 and b_2, corresponding to first and second harmonics, differ of ones in Brambilla's book by the constant $3/8$, which appears after taking into account all terms of order $1/\mu$.

In this way, one can see that every next approximation differs from previous one by the structure of the matrix $\hat{B}$. In the lowest significant approximation, matrix $\hat{B}$ has units in main diagonal, i.e. it is the unit matrix. In the next one it has also single nonzero terms diagonal, which is next right diagonal to the to the main one. In each more deep approximation matrix $\hat{B}$ has also single nonzero diagonal, again, which is next right diagonal to the previous one. The coefficients b_n for μ^{-2} - approximation are:

$$b_n = \left[\frac{3n(n-2)}{2^3} + \frac{15n}{2^4} - \frac{15}{2^7}\right] + \binom{n}{1}\frac{n+1}{2}\left[\frac{n-1}{2} + \frac{3}{2^3}\right] + \binom{n}{2}\frac{(n+1)(n+2)}{2^2}$$

and for μ^{-3} -approximation

$$b_n = \left[\frac{n(n-2)(n-4)}{2^4} + \frac{105n(n-2)}{2^6} + \frac{105n}{2^8} + \frac{105}{2^{10}}\right] + \binom{n}{1}\frac{n+1}{2}\left[\frac{3(n-1)(n-3)}{2^3} + \frac{15(n-1)}{2^4} - \frac{15}{2^7}\right] + \binom{n}{2}\frac{(n+1)(n+2)}{2^2}\left[\frac{n-2}{2} + \frac{3}{2^3}\right] + \binom{n}{3}\frac{(n+1)(n+2)(n+3)}{2^3}$$

Obviously the coefficients b_n can be written for any μ - approximation and with every step they become more and more cumbersome, but the coefficients presented above are enough for laboratory and reactor plasmas.

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