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A NEW CLASS OF NONSTATIONARY MOTIONS OF A SYSTEM OF HEAVY LAGRANGE TOPS WITH A NON-PLANAR CONFIGURATION OF THE SYSTEM'S SKELETON

For a chain consisting of n heavy Lagrange tops coupled by ideal spherical joints, the existence of a class of nonstationary motions with a non-planar configuration of the chain's skeleton is proved. Sufficient conditions for existence of these motions are established, and the equations of motion of the chain are reduced to quadratures. Under the assumption that the mass distribution of the bodies forming the chain is given, it is shown how they have to be coupled so that the motions of interest could be realized. Some properties of the new motions are discussed.

Keywords: *analytical multibody dynamics, Lagrange top, nonstationary motion of a system of coupled rigid bodies*

1. Formulation of the problem. We consider a mechanical system S consisting of heavy Lagrange tops $B_1, B_2, \ldots, B_n$. The bodies B_i and B_{i+1} $(i = 1, 2, \ldots, n-1)$ are coupled by an ideal spherical joint at a common point O_{i+1} so that the system S constitutes a chain of rigid bodies. One of the chain's end links, B_1, is absolutely fixed at one of its points $O_1 (\neq O_2)$. It is assumed that the attachment points of the body B_i lie on its axis of symmetry l_i, i.e., $O_1 \in l_1, O_i \in l_{i-1} \cap l_i (i = 2, 3, \ldots, n-1)$.

Let $\{O_1, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\}$ be a Cartesian reference frame whose vectors are fixed in inertial space so that the vector $\mathbf{e}_3$ is vertically directed. Let also $\{O_i, \mathbf{e}_1^{(i)}\mathbf{e}_2^{(i)}\mathbf{e}_3^{(i)}\}$ be a Cartesian frame that is rigidly embedded in the body B_i such that $\mathbf{e}_3^{(i)} \,||\, l_i$. We determine the position of the body B_i with respect to the reference frame by Euler angles θ_i, ψ_i, and φ_i. The vector equations of motion for a chain of coupled rigid bodies is given in [1]. Projecting these equations on the axes of the corresponding body-fixed frames, one can obtain the following scalar equations of motion of the system S:

$$F_i^{(m)} + a_i \sum_{j=1}^{i-1} s_j G_{ij}^{(m)} + s_i \sum_{j=i+1}^{n} a_j G_{ij}^{(m)} = 0, \tag{1}$$

$$\dot{\varphi}_i + \dot{\psi}_i \cos\theta_i = q_i, \qquad i = 1, 2, \ldots, n, \tag{2}$$

where $m = 1, 2$,

$$F_i^{(1)} = J_i' \left(\ddot{\theta}_i - \dot{\psi}_i^2 \sin\theta_i \cos\theta_i\right) + J_i^s q_i \dot{\psi}_i \sin\theta_i - a_i g \sin\theta_i,$$

$$G_{ij}^{(1)} = \left(\ddot{\theta}_j \sin\theta_j + \dot{\theta}_j^2 \cos\theta_j\right) \sin\theta_i + \left(\ddot{\psi}_j \sin\theta_j + 2\dot{\theta}_j \dot{\psi}_j \cos\theta_j\right) \cos\theta_i \sin(\psi_i - \psi_j) +$$
$$+ \left(\ddot{\theta}_j \cos\theta_j - \left(\dot{\theta}_j^2 + \dot{\psi}_j^2\right) \sin\theta_j\right) \cos\theta_i \cos(\psi_i - \psi_j),$$

$$F_i^{(2)} = J_i' \left(\ddot{\psi}_i \sin\theta_i + 2\dot{\theta}_i\dot{\psi}_i \cos\theta_i\right) - J_i^s q_i \dot{\theta}_i,$$
$$G_{ij}^{(2)} = \left(\ddot{\psi}_j \sin\theta_j + 2\dot{\theta}_j\dot{\psi}_j \cos\theta_j\right) \cos(\psi_i - \psi_j) -$$
$$- \left(\ddot{\theta}_j \cos\theta_j - \left(\dot{\theta}_j^2 + \dot{\psi}_j^2\right) \sin\theta_j\right) \sin(\psi_i - \psi_j)$$

and the dots denote differentiation with respect to time. In equations (1), (2), $s_i = |\mathbf{O}_i\mathbf{O}_{i+1}|, q_i$ is an integration constant, J_i and J_i^s are the moments of inertia of body B_i with respect to O_i about the axes $\mathbf{e}_1^{(i)}$ (or $\mathbf{e}_2^{(i)}$) and $\mathbf{e}_3^{(i)}$, respectively, and

$$J_i' = J_i + \widetilde{m}_i s_i^2, \qquad a_i = m_i c_i + \widetilde{m}_i s_i, \tag{3}$$

where $c_i = |\mathbf{O}_i\mathbf{C}_i|, C_i$ is the center of mass of body $B_i, \widetilde{m}_i = \sum\limits_{j=i+1}^{n} m_j$, and m_i is the mass of body B_i.

The motion of system S is a superposition of the motion of its skeleton, that is composed of the segments of axes l_i bounded by the corresponding suspension points, and the pure rotation of each body about its axis of dynamic symmetry. The former motion is completely determined by all angles θ_i, ψ_i, while the rotation of B_i about l_i is described by the angle φ_i.

When the system S performs P.V. Kharlamov's motion [2], the skeleton belongs to a vertical plane Π rotating about the vertical in accordance with a non-stationary law $\psi(t)$ while its segments change their position with respect to Π identically in time, i.e., all the bodies move similarly. Therefore, these motions of the bodies system are called similar motions. For such motions, it is fulfilled that $\theta_i = \theta(t), \psi_i = \psi(t) + \delta_i\pi$, where $\delta_i \in \{-1, 0, 1\}$ and $i = 1, 2, \dots, n$. Some properties of the system's motion and its generalizations can be found in the works [2–4].

Let n_* be a fixed integer with $1 \le n_* < n$. This integer partitions the set $I = \{1, 2, \dots, n\}$ of indices of all bodies constituting system S into two subsets $I_* = \{1, 2, \dots, n_*\}$ and $I^* = \{n_* + 1, n_* + 2, \dots, n\}$. In what follows, we consider two subsystems of S: $S_* = \{B_j | j \in I_*\}$ and $S^* = \{B_k | k \in I^*\}$. (Clearly, the subsystems are coupled at the point O_{n_*+1}.) In this paper, we seek to find a new class of nonstationary motions of system S with the following properties: the bodies forming the subsystem S_* (S^*) move similar to each other and the planes Π_* and Π^* containing the skeletons of the subsystems S_* and S^*, respectively, in general, do not coincide, i.e.

$$\begin{array}{lll} \theta_j = \theta_1, & \psi_j = \psi_1 + \delta_j\pi, & j \in I_*, \\ \theta_k = \theta_n, & \psi_k = \psi_n + \delta_k\pi, & k \in I^*, \end{array} \tag{4}$$

where $\delta_i \in \{-1, 0, 1\}$ and $\theta_1, \theta_n, \psi_1$, and ψ_n are functions of time to be determined. We also require that

$$\cos\theta_n = \mu \cos\theta_1, \tag{5}$$

where $\mu (\neq 0)$ is a constant.

2. Structure of the solution. In this section we establish sufficient conditions for existence of the class of solution to equations (1), (2) with properties (4), (5). We restrict our study to the case of nonstationary motions of the bodies system, i.e., $\dot\theta_1 \not\equiv 0, \dot\theta_n \not\equiv 0, \dot\psi_1 \not\equiv 0$, and $\dot\psi_n \not\equiv 0$. In all notations used below, we assume that $i \in I, j \in I_*$, and $k \in I^*$.

Introducing the notation

$$\varepsilon_{lm} = \cos\left[(\delta_l - \delta_m)\,\pi\right] = \begin{cases} \;\;1, & \text{if } \delta_l = \delta_m \text{ or } \delta_l = \delta_m \pm 2, \\ -1, & \text{if } \delta_l = \delta_m \pm 1, \end{cases} \qquad l, m \in I, \tag{6}$$

we derive, by virtue of (4), that

$$\begin{gathered} \cos(\psi_l - \psi_m) = \begin{cases} \varepsilon_{lm}, & \text{if } l, m \in I_* \text{ or } l, m \in I^*, \\ \varepsilon_{lm}\cos(\psi_1 - \psi_n), & \text{if } l \in I_*, m \in I^* \text{ or } l \in I^*, m \in I_*, \end{cases} \\ \sin(\psi_l - \psi_m) = \begin{cases} 0, & \text{if } l, m \in I_* \text{ or } l, m \in I^*, \\ \varepsilon_{lm}\sin(\psi_1 - \psi_n), & \text{if } l \in I_*, m \in I^*, \\ \varepsilon_{lm}\sin(\psi_n - \psi_1), & \text{if } l \in I^*, m \in I_*. \end{cases} \end{gathered} \tag{7}$$

Substituting (4) into (1) and taking into account formulas (6) and (7), we obtain

$$\begin{aligned} &P_j\left(\ddot\theta_1 - \dot\psi_1^2 \sin\theta_1\cos\theta_1\right) + \left(J_j^s q_j \dot\psi_1 - a_j g - Q_j\left(\cos\theta_n\right)^{\cdot\cdot} - \widetilde{R}_j\left(\cos\theta_1\right)^{\cdot\cdot}\right)\sin\theta_1 + \\ &\qquad + T_j\cos\theta_1\left(f_n^{(1)}\cos\left(\psi_1 - \psi_n\right) + f_n^{(2)}\sin\left(\psi_1 - \psi_n\right)\right) = 0, \\ &P_j\left(\ddot\psi_1\sin\theta_1 + 2\dot\psi_1\dot\theta_1\cos\theta_1\right) - J_j^s q_j \dot\theta_1 - \\ &\qquad - T_j\left(f_n^{(1)}\sin\left(\psi_1 - \psi_n\right) - f_n^{(2)}\cos\left(\psi_1 - \psi_n\right)\right) = 0, \\ &P_k\left(\ddot\theta_n - \dot\psi_n^2 \sin\theta_n\cos\theta_n\right) + \left(J_k^s q_k \dot\psi_n - a_k g - Q_k\left(\cos\theta_1\right)^{\cdot\cdot} - \widetilde{R}_k\left(\cos\theta_n\right)^{\cdot\cdot}\right)\sin\theta_n + \\ &\qquad + T_k\cos\theta_n\left(f_1^{(1)}\cos\left(\psi_n - \psi_1\right) + f_1^{(2)}\sin\left(\psi_n - \psi_1\right)\right) = 0, \\ &P_k\left(\ddot\psi_n\sin\theta_n + 2\dot\psi_n\dot\theta_n\cos\theta_n\right) - J_k^s q_k \dot\theta_n - \\ &\qquad - T_k\left(f_1^{(1)}\sin\left(\psi_n - \psi_1\right) - f_1^{(2)}\cos\left(\psi_n - \psi_1\right)\right) = 0, \end{aligned} \tag{8}$$

where, for $m = 1, n$,

$$f_m^{(1)} = \ddot\theta_m\cos\theta_m - \left(\dot\theta_m^2 + \dot\psi_m^2\right)\sin\theta_m, \qquad f_m^{(2)} = \ddot\psi_m\sin\theta_m + 2\dot\psi_m\dot\theta_m\cos\theta_m,$$

and

$$
\begin{aligned}
&P_j = J'_j + a_j \sum_{l=1}^{j-1} s_l \varepsilon_{jl} + s_j \sum_{l=j+1}^{n_*} a_l \varepsilon_{jl}, && Q_j = s_j \sum_{l=n_*+1}^{n} a_l, \\
&P_k = J'_k + a_k \sum_{l=n_*+1}^{k-1} s_l \varepsilon_{kl} + s_k \sum_{l=k+1}^{n} a_l \varepsilon_{kl}, && Q_k = a_k \sum_{l=1}^{n_*} s_l, \\
&\widetilde{R}_j = a_j \sum_{l=1}^{j-1} s_l \left(1 - \varepsilon_{jl}\right) + s_j \sum_{l=j+1}^{n_*} a_l \left(1 - \varepsilon_{jl}\right), && T_j = s_j \sum_{l=n_*+1}^{n} a_l \varepsilon_{jl}, \\
&\widetilde{R}_k = a_k \sum_{l=n_*+1}^{k-1} s_l \left(1 - \varepsilon_{kl}\right) + s_k \sum_{l=k+1}^{n} a_l \left(1 - \varepsilon_{kl}\right), && T_k = a_k \sum_{l=1}^{n_*} s_l \varepsilon_{kl}.
\end{aligned} \tag{9}
$$

Equations (8) form an overdetermined system of $2n$ second-order differential equations with respect to four unknowns $\theta_1, \psi_1, \theta_n$, and ψ_n. (Note that θ_1 and θ_n are not independent due to (5).) In the rest of this section, we shall examine the compatibility of the system (8) in the case when

$$
T_i = 0. \tag{10}
$$

Using (5) and (10), we rewrite the system (8) as follows:

$$
\begin{aligned}
&P_j \left(\ddot{\theta}_1 - \dot{\psi}_1^2 \sin\theta_1 \cos\theta_1\right) + \left[J_j^s q_j \dot{\psi}_1 - a_j g - R_j \left(\cos\theta_1\right)^{\cdot\cdot}\right] \sin\theta_1 = 0, \\
&P_j \left(\ddot{\psi}_1 \sin\theta_1 + 2\dot{\psi}_1 \dot{\theta}_1 \cos\theta_1\right) - J_j^s q_j \dot{\theta}_1 = 0,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
&P_k \left(\ddot{\theta}_n - \dot{\psi}_n^2 \sin\theta_n \cos\theta_n\right) + \left[J_k^s q_k \dot{\psi}_n - a_k g - R_k \left(\cos\theta_n\right)^{\cdot\cdot}\right] \sin\theta_n = 0, \\
&P_k \left(\ddot{\psi}_n \sin\theta_n + 2\dot{\psi}_n \dot{\theta}_n \cos\theta_n\right) - J_k^s q_k \dot{\theta}_n = 0,
\end{aligned} \tag{12}
$$

where

$$
R_j = \widetilde{R}_j + \mu Q_j, \qquad R_k = \mu \widetilde{R}_k + Q_k. \tag{13}
$$

For each index $j \in I_*$, the pair of equations (11) has the following first integrals

$$
\begin{aligned}
&P_j \left(\dot{\theta}_1^2 + \dot{\psi}_1^2 \sin^2\theta_1\right) + R_j \dot{\theta}_1^2 \sin^2\theta_1 + 2a_j g \cos\theta_1 = h_j, \\
&P_j \dot{\psi}_1 \sin^2\theta_1 + J_j^s q_j \cos\theta_1 = p_j,
\end{aligned}
$$

where h_j and p_j are constants of integration. Solving the above equations for $\dot{\theta}_1^2$ and $\dot{\psi}_1$ yields

$$
\dot{\theta}_1^2 = \Theta_j(\theta_1), \qquad \dot{\psi}_1 = \Psi_j(\theta_1), \tag{14}
$$

where $\Theta_j(\theta_1) = \left[P_j \sin^2\theta_1 (h_j - 2a_j g\cos\theta_1) - \left(p_j - J_j^s q_j \cos\theta_1\right)^2\right] / [P_j \sin^2\theta_1 \times$ $\times \left(P_j + R_j \sin^2\theta_1\right)], \Psi_j(\theta_1) = \left(p_j - J_j^s q_j \cos\theta_1\right) / \left(P_j \sin^2\theta_1\right)$.

Similarly, for each index $k \in I^*$, the pair of equations (12) leads to the equations

$$\dot\theta_n^2 = \Theta_k(\theta_n), \qquad \dot\psi_n = \Psi_k(\theta_n), \tag{15}$$

where $\Theta_k(\theta_n) = \left[P_k \sin^2\theta_n (h_k - 2a_k g\cos\theta_n) - (p_k - J_k^s q_k \cos\theta_n)^2\right] / [P_k \sin^2\theta_n \times$ $\times \left(P_k + R_k \sin^2\theta_n\right)], \Psi_k(\theta_n) = (p_k - J_k^s q_k \cos\theta_n) / \left(P_k \sin^2\theta_n\right), h_k$ and p_k are constants of integration.

One can check that if the conditions

$$\frac{P_j}{P_1} = \frac{R_j}{R_1} = \frac{J_j^s q_j}{J_1^s q_1} = \frac{a_j}{a_1} = \frac{h_j}{h_1} = \frac{p_j}{p_1}, \qquad \frac{P_k}{P_n} = \frac{R_k}{R_n} = \frac{J_k^s q_k}{J_n^s q_n} = \frac{a_k}{a_n} = \frac{h_k}{h_n} = \frac{p_k}{p_n} \tag{16}$$

are fulfilled, then $\Theta_1(\theta_1) \equiv \Theta_2(\theta_1) \equiv \ldots \equiv \Theta_{n_*}(\theta_1), \Psi_1(\theta_1) \equiv \Psi_2(\theta_1) \equiv \ldots \equiv$ $\equiv \Psi_{n_*}(\theta_1), \Theta_{n_*+1}(\theta_n) \equiv \Theta_{n_*+2}(\theta_n) \equiv \ldots \equiv \Theta_n(\theta_n)$, and $\Psi_{n_*+1}(\theta_n) \equiv$ $\equiv \Psi_{n_*+2}(\theta_n) \equiv \ldots \equiv \Psi_n(\theta_n)$. Hence, in this case, the system of equations (11), (12) reduces to the four equations

$$\dot\theta_1^2 = \Theta_1(\theta_1), \quad \dot\theta_n^2 = \Theta_n(\theta_n), \tag{17}$$

$$\dot\psi_1 = \Psi_1(\theta_1), \quad \dot\psi_n = \Psi_n(\theta_n) = \widetilde\Psi_n(\theta_1), \tag{18}$$

where $\widetilde\Psi_n(\theta_1) = (p_n - J_n^s q_n \mu \cos\theta_1) / \left[P_n\left(1 - \mu^2\cos^2\theta_1\right)\right]$.

It follows from (5) that $\dot\theta_n^2 \sin^2\theta_n = \mu^2 \dot\theta_1^2 \sin^2\theta_1$. Therefore, the right-hand sides of equations (17) are related to each other by the formula

$$\mu^2 \Theta_1(\theta_1)\sin^2\theta_1 - \Theta_n(\theta_n)\sin^2\theta_n = 0. \tag{19}$$

Since $\Theta_m(\theta_m)\sin^2\theta_m$ $(m = 1, n)$ is a polynomial in $\cos\theta_m$, we can eliminate $\cos\theta_n$ from (19) by means of (5). This results in a polynomial in $\cos\theta_1$ which needs to be satisfied identically in $\cos\theta_1$. Equating the coefficients of all powers of $\cos\theta_1$ to zero leads to the following conditions:

$$\begin{aligned}
&\widetilde a_n \widetilde R_1 = \mu \widetilde a_1 \widetilde R_n,\\
&\left[\widetilde h_n + \left(\widetilde J_n^s\right)^2 q_n^2\right]\widetilde R_1 = \mu^2\left[\widetilde h_1 + \left(\widetilde J_1^s\right)^2 q_1^2\right]\widetilde R_n,\\
&\mu^2 \widetilde a_n g - \widetilde p_n \widetilde J_n^s q_n \widetilde R_1 = \mu\left(\widetilde a_1 g - \mu^2 \widetilde p_1 \widetilde J_1^s q_1 \widetilde R_n\right),\\
&\mu^2\left[\widetilde h_n + \left(\widetilde J_n^s\right)^2 q_n^2\right]\left(1 + \widetilde R_1\right) + \left(\widetilde h_n - \widetilde p_n^2\right)\widetilde R_1 =\\
&\qquad = \mu^2\left\{\left[\widetilde h_1 + \left(\widetilde J_1^s\right)^2 q_1^2\right]\left(1 + \widetilde R_n\right) + \mu^2\left(\widetilde h_1 - \widetilde p_1^2\right)\widetilde R_n\right\},\\
&\widetilde a_n g - \widetilde p_n \widetilde J_n^s q_n\left(1 + \widetilde R_1\right) = \mu\left[\widetilde a_1 g - \widetilde p_1 \widetilde J_1^s q_1\left(1 + \widetilde R_n\right)\right],
\end{aligned} \tag{20}$$

$$\left(\widetilde{h}_n - \widetilde{p}_n^2\right)\left(1 + \widetilde{R}_1\right) = \mu^2\left(\widetilde{h}_1 - \widetilde{p}_1^2\right)\left(1 + \widetilde{R}_n\right),$$

where $\widetilde{a}_m = a_m/P_m, \widetilde{h}_m = h_m/P_m, \widetilde{J}^s_m = J^s_m/P_m, \widetilde{p}_m = p_m/P_m$, and $\widetilde{R}_m = R_m/P_m \quad (m = \overline{1, N})$.

We can now state the following:

Proposition 1. *If the conditions* (10), (16), *and* (20) *are fulfilled, the system of equations* (1), (2) *has a class of exact solutions with properties* (4), (5).

Proof. Indeed, we infer from the previous discussion that, under the assumption of the Claim, the system (8) is compatible. To find the dependence of the variables θ_i, ψ_i, and $\varphi_i, i \in I$, on time, we proceed as follows. We find θ_1 as a function of time by integrating the first equation in (17). Next, we determine $\psi_1(t)$ and $\psi_n(t)$ from (18). We can now obtain $\theta_n(t)$ from (5) (or the second equation in (17)) and $\theta_2(t), \theta_3(t), \ldots, \theta_{n-1}(t), \psi_2(t), \psi_3(t), \ldots, \psi_{n-1}(t)$ from (4). Finally, the remaining variables φ_i can be found from (2). This competes the proof of the proposition. □

Based on the quadratures (17) and (18), geometry of the motion of each body in the system can be analyzed by means of the methods that are usually used for studying the motion of a symmetric top.

3. On Compatibility of the Conditions (10), (16), and (20) . As stated in Proposition 1, the system of equations (1), (2) has exact solutions with properties (4), (5) if the conditions (10), (16), and (20) are fulfilled. In this section we show that there exist physically meaningful values of the multibody chain parameters making these conditions compatible in the case when

$$s_i \neq 0, \qquad a_i \neq 0. \tag{21}$$

In this case, relations (10) are equivalent to n_* conditions $\sum\limits_{l=n_*+1}^{n} a_l \varepsilon_{jl} = 0$ and $n - n_*$ conditions $\sum\limits_{l=1}^{n_*} s_l \varepsilon_{kl} = 0$. Using (6), one can verify that, for $i, l, m \in I$, $\varepsilon_{li} = \varepsilon_{lm}\varepsilon_{mi}$ and

$$\sum_{l=n_*+1}^{n} a_l \varepsilon_{jl} = \sum_{l=n_*+1}^{n} a_l \varepsilon_{jn}\varepsilon_{nl} = \varepsilon_{jn} \sum_{l=n_*+1}^{n} a_l \varepsilon_{nl},$$
$$\sum_{l=1}^{n_*} s_l \varepsilon_{kl} = \sum_{l=1}^{n_*} s_l \varepsilon_{k1}\varepsilon_{1l} = \varepsilon_{k1} \sum_{l=1}^{n_*} s_l \varepsilon_{1l}.$$

Hence, conditions (10) can be replaced with the two equalities:

$$\sum_{l=n_*+1}^{n} a_l \varepsilon_{nl} = 0, \qquad \sum_{l=1}^{n_*} s_l \varepsilon_{1l} = 0. \tag{22}$$

We also note that, if either of the subsystems S_* and S^* consists of a single top only, one of the above equalities contradicts one of the assumptions (21). Therefore, the system S can perform the motion of interest only when each of its subsystems S_* and S^* consists of at least two tops, making the total number of bodies in S not less than four.

The case of a four-body system. Below we consider the simplest possible case of a four-body system assuming that $S_* = \{B_1, B_2\}, S^* = \{B_3, B_4\}, \delta_2 = \delta_3 = 1$, and $R_1 = R_2 = R_3 = R_4 = 0$. Then, the last conditions and relations (22) imply

$$s_1 = s_2, \qquad s_1 + \mu s_3 = 0, \qquad a_3 = a_4, \qquad a_2 + \mu a_4 = 0, \tag{23}$$

the conditions (16) become

$$\frac{J_2' - s_1 a_2}{J_1' - s_1 a_2} = \frac{J_2^s q_2}{J_1^s q_1} = \frac{a_2}{a_1} = \frac{h_2}{h_1} = \frac{p_2}{p_1}, \tag{24}$$

$$J_3' = J_4', \qquad J_3^s q_3 = J_4^s q_4, \qquad p_3 = p_4, \qquad h_3 = h_4, \tag{25}$$

and the relations (20) reduce to

$$\widetilde{a}_1 = \mu \widetilde{a}_4, \tag{26}$$

$$\widetilde{h}_4 + \left(\widetilde{J}_4^s\right)^2 q_4^2 = \widetilde{h}_1 + \left(\widetilde{J}_1^s\right)^2 q_1^2, \tag{27}$$

$$\widetilde{a}_4 g - \widetilde{p}_4 \widetilde{J}_4^s q_4 = \mu \left(\widetilde{a}_1 g - \widetilde{p}_1 \widetilde{J}_1^s q_1\right), \tag{28}$$

$$\widetilde{h}_4 - \widetilde{p}_4^2 = \mu^2 \left(\widetilde{h}_1 - \widetilde{p}_1^2\right). \tag{29}$$

The relations (23)–(29) form an algebraic system of 16 equations with respect to 32 unknowns $J_i^0(> 0), J_i^s(> 0), m_i(> 0), c_i, p_i, h_i, q_i \quad (i = 1, 2, 3, 4)$, s_i $(i = 1, 2, 3)$, and μ. Here J_i^0 is the central equatorial moment of inertia of the body B_i and

$$J_i = J_i^0 + m_i c_i^2. \tag{30}$$

In the rest of this section we solve the following problem: if the chain parameters defining the mass distribution of its bodies and the parameter μ are known, find possible ways for coupling the bodies as well as the initial conditions of their motion. In other words, given the values of $J_i^0,\ J_i^s,\ m_i$, and μ, we seek to find the quantities $c_i,\ s_i,\ p_i,\ h_i$, and q_i so that the system of relations (23)–(29) is compatible.

We start our analysis of the abovementioned system with relations (23). From the first two relation in (23), we have

$$s_2 = s_1 \qquad \text{and} \qquad s_3 = -s_1/\mu. \tag{31}$$

By virtue of (3) and (31), the remaining pair of equations in (23) can be solved for c_2 and c_3 as follows:

$$c_2 = -\left(\mu m_4 c_4 + \widetilde{m}_2 s_1\right)/m_2, \tag{32}$$

$$c_3 = m_4 \left(\mu c_4 + s_1\right) / \left(\mu m_3\right). \tag{33}$$

Next, we observe that equation (26) can be transformed, by means of (24), into

$$\mu a_4 \left(J_2' - s_1 a_2\right) = a_2 \left(J_4' - s_3 a_4\right). \tag{34}$$

After successive substitutions of the expressions for J_i', a_i, s_2, s_3, c_2, and c_3 from (3) and (30)–(32) into (34) and the first equation in (25), we arrive at the system of two equations with respect to c_4 and s_1:

$$\mu^2 m_4 \left(m_4 - m_3\right) c_4^2 + 2\mu m_4^2 c_4 s_1 + \widetilde{m}_2 m_4 s_1^2 + \mu^2 m_3 \left(J_3^0 - J_4^0\right) = 0, \tag{35}$$

$$\begin{aligned}&\mu m_4 \left(m_2 + \mu^2 m_4\right) c_4^2 + m_4 \left[m_2 + \mu^2 \left(\widetilde{m}_1 + \widetilde{m}_2\right)\right] c_4 s_1 + \mu \widetilde{m}_1 \widetilde{m}_2 s_1^2 + \\ &\quad + \mu m_2 \left(J_2^0 + J_4^0\right) = 0.\end{aligned} \tag{36}$$

For the sake of brevity, below we consider a special case when

$$J_3^0 = J_4^0. \tag{37}$$

Then, equation (35) decomposes into the following two cases: $s_1 = -\mu c_4$ or

$$s_1 = \mu c_4 (m_3 - m_4)/\widetilde{m}_2. \tag{38}$$

In the first case, equation (36) reduces to the form $c_4^2 = -m_2 \left(J_2^0 + J_4^0\right) \times \times \left[\mu^2 m_3 \left(m_2 + m_3\right)\right]^{-1}$ and cannot be satisfied by the acceptable values of the quantities it contains.

In the second case, when s_1 and c_4 are related by (38), equation (36) assumes the form

$$c_4^2 = \alpha_1, \tag{39}$$

where $\alpha_1 = m_2 \widetilde{m}_2 \left(J_2^0 + J_4^0\right) / \left(m_3 \alpha_2\right)$, $\alpha_2 = \left(\mu^2 - 2\right) m_2 m_4 - \mu^2 \widetilde{m}_1 m_3$. Clearly, one can find a real-valued c_4 from (39) only when $\alpha_2 > 0$. This inequality is true if the parameters μ, m_3, and m_2 are selected such that

$$\mu^2 > 2, \tag{40}$$

$$m_3 < \left(1 - 2/\mu^2\right) m_4, \qquad m_2 > m_3 \left(m_3 + m_4\right) / \left[\left(1 - 2/\mu^2\right) m_4 - m_3\right]. \tag{41}$$

(Note that it follows from (41) that

$$m_3 - m_4 < 0 \qquad \text{and} \qquad \widetilde{m}_1 m_3 - m_2 m_4 < 0, \tag{42}$$

respectively.)

We now consider the equation (see (24)) $a_1 \left(J_2' - s_1 a_2\right) = a_2 \left(J_1' - s_1 a_2\right)$. Using (3), (30)–(32), (38), and (39), we can write it in the form $\beta_1 c_1^2 + \beta_2 c_1 + \beta_3 = 0$, where

$$\begin{aligned}
\beta_1 &= -\alpha_2 \mu m_1 \widetilde{m}_2 m_3 c_4, \\
\beta_2 &= m_1 \widetilde{m}_2 m_3 \left[2 J_2^0 m_2 m_4 - \mu^2 J_4^0 (\widetilde{m}_1 m_3 - m_2 m_4)\right], \\
\beta_3 &= \mu c_4 \Big[\widetilde{m}_2 m_3 m_4 \left(\mu^2 \widetilde{m}_1 m_3 + \left(2 - \mu^2\right) m_2 m_4\right) J_1^0 + \\
&\quad + m_2 m_4 (m_3 - m_4) \left(\left(2 - \mu^2\right) \widetilde{m}_1 m_3 + \mu^2 m_2 m_4\right) J_2^0 + \\
&\quad + \mu^2 \left(m_4^2 - m_3^2\right) \left(m_2 + m_3\right) \left(\widetilde{m}_1 m_3 - m_2 m_4\right) J_4^0\Big].
\end{aligned}$$

The discriminant of the last equation $D = m_1\widetilde{m}_2^2\alpha_3$ is nonnegative only when

$$\alpha_3 = -\alpha_4 J_1^0 + \alpha_5 \geq 0. \tag{43}$$

Here $\alpha_4 = 4\alpha_2\mu^2 m_2\widetilde{m}_2 m_3 m_4^2\left(J_2^0 + J_4^0\right)$,

$$\begin{aligned}
\alpha_5 &= \left(J_4^0\right)^2\left[\alpha_6\left(J_2^0/J_4^0\right)^2 + \alpha_7\left(J_2^0/J_4^0\right) + \alpha_8\right], \qquad (44)\\
\alpha_6 &= 4m_2^2m_4^2\left[m_1m_3^2 + \mu^2(m_3 - m_4)\left(\left(2-\mu^2\right)\widetilde{m}_1m_3 + \mu^2m_2m_4\right)\right],\\
\alpha_7 &= 4\mu^2m_2m_4\big[m_1m_3^2\left(m_2m_4 - \widetilde{m}_1m_3\right) + 2\widetilde{m}_1m_2m_3m_4(m_3-m_4)+\\
&\qquad + \mu^2(m_4-m_3)\left(\widetilde{m}_1\widetilde{m}_2m_3(m_2+m_3) - 2m_2^2m_4^2\right)\big],\\
\alpha_8 &= \mu^4\left(\widetilde{m}_1m_3 - m_2m_4\right)\big[4m_2m_4(m_2+m_3)\left(m_4^2 - m_3^2\right)+\\
&\qquad + m_1m_3^2\left(\widetilde{m}_1m_3 - m_2m_4\right)\big].
\end{aligned}$$

Note that $\alpha_4 > 0$ for all acceptable values of the mechanical parameters. Therefore, in order to satisfy (43), one should require that

$$J_1^0 \leq \alpha_5/\alpha_4. \tag{45}$$

A meaningful value of J_1^0 satisfying (45) can only be selected when $\alpha_5 > 0$. The sign of α_5 coincides with the sign of the quadratic polynomial (in J_2^0/J_4^0) in (44). The leading coefficient of this polynomial, α_6, is positive when

$$m_1 > \mu^2(m_3 - m_4)\left[(\mu^2-2)\widetilde{m}_1m_3 - \mu^2m_2m_4\right]/m_3^2 \tag{46}$$

and, in accordance with (42), its discriminant $D_* = 16\alpha_2^2\mu^4m_2^2m_3^2m_4^2\left(m_3-m_4\right)\times$ $\times\left[m_1(\widetilde{m}_1m_3 - m_2m_4) + \widetilde{m}_1^2\left(m_3-m_4\right)\right]$ is always positive. Hence, the polynomial in J_2^0/J_4^0 in (44) has two real roots and, by choosing

$$J_2^0 > J_4^0\left(-\alpha_7 + \sqrt{D_*}\right)/(2\alpha_6), \tag{47}$$

we obtain $\alpha_5 > 0$.

Based on the above analysis, we can suggest the following algorithm for selecting the multibody system parameters satisfying the system of equations (23)–(29).

First, we assume that the moments $J_i^c\,(>0)\,(i = 1,2,3,4)$, J_4^0 and the mass m_4 are chosen arbitrarily. Then, $J_3^0 = J_4^0$ (see (37)). We also assume that $\mu > \sqrt{2}$ or $\mu < -\sqrt{2}$ (see (40)) and the masses m_3, m_2, and m_1 as well as the moments J_2^0 and J_1^0 are successively selected to satisfy the inequalities (41), (46), (47), and (45). Now, one can find c_4 from (39): $c_4 = \pm\sqrt{\alpha_1}$. Selecting one of the two possible values of c_4, we obtain s_1 from (38) and, then, s_2, s_3, c_2, and c_3 from (31)–(33). We can also compute c_1 by the formula $c_1 = \left(-\beta_2 \pm \sqrt{D}\right)/(2\beta_1)$.

Note that, by this moment, we have already obtained the values of all parameters of interest, except for p_i, q_i, and $h_i\,(i = 1,2,3,4)$. To find the remaining parameters, we assign arbitrary values to p_1, p_4, and q_4. This immediately gives us the value of p_3 (see (25)). We can also find p_2 from (24):

$p_2 = a_2 p_1/a_1$. Next, by means of (28), (24), and (25), we derive that $q_1 = \left(p_4 \widetilde{J}_4^s q_4 - \widetilde{a}_4 g + \mu \widetilde{a}_1 g\right) / \left(\mu \widetilde{p}_1 \widetilde{J}_1^s\right), q_2 = a_2 J_1^s q_1 / \left(a_1 J_2^s\right)$, and $q_3 = J_4^s q_4 / J_3^s$. Solving the system of equations (27) and (29) with respect to h_1 and h_4, we obtain

$$h_1 = \left(J_1' - s_1 a_2\right) \left[\widetilde{p}_4^2 + \left(\widetilde{J}_4^s\right)^2 q_4^2 - \left(\widetilde{J}_1^s\right)^2 q_1^2 - \mu^2 \widetilde{p}_1^2\right] / (1 - \mu^2),$$

$$h_4 = \left(J_4' - s_3 a_4\right) \left[\widetilde{p}_4^2 + \mu^2 \left(\left(\widetilde{J}_4^s\right)^2 q_4^2 - \left(\widetilde{J}_1^s\right)^2 q_1^2 - \widetilde{p}_1^2\right)\right] / (1 - \mu^2).$$

Finally, according to (25), $h_3 = h_4$ and h_2 can be found from (24): $h_2 = a_2 h_1 / a_1$. Knowing the values of the integration constants, one can determine the initial conditions of motion, using (2), (4), (5), (17), and (18).

Thus, we have proven that it is possible to select physically meaningful values of parameters characterizing the chain of rigid bodies under consideration so that the relations (10), (16), and (20) will be fulfilled. This completes our proof on the existence of the motions of interest for the chain of Lagrange tops.

4. Some properties of the motions of interest. In this section we give a mechanical interpretation of conditions (22). We recall that O_{n_*+1} is the point where the subsystems S_* and S^* are coupled to each other. Let also C^* denote the center of mass of S^*.

Proposition 2. *When the system S performs the motion of interest, the points O_{n_*+1} and C^* move along the vertical line L passing through O_1.*

We denote $\mathbf{s}_* = \mathbf{O}_1\mathbf{O}_{n_*+1}, \mathbf{c}^* = \mathbf{O}_1\mathbf{C}^*, m^* = \widetilde{m}_{n_*}$ and observe that

$$\mathbf{s}_* = \sum_{j=1}^{n_*} \mathbf{s}_j = \sum_{j=1}^{n_*} s_j \mathbf{e}_3^{(j)},$$

$$m^* \mathbf{c}^* = \sum_{k=n_*+1}^{n} m_k \mathbf{O}_1 \mathbf{C}_k = \sum_{k=n_*+1}^{n} m_k \left(\mathbf{c}_k + \mathbf{s}_* + \sum_{i=n_*+1}^{k-1} \mathbf{s}_i\right) =$$

$$= m^* \mathbf{s}_* + \sum_{k=n_*+1}^{n} \left(m_k \mathbf{c}_k + \widetilde{m}_k \mathbf{s}_k\right) = m^* \mathbf{s}_* + \sum_{k=n_*+1}^{n} a_k \mathbf{e}_3^{(k)}. \tag{48}$$

As was mentioned in section 1, when the system S moves so that the properties (4) are fulfilled, the skeleton of the subsystem $S_*(S^*)$ remains in the vertical plane $\Pi_*(\Pi^*)$. Clearly, the point O_{n_*+1} belongs to the plane Π_* which rotates about L with the speed $\dot{\psi}_1(t)$. Introducing the Cartesian frame $\{O_1, \mathbf{e}_3 \mathbf{e}_*\}$ rigidly embedded in Π_* (the unit vector $\mathbf{e}_*$ is chosen such that $\mathbf{e}_* \cdot \mathbf{s}_1 = s_1 \sin\theta_1$), we obtain, using the second relation in (22), that $\mathbf{s}_* \cdot \mathbf{e}_* = \sum\limits_{j=1}^{n_*} s_j \varepsilon_{1j} \sin\theta_1 = 0$, i.e. $\mathbf{s}_* || \, \mathbf{e}_3 \, || \, L$.

Since the attachment point of the subsystem S^* moves along L, we conclude that the plane Π^*, containing the point C^*, rotates about L with the speed

$\dot{\psi}_n(t)$. Now, in order to prove that $\mathbf{c}^* || L$, it is sufficient to show that the second term in (48) is parallel to L. Introducing the Cartesian frame $\{O_{n_*+1}, \mathbf{e}_3\mathbf{e}^*\}$ rigidly embedded in Π^* (the unit vector $\mathbf{e}^*$ is chosen such that $\mathbf{e}^* \cdot \mathbf{s}_n = s_n \sin\theta_n$), we obtain, using the first relation in (22), that $\left(\sum\limits_{k=n_*+1}^{n} a_k \mathbf{e}_3^{(k)}\right) \cdot \mathbf{e}^* = \sum\limits_{k=n_*+1}^{n} a_k \varepsilon_{nk} \sin\theta_n = 0$, which implies $\mathbf{c}^* || \, \mathbf{e}_3 \, || \, L$.

Thus, we have proven that both points O_{n_*+1} and C^* move along L.

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Новый класс нестационарных движений системы тяжелых гироскопов Лагранжа с неплоской конфигурацией остова системы

Для цепочки n тяжелых гироскопов Лагранжа, соединенных идеальными сферическими шарнирами, установлено существование класса нестационарных движений, при которых остов системы имеет неплоскую конфигурацию. Получены достаточные условия существования таких движений. Найдена зависимость основных переменных от времени. При заданном распределении масс в телах для цепочки, состоящей из четырех тел, определены способы их сочленения, при которых установленные движения возможны. Указаны некоторые свойства новых движений.

Ключевые слова: *аналитическая динамика систем тел, гироскоп Лагранжа, нестационарное движение системы твердых тел*

Д.О. Чебанов

Новий клас нестаціонарних рухів системи важких гіроскопів Лагранжа з неплоскою конфігурацією остова системи

Для ланцюжка n важких гіроскопів Лагранжа, з'єднаних ідеальними сферичними шарнірами, встановлено існування класу нестаціонарних рухів, при яких остів системи має неплоску конфігурацію. Отримано достатні умови існування таких рухів. Знайдено залежність основних змінних від часу. При заданому розподілі мас в тілах для ланцюжка, що складається з чотирьох тіл, визначено способи їх зчленування, при яких встановлені рухи можливі. Указано деякі властивості нових рухів.

Ключові слова: *аналітична динаміка систем тіл, гіроскоп Лагранжа, нестаціонарний рух системи твердих тіл*

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