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On the efficient method of solving ill-posed problems by adaptive discretization

To solve ill-posed problems $Ax = f$ is used the Fakeev-Lardy regularization, using an adaptive discretization strategy. It is shown that for some classes of finitely smoothing operators proposed algorithm achieves the optimal order of accuracy and is more economical in the sense of amount of discrete information then standard methods.

Keywords: *optimal approximations, order of accuracy, ill-posed problems, Fakeev-Lardy method, discrepancy principle*

1. Introduction. Statement of the problem

In a Hilbert space X with inner product $(\cdot\,,\cdot)$ and norm

$$\|x\| = \sqrt{(x,x)}$$

we consider the operator equation of the first kind

$$Ax = f, \tag{1}$$

where A is a compact linear operator in X and $f \in \mathrm{Range(A)}$. Suppose that instead of the exact right-hand side of (1) some its perturbation $f_\delta : \|f - f_\delta\| \le \delta$, $\delta > 0$ is available only.

We will construct approximations to minimal-norm solution $x^\dagger$ of (1) that satisfies the Holder-type source condition, i.e.

$$(2)\quad x^\dagger \in M_{\nu,\rho}(A) = \{u : u = |A|^\nu v, \|v\| \leq \rho\},\qquad |A| = (A^*A)^{1/2}, \quad \rho \geq 1,$$

where A^* is the adjoint of A and the parameter $\nu > 0$ is unknown.

Consider a class $\mathcal{H}^r$, $r = 1, 2, \ldots$, of compact linear operators A, $\|A\| \leq 1$, such that for any $m = 1, 2, \ldots$ the conditions

$$(3)\qquad \| (I - P_m)A \| \leq m^{-r}, \qquad \| A(I - P_m) \| \leq m^{-r}$$

are satisfied, where P_m is the orthoprojector onto linear span of the first m elements of some orthonormal basis $E = \{e_i\}_{i=1}^\infty$ in space X.

As an example of equation (1) with operator $A \in \mathcal{H}^r$ in the space $X = L_2(0,1)$ one can take Fredholm integral equation of the first kind

$$Ax(t) \equiv \int_0^1 k(t,\tau)x(\tau)d\tau = f(t),$$

where $\max\limits_{0\leq t,\tau\leq 1} |k(t,\tau)| \leq 1$, operators A and A^* act from $L_2(0,1)$ into the Sobolev space $W_2^r[0,1]$ and as basis E is selected the orthonormal system of Legendre polynomials or (if $r = 1$) the orthonormal system of Haar functions.

It is known (see [11, p. 14]) that the best accuracy of recovering minimal-norm solutions of (1) that fill up set $M_{\nu,\rho}(A)$ can be lower estimated by

$$\rho^{1/(\nu+1)}\delta^{\nu/(\nu+1)}.$$

This is because every method guaranteeing approximation accuracy $O(\delta^{\nu/(\nu+1)})$ on the set of solutions (2) is referred to as order-optimal approximate method for solving (1).

In this paper we investigate projection methods of solving (1) that using Galerkin information as discrete information about (1).

Remind that by Galerkin information about equation (1) one usually mean a set of inner products

$$(Ae_j, e_i), \qquad (f_\delta, e_i). \tag{4}$$

The volume of inner products (4) used to approximate solve (1) characterizes economical properties of corresponding projection methods.

Obviously that to construct economical projection method special attention must be put to effective choice of set Ω of indices (i, j) for inner products (Ae_j, e_i) which form discrete operator A_Ω.

In the first time the problem of constructing economical projection methods for solving (1) was studied in [3] in the framework of traditional Galerkin discretization scheme with $\Omega = [1, m] \times [1, n]$. From [3] it is follows that to guarantee the optimal order of accuracy we need to choose $n \asymp m \asymp O(\delta^{-1/r})$, i.e. to compute at least $O(\delta^{-2/r})$ inner products (4).

Statement of the problem. Our aim is to construct an algorithm of solving (1) on class of operators $\mathcal{H}^r$ such that, firstly, guarantees the optimal order of accuracy for solutions $x^\dagger$ of the form (2) and, secondly, is more economical in the sense of using Galerkin information compare with methods considered in [3].

To construct such algorithm we will use an adaptive approach to discretization that earlier was studied in [1]. To reduce the volume of Galerkin information for this approach it will apply so-called hyperbolic cross (see Section 4) as the area Ω and the discretization level will be selected during computations as following

$$\| A^*A - A^*_\Omega A_\Omega \| = O(\sqrt{\alpha}\delta), \tag{5}$$

where α is a regularization parameter.

For the first time such adaptive discretization scheme was studied in [1] for the standard Tikhonov method. In [6], [8] it was investigated the optimality of the adaptive approach for the stationary iterated Tikhonov method, in [4] for the nonstationary iterated Tikhonov method, in [10] for the generalized Tikhonov method, in [8], [9] for the Landweber method and in [7] for the method of asymptotical regularization.

It turn out that discretization strategy (1.5) allows to solve the problem formulated for all mentioned above regularization methods. Let us continue these investigations and verify efficiency of adaptive discretization for the Fakeev-Lardy regularizator.

In conclusion we want mention one more adaptive discretization scheme proposed in [2]. In the framework of this scheme the discretization level is chosen as $\| A - A_\Omega \| = O(\sqrt{\alpha}\sqrt{\delta})$, and as area Ω is selected rectangle. It turn out that such approach is not order-optimal and is less economical with compare both nonadaptive scheme in [3] and adaptive scheme in the present paper.

2. Fakeev-Lardy method

The Fakeev-Lardy method is an iterative procedure of the following type:

$$(6) \quad x_0 = 0; \quad \mu x_l + A^* A x_l = \mu x_{l-1} + A^* f_\delta, \qquad l = 1, 2, \dots, \mu = \text{const} > 2/\rho.$$

For generating function of this method

$$g_l(\lambda) = \frac{1}{\lambda}\left(1 - \left(\frac{\mu}{\mu+\lambda}\right)^l\right) = \sum_{j=0}^{l-1} \frac{\mu^j}{(\mu+\lambda)^{j+1}}, \quad \lambda \neq 0,$$

the following estimates (see [11, p. 22])

$$\sup_{0\le\lambda<\infty} g_l(\lambda) = l/\mu; \qquad \sup_{0\le\lambda<\infty} \lambda g_l(\lambda) \le 1;$$
$$(7) \quad \sup_{0\le\lambda<\infty} \lambda^{1/2} g_l(\lambda) = (l/\mu)^{1/2}; \qquad \sup_{0\le\lambda<\infty} \lambda^p (1 - \lambda g_l(\lambda)) \le \kappa_p l^{-p};$$
$$0 \le p \le l, \quad \kappa_0 = 1, \quad \kappa_p = (\mu p)^p, \quad p > 0.$$

are true.

Let λ_k are singular values of operator A and ϕ_k, ψ_k are corresponding singular elements. Then operator A can be written as

$$A = \sum_k \lambda_k \phi_k(\cdot\,, \psi_k)$$

and following relations

$$\begin{aligned} x^\dagger &= |A|^\nu v = (A^*A)^{\nu/2} v = \sum_k |\lambda_k|^\nu \psi_k(\psi_k, v), \\ f &:= Ax^\dagger = A|A|^\nu v = \sum_k \lambda_k |\lambda_k|^\nu \phi_k(\psi_k, v). \end{aligned} \tag{8}$$

hold.

Then the elements x_l and Ax_l can be written as

$$x_l = g_l(A^*A)|A|^{\nu+2} v = g_l(A^*A) \sum_k |\lambda_k|^{\nu+2} \psi_k(\psi_k, v) = \\ = \sum_k g_l(|\lambda_k|^2)|\lambda_k|^{\nu+2} \psi_k(\psi_k, v),$$

$$\begin{aligned} Ax_l &= \sum_k \lambda_k \phi_k \left(\psi_k, \sum_m g_l(|\lambda_m|^2)|\lambda_m|^{\nu+2} \psi_m(\psi_m, v) \right) = \\ &= \sum_k \lambda_k \phi_k \sum_m |\lambda_m|^{\nu+2} (\psi_m, v) g_l(|\lambda_m|^2)(\psi_k, \psi_m) = \\ &= \sum_k \lambda_k |\lambda_k|^{\nu+2} g_l(|\lambda_k|^2) \phi_k(\psi_k, v). \end{aligned}$$

As it follows from (5) in our approximate method discretized operator can be changed in every step of iterations. Denote as A_l, $l = 1, 2, \ldots$, discretized operator corresponding l-th step of iterative process. More detailed this discretization scheme will be considered in Section 4.

Thus a finite-dimensional version of the method (6) has the form

$$\hat{x}_1 = (\mu I + A_1^* A_1)^{-1} A_1^* f_\delta,$$
$$\hat{x}_2 = (\mu I + A_2^* A_2)^{-1}(\mu(\mu I + A_1^* A_1)^{-1} A_1^* + A_2^*) f_\delta,$$
$$\dots$$
$$\hat{x}_l = \sum_{k=0}^{l-1} \mu^k \left[\prod_{j=0}^{k} (\mu I + A_{l-j}^* A_{l-j})^{-1} \right] A_{l-k}^* f_\delta.$$

To prove the optimality of the method we have to estimate error of approximation of minimal-norm solution $x^\dagger$ by elements $\hat{x}_l$. So in l-th step of iterative process it's holds

$$x^\dagger - \hat{x}_l = g_l(A^*A)A^*(f - f_\delta) + \\ + (x^\dagger - g_l(A^*A)A^*f) + (g_l(A^*A)A^*f_\delta - \hat{x}_l)$$

and hence the error can be upper estimated:

$$\|x^\dagger - \hat{x}_l\| \le \|g_l(A^*A)A^*(f - f_\delta)\| + \\ + \|x^\dagger - x_l\| + \|g_l(A^*A)A^*f_\delta - \hat{x}_l\|. \tag{9}$$

Let us estimate now the right-hand side of (9) term by term. Due to conditions (7) on generating function it is immediately follows that for the first term

$$\|g_l(A^*A)A^*(f - f_\delta)\| \le \|g_l(A^*A)A^*\| \|f - f_\delta\| \le \\ \le \delta \sup_\lambda \lambda^{1/2} g_l(\lambda) \le \delta \left(\frac{l}{\mu}\right)^{1/2}. \tag{10}$$

The second term can be represented by (8) as

$$x^\dagger - x_l = (I - g_l(A^*A)A^*A)x^\dagger = \sum_k \left(\frac{\mu}{\mu + \lambda_k^2}\right)^l |\lambda_k|^\nu \psi_k(v, \psi_k). \tag{11}$$

Thus,

$$\|x^\dagger - x_l\|^2 = \sum_k \left(\frac{\mu}{\mu+\lambda_k^2}\right)^{2l} |\lambda_k|^{2\nu}(v,\psi_k)^2,$$

or

$$\|x^\dagger - x_l\|^2 = |c_{\nu,l}(v)|^2 l^{-\nu}, \tag{12}$$

where $|c_{\nu,l}(v)|^2 := l^\nu \sum_k \left(\frac{\mu}{\mu+\lambda_k^2}\right)^{2l} |\lambda_k|^{2\nu}(v,\psi_k)^2$.

To estimate (11) we need to estimate $\|Ax_l - f\|$ too. Taking into account (8) and relation

$$1 - \lambda^2 g_l(\lambda^2) = \left(\frac{\mu}{\mu+\lambda^2}\right)^l,$$

we have

$$\|Ax_l - f\|^2 = \mu^{2l} \sum_k |\lambda_k|^{2(\nu+1)} \frac{(v,\psi_k)^2}{(\mu+\lambda_k^2)^{2l}}.$$

and hence

$$\|Ax_l - f\|^2 = |d_{\nu,l}(v)|^2 l^{-(\nu+1)} \tag{13}$$

with $|d_{\nu,l}(v)|^2 := \mu^{2l} \sum_k |\lambda_k|^{2(\nu+1)} \frac{(v,\psi_k)^2}{(\mu+\lambda_k^2)^{2l}} l^{\nu+1}$.

To estimate $\|x^\dagger - x_l\|$ we need the following auxiliary statement.

Lemma 1. *For functions $c_{\nu,l}(v)$ and $d_{\nu,l}(v)$ the bounds*

$$|c_{\nu,l}(v)| \le \rho \kappa_{(\nu+1)/2}^{\nu/(\nu+1)}, \qquad |d_{\nu,l}(v)| \le \rho \kappa_{\frac{\nu+1}{2}}$$

hold.

Using Holder's inequality we have

$$|c_{\nu,l}(v)|^2 =$$

$$\sum_k \left(\frac{l^{\nu+1}\mu^{2l}\lambda_k^{2(\nu+1)}}{(\mu+\lambda_k^2)^{2l}}(v,\psi_k)^2 \right)^{\frac{\nu}{\nu+1}} \left(\frac{\mu^{2l}}{(\mu+\lambda_k^2)^{2l}}(v,\psi_k)^2 \right)^{\frac{1}{\nu+1}} \leq$$

$$\leq \sum_k \left(\frac{l^{\nu+1}\mu^{2l}\lambda_k^{2(\nu+1)}}{(\mu+\lambda_k^2)^{2l}}(v,\psi_k)^2 \right)^{\frac{\nu}{\nu+1}} \|v\|^{\frac{2}{\nu+1}} =$$

$$= |d_{\nu,l}(v)|^{\frac{2\nu}{\nu+1}} \rho^{\frac{2}{\nu+1}}.$$

For the second inequality we obtain

$$|d_{\nu,l}(v)|^2 \leq l^{\nu+1} \sup_\lambda \lambda^{2(\nu+1)} \left(\frac{\mu}{\mu+\lambda^2} \right)^{2l} \sum_k (v,\psi_k)^2 \leq$$

$$\leq \kappa^2_{(\nu+1)/2} \|v\|^2 = \rho^2 \kappa^2_{(\nu+1)/2}.$$

Substitution of this estimate into previous inequality completes the proof of Lemma.

Thus due to (12) and to the first estimate in Lemma 1 we have

$$\|x^\dagger - x_l\| \leq \rho \kappa_{(\nu+1)/2}^{\nu/(\nu+1)} l^{\frac{-\nu}{2}}. \tag{14}$$

To estimate the last term in (9) we consider the auxiliary operator

$$B_l = \sum_{k=0}^{l-1} \left(\mu^k (\mu I + A^*A)^{-(k+1)} A^* - G_{k,l} A^*_{l-k} \right)$$

with $G_{k,l} = \mu^k \prod_{j=0}^k (\mu I + A^*_{l-j} A_{l-j})^{-1}$.

Then for the third item in the right-hand side of (9) we obtain

$$g_l(A^*A)A^* f_\delta - \hat{x}_l = B_l f_\delta. \tag{15}$$

To estimate norm of the element $B_l f_\delta$ we write down B_l in more suitable form that will be shown in next statement.

Lemma 2. *For any $l = 2, 3, \dots$ it holds*

(16)
$$B_l = \sum_{k=0}^{l-1} \mu^k (\mu I + A^* A)^{-(k+1)} (A^* - A^*_{l-k}) - \sum_{k=1}^{l} F_k,$$

where

(17)
$$F_k = \sum_{j=0}^{l-k} (\mu I + A^* A)^{-j} T_{j,k}, \quad k = 1, \dots, l;$$
$$T_{j,k} = D_j \sum_{i=j+1}^{l} (\mu I + A^* A)^{-(i-j)} T_{i,k-1},$$
$$j = 0, \dots, l-k, \quad k \geq 2;$$
$$D_j = (\mu I + A^*_{l-j} A_{l-j})^{-1} (A^* A - A^*_{l-j} A_{l-j}), \quad j = 0, \dots, l-1;$$
$$T_{j,1} = D_j \sum_{i=j+1}^{l} \mu^{i-1} (\mu I + A^* A)^{-(i-j)} A^*_{l-i+1}, \quad j = 0, \dots, l-1.$$

To reduct computation we introduce into consideration some denotations:

$$I_\mu := \mu I + A^* A; \qquad J_\mu := I_\mu^{-1};$$
$$U_j := A^* A - A^*_{l-j} A_{l-j}; \qquad H_j := D_j J_\mu = (I_\mu - U_j)^{-1} U_j J_\mu.$$

Quite easy to check that

$$(\mu I + A^*_{l-j} A_{l-j})^{-1} = J_\mu + H_j$$

and I_μ, J_μ, U_j, $H_j \in \mathcal{L}(X)$, where $\mathcal{L}(X)$ is the space of linear continuous operators in X.

Further we need to introduce a special operation of substituting operators. Thus suppose that we have sequence of operators

$\{\Phi_i\}, i = 1, 2, \ldots,\ \Phi_i \in \mathcal{L}(X)$, and operator $\Psi \in \mathcal{L}(X)$. The operation of substitution we will note as

$$\Phi \bigoplus_{N}^{M} \Psi^{(p)},$$

where $M \geq N \geq 1,\ \ p \leq M - N + 1$. This operation affects on product of $M - N + 1$ operators $\Phi_N, \Phi_{N+1}, \ldots, \Phi_M$. The main point of the operation consists in replacement of all possible combinations from p distinct operators Φ_i of initial product by the operator Ψ with preserved order of remained $(M - N - p + 1)$ multipliers Φ_i. Thus, as result of described operation we obtain a sum of $\frac{(M-N+1)!}{p!(M-N-p+1)!}$ (distinct!) replacement in such way operators, every of it is the product of p operators Ψ and $(M-N-p+1)$ operators Φ_i.

The above operation has some properties that will be used in further reasoning. Namely,

$$\Phi \bigoplus_{N}^{M} \Psi^{(M-N+1)} = \prod_{j=N}^{M} \Psi_j,$$

$$\Phi \bigoplus_{N}^{M} \Psi^{(M-N)} = \sum_{q=0}^{M-N} \left(\prod_{i=0}^{q-1} \Psi_{i+N} \right) \Phi_{N+q} \left(\prod_{s=q+1}^{M-N} \Psi_{s+N} \right),$$

$$\Phi \bigoplus_{N}^{M} \Psi^{(p)} = \sum_{q=0}^{p} \left(\prod_{i=0}^{q-1} \Psi_{i+N} \right) \Phi_{N+q} \left(\Phi \bigoplus_{N+q+1}^{M} \Psi^{(p-q)} \right), \qquad p < M - N.$$

Let's note operator $G_{k,l}$ in new form with help of above operation:

$$G_{k,l} = \mu^k \prod_{j=0}^{k} (I_\mu - U_j)^{-1} = \mu^k \prod_{j=0}^{k} (H_j + J_\mu) =$$

$$= \mu^k \sum_{i=0}^{k+1} H \bigoplus_0^k J_\mu^{(k-i+1)} =$$

$$= \mu^k \left(J_\mu^{k+1} + \sum_{i=1}^{k+1} H \bigoplus_0^k J_\mu^{k-i+1} \right) = \mu^k \left(J_\mu^{k+1} + \sum_{i=1}^{k+1} S_{k,i} \right)$$

with $S_{k,i} = H \bigoplus\limits_0^k J_\mu^{(k-i+1)}$.

Then

$$B_l := \sum_{k=0}^{l-1} \left(\mu^k J_\mu^{k+1} A^* - \mu^k (J_\mu^{k+1} + \sum_{i=1}^{k+1} S_{k,i}) A_{l-k}^* \right) =$$
$$= \sum_{k=0}^{l-1} \mu^k J_\mu^{k+1} (A^* - A_{l-k}^*) - \sum_{k=0}^{l-1} \mu^k (\sum_{i=1}^{k+1} S_{k,i}) A_{l-k}^* =$$
$$= \sum_{k=0}^{l-1} \mu^k J_\mu^{k+1} (A^* - A_{l-k}^*) - \sum_{j=1}^{l} \sum_{k=j-1}^{l-1} \mu^k S_{k,j} A_{l-k}^*.$$

Denote

$$\hat{F}_j := \sum_{j=1}^{l} \sum_{k=j-1}^{l-1} \mu^k S_{k,j} A_{l-k}^*$$

and establish that $F_j = \hat{F}_j$.

At first we consider the case $j = 1$

$$\hat{F}_1 = \sum_{k=0}^{l-1} (\mu^k S_{k,1}) A_{l-k}^* = \sum_{k=0}^{l-1} \mu^k \left(H \bigoplus_0^k J_\mu^{(k)} \right) A_{l-k}^* =$$

$$= \sum_{k=0}^{l-1} \mu^k (\sum_{q=0}^{k} J_\mu^q H_q J_\mu^{k-q}) A^*_{l-k} = \sum_{q=0}^{l-1} J_\mu^q H_q \sum_{k=q}^{l-1} \mu^k J_\mu^{k-q} A^*_{l-k} =$$

$$= \sum_{j=0}^{l-1} J_\mu^j H_j \sum_{i=j+1}^{l} \mu^{i-1} J_\mu^{i-j-1} A^*_{l-i+1} =$$

$$= \sum_{j=0}^{l-1} J_\mu^j H_j J_\mu^{-1} \sum_{i=j+1}^{l} \mu^{i-1} J_\mu^{i-j} A^*_{l-i+1} = \sum_{j=0}^{l-1} J_\mu^j T_{j,1}.$$

Let now $j \geq 2$. Then

$$\hat{F}_j = \sum_{k=j-1}^{l-1} \mu^k S_{k,j} A^*_{l-k} = \sum_{p=0}^{l-j} \mu^{p+j-1} S_{p+j-1,j} A^*_{l-(p+j-1)} =$$

$$= \sum_{p=0}^{l-j} \mu^{p+j-1} \left(H \bigoplus_{0}^{p+j-1} J_\mu^{(p)} \right) A^*_{l-(p+j-1)} =$$

$$= \sum_{p=0}^{l-j} \mu^{p+j-1} \left(\sum_{q=0}^{p} J_\mu^q H_q (H \bigoplus_{q+1}^{p+j-1} J_\mu^{(p-q)}) \right) A^*_{l-(p+j-1)} =$$

$$= \sum_{q=0}^{l-j} J_\mu^q H_q \sum_{p=q}^{l-j} \mu^{p+j-1} \left(H \bigoplus_{q+1}^{p+j-1} J_\mu^{(p-q)}) \right) A^*_{l-(p+j-1)} =$$

$$= \sum_{k=0}^{l-j} J_\mu^k \hat{T}_{k,j},$$

where $\hat{T}_{k,j} = H_k \sum_{p=k}^{l-j} \mu^{p+j-1} \left(H \bigoplus_{q+1}^{p+j-1} J_\mu^{(p-q)} \right) A^*_{l-p-j+1}$.

We need to prove that $T_{k,j} = \hat{T}_{k,j}$ if $j \geq 2$. For $j = 2, k = 0, \ldots, l-2$, it holds

$$
\begin{aligned}
\hat{T}_{k,2} = H_k \sum_{p=k}^{l-2} \mu^{p+1} \left(H \bigoplus_{k+1}^{p+1} J_\mu^{(p-k)} \right) A^*_{l-p-1} = \\
= H_k \sum_{p=k}^{l-2} \mu^{p+1} \left(\sum_{q=0}^{p-k} J_\mu^q H_{k+q+1} J_\mu^{(p-q-k)}) \right) A^*_{l-p-1} = \\
= H_k \sum_{q=0}^{l-k-2} J_\mu^q H_{k+q+1} \left(\sum_{p=q+k}^{l-2} (\mu^{p+1} J_\mu^{p-q-k}) \right) A^*_{l-p-1} = \\
= H_k \sum_{i=k+1}^{l-1} J_\mu^{i-(k+1)} H_i \sum_{p=i-1}^{l-2} (\mu^{p+1} J_\mu^{p-i+1}) A^*_{l-p-1} = \\
= H_k J_\mu^{-1} \sum_{i=k+1}^{l-1} J_\mu^{i-k} H_i \sum_{m=i+1}^{l} \mu^{m-1} J_\mu^{m-i-1} A^*_{l-m-1} = \\
= D_k \sum_{j=k+1}^{l-1} J_\mu^{j-k} T_{i,1}
\end{aligned}
$$

Finally, for $j \geq 3$, $k = 0, \dots, l-j$, we have:

$$
\begin{aligned}
\hat{T}_{k,j} = H_k \sum_{p=k}^{l-j} \mu^{p+j-1} \left(H \bigoplus_{k+1}^{p+j-1} J_\mu^{(p-k)} \right) A^*_{l-p-j+1} = \\
= H_k \sum_{p=k}^{l-j} \mu^{p+j-1} \sum_{q=0}^{p-k} J_\mu^q H_{k+q+1} \left(H \bigoplus_{k+1+q+1}^{p+j-1} J_\mu^{(p-k-q)} \right) A^*_{l-p-j+1} = \\
= H_k \sum_{q=0}^{l-j-k} J_\mu^q H_{k+q+1} \sum_{p=k+q}^{l-j} \mu^{p+j-1} \left(H \bigoplus_{k+q+2}^{p+j-1} J_\mu^{(p-k-q)} \right) A^*_{l-p-j+1} = \\
= H_k \sum_{i=k+1}^{l-j+1} J_\mu^{i-k} J_\mu^{-1} H_i \sum_{p=i-1}^{l-j} \mu^{p+j-1} \left(H \bigoplus_{i+1}^{p+j-1} J_\mu^{(p-i+1)} \right) A^*_{l-p-j+1} =
\end{aligned}
$$

$$= H_k J_\mu^{-1} \sum_{i=k+1}^{l-j+1} J_\mu^{i-k} H_i \sum_{m=i}^{l-j+1} \mu^{m+j-2} \left(H \bigoplus_{i+1}^{m+j-2} J_\mu^{(m-i)} \right) A^*_{l-m-j+2} =$$

$$= D_k \sum_{i=k+1}^{l-(j-1)} J_\mu^{i-k} H_i \times$$

$$\times \sum_{m=i}^{l-(j-1)} \mu^{m+(j-1)-1} \left(H \bigoplus_{i+1}^{m+(j-1)-1} J_\mu^{(m-i)} \right) A^*_{l-m-(j-1)+1} =$$

$$= D_k \sum_{i=k+1}^{l-j+1} J_\mu^{i-k} \hat{T}_{i,j-1}.$$

The lemma is proved completely.

3. Error bound

Concrete representation of discrete operator A_l, $l = 1, 2, \ldots$, will shown in (23). To prove further statements we restrict ourselves some additional conditions to A_l. Namely, we will consider discretization which satisfies the following conditions:

$$\begin{aligned} &\|A^*A - A_l^* A_l\| \le \frac{\delta}{\rho\sqrt{l}}; \qquad \|A - A_l\| \le \left(\frac{\delta}{\sqrt{l}}\right)^{1/2}; \\ &\|(A - A_l)A^*\| \le \frac{\delta}{\rho\sqrt{l}}; \qquad \|(A^* - A_l^*)A\| \le \frac{\delta}{\rho\sqrt{l}}. \end{aligned} \tag{18}$$

It is not difficult to notice that first of inequalities (18) corresponds to requirement of adaptive discretization strategy (5) with $\alpha = 1/l$.

Without lost of generality we will consider that number L of steps of iterative process satisfies to the condition:

$$\delta\sqrt{L} \le 1. \tag{19}$$

Remind that to estimate error $\| x^\dagger - \hat{x}_l \|$ we have to estimate the last term in (9). To do this we estimate right-hand side of expansion (16) term by term. It's easy to see that for the first item

$$\| \sum_{k=0}^{l-1} \mu^k (\mu I + A^* A)^{-(k+1)} (A^* - A^*_{l-k}) f_\delta \| =$$
$$= \frac{1}{\mu} \sum_{k=0}^{l-1} \| \mu^{k+1} (\mu I + A^* A)^{-(k+1)} \| \| (A^* - A^*_{l-k}) f_\delta \| \leq$$
$$\leq \frac{1}{\mu} \sum_{k=0}^{l-1} \frac{2\delta}{\sqrt{l-k}}.$$

Next statement gives bound for second term in the right-hand side of (16).

Lemma 3. *For any $l = 1, 2, \ldots, L$ there is a constant $c_1 < \infty$ such that*

$$\sum_{k=1}^{l} \|F_k f_\delta\| \leq c_1 \delta \sqrt{l}.$$

First of all by (18) we can write inequality

$$\| (A^* - A^*_l) f_\delta \| \leq \| (A^* - A^*_l) A x^\dagger \| + \| (A^* - A^*_l)(f - f_\delta) \| \leq$$
$$\leq \frac{\delta}{\sqrt{l}} + \frac{\delta^{3/2}}{l^{1/4}} \leq \frac{2\delta}{\sqrt{l}}.$$

Due to (17) obviously equality

$$\sum_{k=1}^{l} \|F_k f_\delta\| = \sum_{k=1}^{l} \| \sum_{j=0}^{l-k} J_\mu^{-j} T_{j,k} f_\delta \|$$

is true. Now let us estimate norm of $T_{j,k}f_\delta$ and J_μ. Firstly we find a bound of element $T_{j,1}f_\delta$:

$$T_{j,1}f_\delta := D_j \sum_{i=j+1}^{l} \mu^{i-1} J_\mu^{i-j} A^*_{l-i+1} f_\delta =$$
$$= D_j \sum_{i=j+1}^{l} \mu^{i-1} J_\mu^{i-j} A^* f_\delta - D_j \sum_{i=j+1}^{l} \mu^{i-1} J_\mu^{i-j} (A^* - A^*_{l-i+1}) f_\delta.$$

Remind that

$$g_l(\lambda) = \sum_{i=0}^{l-1} \frac{\mu^i}{(\mu+\lambda)^{i+1}}.$$

Change order of summation in the last equality

$$g_{l-j}(A^*A) = \sum_{i=0}^{l-j-1} \mu^i J_\mu^{i+1} =$$
$$= \mu^0 J_\mu^1 + \ldots + \mu^{l-j-1} J_\mu^{l-j} = \sum_{i=j+1}^{l} \mu^{l-i} J_\mu^{l-i+1}.$$

Hence

$$\sum_{i=j+1}^{l} \mu^{i-1} J_\mu^{i-j} = \mu^j J_\mu^1 + \mu^{j+1} J_\mu^2 + \ldots + \mu^{l-1} J_\mu^{l-j} =$$
$$= \sum_{i=j+1}^{l} \mu^{l+j-i} J_\mu^{l-i+1} = \mu^j \sum_{i=j+1}^{l} \mu^{l-i} J_\mu^{l-i+1} = \mu^j g_{l-j}(A^*A).$$

By this relation $T_{j,1}f_\delta$ can be written as

$$\begin{aligned}
&T_{j,1}f_\delta = \\
&= D_j\mu^j g_{l-j}(A^*A)A^*f_\delta - D_j\sum_{i=j+1}^{l}\mu^{i-1}J_\mu^{i-j}(A^*-A^*_{l-i+1})f_\delta = \\
&\quad D_j\mu^j g_{l-j}(A^*A)A^*Ax^\dagger - D_j\mu^j g_{l-j}(A^*A)A^*(f-f_\delta) - \\
&\qquad - D_j\sum_{i=j+1}^{l}\mu^{i-1}J_\mu^{i-j}(A^*-A^*_{l-i+1})f_\delta.
\end{aligned}$$

Taking into account estimations (cf. (7))

$$\begin{aligned}
&\|g_{l-j}(A^*A)A^*A\| \le 1, \qquad \|g_{l-j}(A^*A)A^*\| \le \sqrt{\frac{l-j}{\mu}}, \\
&\qquad\qquad \|\mu^{i-1}J_\mu^{i-j}\| \le \mu^{j-1},
\end{aligned} \tag{20}$$

we have

$$\begin{aligned}
&\|T_{j,1}f_\delta\| \le \|D_j\|\mu^j\left(\rho + \frac{\delta}{\sqrt{\mu}}\sqrt{l-j}\right) + \\
&\qquad + \|D_j\|\mu^{j-1}2\delta\sum_{i=j+1}^{l}\frac{1}{\sqrt{l-i+1}}.
\end{aligned}$$

Using the last relation and estimation

$$\sum_{i=j+1}^{l}\frac{1}{\sqrt{l-i+1}} \le \int_0^{l-j}\frac{dx}{\sqrt{x}} = 2\sqrt{l-j},$$

we find

$$\|T_{j,1}f_\delta\| \le \mu^{j-1}\|D_j\|(\mu\rho + \delta\sqrt{\mu}\sqrt{l-j} + 4\delta\sqrt{l-j}) \le$$
$$\le \frac{\delta}{\mu\rho\sqrt{l-j}}\mu^{j-1}(\mu\rho + (4+\sqrt{\mu})\delta\sqrt{l-j}) \le$$
$$\le \frac{\delta}{\rho\sqrt{l-j}}\mu^{j-2}(4+\sqrt{\mu}+\mu\rho) = \frac{c_2\delta}{\sqrt{l-j}}\mu^{j-1},$$

where $c_2 = \rho + 1/\sqrt{\mu} + 4/\mu$.

Now we have

$$\|T_{j,2}f_\delta\| = \|D_j\sum_{i=j+1}^{l-1}(\mu I + A^*A)^{-(i-j)}T_{i,1}f_\delta\| \le$$
$$\le \mu^{j-1}\|D_j\|c_2\delta\sum_{i=j+1}^{l-1}\frac{1}{\sqrt{l-i}} \le$$
$$\le \mu^{j-1}\|D_j\|c_2 2\delta\sqrt{l-j-1} \le \frac{2c_2}{\rho}\mu^{j-2}\frac{\delta}{\sqrt{l-j}}.$$

In a like manner for every $k = 1, 2, \ldots$ one can find

$$\|T_{j,k}f_\delta\| \le \left(\frac{2}{\rho}\right)^{k-1}c_2\mu^{j-k}\frac{\delta}{\sqrt{l-j}}.$$

Thus we have

$$\|F_k f_\delta\| = \|\sum_{j=0}^{l-k}(\mu I + A^*A)^{-j}T_{j,k}f_\delta\| \le \left(\frac{2}{\rho}\right)^{k-1}\frac{2c_2}{\mu^k}\delta\sqrt{l},$$

$$\sum_{k=1}^{l}\|F_k f_\delta\| \le \frac{2c_2\delta\sqrt{l}}{\mu}\sum_{k=1}^{l}\left(\frac{2}{\rho\mu}\right)^{k-1} \le \frac{2c_2\delta\sqrt{l}}{\mu - 2/\rho}.$$

We obtain assertion of Lemma for $c_1 = \frac{2c_2}{\mu-2/\rho}$.

Next statement contains finally estimation of third item in the right-hand side of (9).

Lemma 4. *For every $l \le L$ it holds*

$$\|B_l f_\delta\| \le (4/\mu + c_1)\delta\sqrt{l}. \tag{21}$$

Taking into account Lemmas 2 and 3 we find

$$\begin{aligned}\|B_l f_\delta\| \le \| \sum_{k=0}^{l-1} \mu^k J_\mu^{k+1}(A^* - A^*_{l-k}) f_\delta - \sum_{k=1}^{l} F_k f_\delta \| \le \\ \le \sum_{k=0}^{l-1} \|\mu^k J_\mu^{k+1}(A^* - A^*_{l-k}) f_\delta\| + \sum_{k=1}^{l} \|F_k f_\delta\| \le \\ \le \sum_{k=0}^{l-1} \|\mu^k J_\mu^{k+1}(A^* - A^*_{l-k}) f_\delta\| + c_1\delta\sqrt{l}.\end{aligned}$$

Using (18) and (20) we estimate first item:

$$\begin{aligned}\sum_{k=0}^{l-1} \|\mu^k J_\mu^{k+1}\| \|(A^* - A^*_{l-k}) f_\delta\| = \\ = \frac{1}{\mu} \sum_{k=0}^{l-1} \|\mu^{k+1} J_\mu^{k+1}\| \|(A^* - A^*_{l-k}) f_\delta\| \le \frac{1}{\mu} \sum_{k=0}^{l-1} \frac{2\delta}{\sqrt{l-k}}.\end{aligned}$$

As a result we have:

$$\|B_l f_\delta\| \le \frac{2\delta}{\mu} \sum_{k=0}^{l-1} \frac{1}{\sqrt{l-k}} + c_1\delta\sqrt{l} \le \left(\frac{4}{\mu} + c_1\right)\delta\sqrt{l}.$$

The lemma is proved.

Final bound for method's accuracy (6) is contained in next statement.

Lemma 5. *For every $l \le L$ there exists a constant $c_3 > 0$ such that*

$$\|x^\dagger - \hat{x}_l\| \le \rho\kappa^{\frac{\nu}{\nu+1}}_{\frac{\nu+1}{2}} l^{-\nu/2} + c_3\delta\sqrt{l}. \tag{22}$$

Taking into account (10), (14) and (21) from relation (9) we have

$$\|x^\dagger - \hat{x}_l\| \le \rho\kappa_{\frac{\nu+1}{2}}^{\frac{\nu}{\nu+1}} l^{-\nu/2} + \delta\frac{\sqrt{l}}{\sqrt{\mu}} + (4/\mu + c_1)\delta\sqrt{l} =$$
$$= \rho\kappa_{\frac{\nu+1}{2}}^{\frac{\nu}{\nu+1}} l^{-\nu/2} + c_3\delta\sqrt{l}.$$

We obtain assertion of Lemma for $c_3 = 1/\sqrt{\mu} + 4/\mu + c_1$.

4. Algorithm of solving

First of all we describe adaptive discretization scheme used in this paper for solving (1) with operators $A \in \mathcal{H}^r$. Let the discretization level n depends on step of iteration process: $n = n(l)$.

Denote as Γ_n area

$$\Gamma_n := \cup_{k=1}^{2n(l)} (2^{k-1}, 2^k] \times [1, 2^{2n(l)-k}) \cup \{1\} \times [1, 2^{2n(l)}].$$

of coordinate plane corresponding to the basis E that appear in the definition of class $\mathcal{H}^r$.

Operators A_l, $l = 1, 2, \ldots$, will be constructed in the following way:

$$A_{n(l)} = A_l = \sum_{k=1}^{2n(l)} (P_{2^k} - P_{2^{k-1}}) A P_{2^{2n(l)-k}} + P_1 A P_{2^{2n(l)}}. \tag{23}$$

Next statement characterizes some approximation properties of the operator $A_{n(l)}$.

Lemma 6. *If parameter $n = n(l)$ is chosen by relation*
(24)
$$c_4 n 2^{-2nr} = \frac{\delta}{\rho\sqrt{l}}, \quad c_4 = \max\left\{1 + 2^{r+3}; 3 * 2^r; 2^r + \frac{2^{r+1}}{\sqrt{2^{2r}-1}}\right\},$$

then for operator $A_{n(l)} = A_l$ (23) and any operator $A \in \mathcal{H}^r$ it holds estimates (18).

This lemma can be proved in the same way as Lemma 1 [1].

Denote

$$c_5 := 1 + \frac{1}{\sqrt{\mu}} + \frac{1}{\mu}\left(4 + \frac{1}{\rho} + 2\pi\kappa_{1/2}\right) + \frac{c_2(2 + (1+\pi)\kappa_{1/2})}{\mu - 2/\rho}.$$

Now we describe algorithm that consists of Fakeev-Lardy regularization method and proposed adaptive discretization strategy.

(1) Given data: $A \in \mathcal{H}^r$, f_δ, δ, ρ.
(2) Initialization: $\hat{x}_0 = 0$, $b > c_5 + 2$.
(3) Iteration by $l = 1, 2, \ldots$
  (a) choosing of discretization level $n = n(l, \delta)$:

$$c_4 n 2^{-2nr} = \frac{\delta}{\rho\sqrt{l}}; \tag{25}$$

  (b) computation of Galerkin functionals:

$$\begin{aligned} &(f_\delta, e_i), \quad i \in (2^{2n(l-1)}, 2^{2n(l)}] \\ &(Ae_j, e_i), \quad (i,j) \in \Gamma_{n(l)} \setminus \Gamma_{n(l-1)}; \end{aligned} \tag{26}$$

  (c) solving equation

$$\mu\hat{x}_l + A^*_{n(l)} A_{n(l)} \hat{x}_l = \mu\hat{x}_{l-1} + A^*_{n(l)} f_\delta; \tag{27}$$

  (d) stop rule by discrepancy principle

$$\begin{aligned} &\| A_{n(L)}\hat{x}_L - P_{2^{2n(L)}} f_\delta \| \le b\delta, \\ &\| A_{n(l)}\hat{x}_l - P_{2^{2n(l)}} f_\delta \| > b\delta, \qquad l < L. \end{aligned} \tag{28}$$

(4) Approximate solution: $\hat{x}_L$.

To establish optimality of the algorithm we need two assertions.

Lemma 7. *For any $l \le L$ the inequality*

$$\|Ax_l - f\| \le \|A_l\hat{x}_l - f\| + c_5\delta$$

is true.

Denote expression $Ax_l - f$ as:

$$Ax_l - f := Ag_l(A^*A)A^*f - f = Z_1 + Z_2 + Z_3 + Z_4 + Z_5, \tag{29}$$

where

$$\begin{aligned}
Z_1 &= Ag_l(A^*A)A^*(f - f_\delta);\\
Z_2 &= (A - A_l)A^*g_l(AA^*)f_\delta;\\
Z_3 &= -(A - A_l)(g_l(A^*A)A^*f_\delta - \hat{x}_l);\\
Z_4 &= A(g_l(A^*A)A^*f_\delta - \hat{x}_l);\\
Z_5 &= A_l\hat{x}_l - f.
\end{aligned}$$

Let's estimate all elements $Z_1 - Z_4$. By (7) we obtain

$$\|Z_1\| \leq \|AA^*g_l(A^*A)\|\|f - f_\delta\| \leq \delta.$$

Taking into account (7) and (18) we find

$$\|Z_2\| \leq \|(A - A_l)A^*\|(\|g_l(A^*A)Ax^\dagger\| + \|g_l(A^*A)\|\|f - f_\delta\ \| \leq$$
$$\leq \frac{\delta}{\rho\sqrt{l}}\left(\rho\sqrt{\frac{l}{\mu}} + \delta\frac{l}{\mu}\right) \leq \delta\left(\frac{1}{\sqrt{\mu}} + \frac{1}{\rho\mu}\right).$$

Using Lemma 4 and (18) we have

$$\|Z_3\| \leq \left(\frac{\delta}{\sqrt{l}}\right)^{1/2}(4/\mu + c_1)\delta\sqrt{l} \leq (4/\mu + c_1\delta).$$

To estimate Z_4 we use Lemma 2 and (17)

$$Z_4 = AB_lf_\delta = \sum_{k=0}^{l-1} A\mu^k J_\mu^{k+1}(A^* - A^*_{l-k}) - \sum_{k=1}^{l} AF_kf_\delta.$$

By inequality

$$\|\mu^kA(\mu I + A^*A)^{-(k+1)}\| \leq \frac{1}{\mu}\sup_\lambda \lambda^{1/2}(1 - \lambda g_{k+1}(\lambda)) \leq$$
$$\leq \frac{\kappa_{1/2}}{\mu}(k+1)^{-1/2},$$

we have

$$\|Z_4\| \leq \frac{2\delta\kappa_{1/2}}{\mu}\sum_{k=0}^{l-1}\frac{1}{\sqrt{(k+1)(l-k)}} + \sum_{j=1}^{l}\|AF_jf_\delta\|.$$

We estimate both items in the right-hand side of last relation. So

$$\sum_{k=0}^{l-1} \frac{1}{\sqrt{(k+1)(l-k)}} = \sum_{j=1}^{N} \frac{1}{\sqrt{j(N-j)}} \le \int_0^N \frac{dx}{x(N-x)} = \pi.$$

Now

$$\|AF_j f_\delta\| \le \sum_{i=0}^{l-j} \|A(\mu I + A^*A)^{-i} T_{i,j} f_\delta\| \le$$

$$\le \sum_{i=0}^{l-j} \mu^{-i} \|\mu^i A(\mu I + A^*A)^{-i}\| \|T_{i,j} f_\delta\| \le$$

$$\le \left(1 + \sum_{i=1}^{l-j} \frac{\mu^{-i}}{\sqrt{i}} \frac{\mu^{i-j}}{\sqrt{l-j}}\right) \kappa_{1/2} c_2 \left(\frac{2}{\rho}\right)^{j-1} \delta =$$

$$= \frac{\kappa_{1/2} c_2 \left(\frac{2}{\rho}\right)^{j-1} \delta}{\mu^j} \left(1 + \sum_{i=1}^{l-j} \frac{1}{\sqrt{i(l-i)}}\right) \le c_6 \left(\frac{2}{\rho\mu}\right)^{j-1} \delta,$$

where $c_6 = \frac{(1+\pi) c_2 \kappa_{1/2}}{\mu}$.

Then

$$\sum_{j=1}^{l} \|AF_j f_\delta\| \le c_6 \delta \sum_{j=0}^{l-1} \left(\frac{2}{\rho\mu}\right)^j \le \frac{(1+\pi) c_2 \kappa_{1/2}}{\mu - 2/\rho} \delta.$$

Finally we obtain

$$\|Z_4\| \le \kappa_{1/2} \left(\frac{2\pi}{\mu} + \frac{(1+\pi) c_2}{\mu - 2/\rho}\right) \delta.$$

By combining received estimates we obtain the statement of Lemma.

Lemma 8. *Let L satisfy to discrepancy principle (28), where $b > 2 + c_5$, $A \in \mathcal{H}^r$ and discretization parameter is chosen as (24).*

Then there are constants $b_1, b_2 > 0$ *exist such that*

$$b_1\delta \leq \|Ax_L - f\| \leq b_2\delta.$$

According to (24) for any $l \leq L$ it holds

$$\|(I - P_l)f\| \leq \delta.$$

Using (28) we have

$$\|A_L\hat{x}_L - f\| \leq \|A_L\hat{x}_L - P_L f_\delta\| + \|P_L(f - f_\delta)\| + \|(I - P_L)f\| \leq \\ \leq (b+2)\delta.$$

Then by Lemma 7 we find

$$\|(Ax_L - f)\| \leq b_2\delta$$

with $b_2 = b + c_5 + 2$. On the other hand, in $(L-1)$-th step according to (28)

$$\|A_{L-1}\hat{x}_{L-1} - P_{L-1}f_\delta\| > b\delta.$$

Using triangle inequality and Lemma 7 we find from (29) with $l = L - 1$

$$\|Ax_{L-1} - f\| \geq \|A_{L-1}\hat{x}_{L-1} - P_{L-1}f_\delta\| - (c_5 + 2)\delta.$$

Let's estimate

$$\|Ax_L - f\|^2 = \mu^{2L} \sum_k |\lambda_k|^{2(\nu+1)} \frac{(v, \psi_k)^2}{(\mu + \lambda_k^2)^{2L}} =$$

$$= \mu^2 \left(\mu^{2(L-1)} \sum_k |\lambda_k|^{2(\nu+1)} \frac{(v, \psi_k)^2}{(\mu + \lambda_k^2)^{2(L-1)}} (\mu + \lambda_k^2)^{-2} \right) \geq$$

$$\geq \left(\frac{\mu}{\mu + j^2} \right)^2 \left(\mu^{2(L-1)} \sum_k |\lambda_k|^{2(\nu+1)} \frac{(v, \psi_k)^2}{(\mu + \lambda_k^2)^{2(L-1)}} \right).$$

Consequently

$$\|Ax_L - f\| \geq \frac{\mu}{\mu + 1} \|Ax_{L-1} - f\|.$$

Finally we have

$$\|Ax_L - f\| \geq b_1\delta,$$

where $b_1 = \frac{\mu}{\mu+1}(b-2-c_5)$. Thus Lemma is completely proved.

5. Optimality of the algorithm. Amount of Galerkin information

In the following statement we will show that described algorithm (23)-(28) guarantees the optimal order of accuracy $O(\delta^{\frac{\nu}{\nu+1}})$ on the whole class of the considered equations.

Theorem 1. *Algorithm (23)-(28) achieves the optimal order of accuracy $O(\delta^{\frac{\nu}{\nu+1}})$ on the class of equations with operator $A \in \mathcal{H}^r$ and minimal-norm solutions $x^\dagger \in M_{\nu,\rho}(A)$, $\nu > 0$.*

From Lemmas 1, 8 and relation (13) it follows that

$$\delta\sqrt{L} = \delta\left(\frac{|d_{\nu,L}(v)|}{\|Ax_L - f\|}\right)^{\frac{1}{\nu+1}} \le \delta\left(\frac{\rho\kappa_{(\nu+1)/2}}{b_1\delta}\right)^{\frac{1}{\nu+1}} \le \left(\frac{\rho}{b_1}\right)^{\frac{1}{\nu+1}}\sqrt{\frac{\mu(\nu+1)}{2}}\delta^{\frac{\nu}{\nu+1}}, \tag{30}$$

$$|c_{\nu,L}(v)|L^{\frac{-\nu}{2}} = |c_{\nu,L}(v)|\left(\frac{\| Ax_L - f \|}{|d_{\nu,L}(v)|}\right)^{\frac{\nu}{\nu+1}} \le \rho^{\frac{1}{\nu+1}}(b_2\delta)^{\frac{\nu}{\nu+1}}.$$

Substituting the estimates into (22), we have

$$\|x^\dagger - \hat{x}_L \| \le \xi\delta^{\frac{\nu}{\nu+1}},$$

where $\xi = \rho^{\frac{1}{\nu+1}}\left(b_2^{\frac{\nu}{\nu+1}} + c_3 b_1^{-\frac{1}{\nu+1}}\sqrt{\frac{\mu(\nu+1)}{2}}\right)$.

The theorem is proved.

Corollary. *To achieve the optimal order of the accuracy on the considered class of equations in the framework of algorithm (23)-(28) it is enough to calculate*

$$O(\delta^{-\frac{\nu+2}{(\nu+1)r}}\log^{1+1/r}\delta^{-1}) \tag{31}$$

of Galerkin functionals (26).

To proof this statement it is sufficiently to estimate volume of the inner products that is equivalent to square of figure Γ_n, which is equal to $(n+1)2^{2n}$. Using (24) and (30) in this expression we have estimate (31).

Remind (see Section 1) that to achieve the optimal order of accuracy in traditional Galerkin discretization scheme it is necessary to calculate $O(\delta^{-2/r})$ inner products (26). Thus for any $\nu > 0$ algorithm (23)-(28) is more economical then methods using in [3] with traditional Galerkin discretization scheme.

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