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On conjugate pseudo-harmonic functions

We prove the following theorem. Let U be a pseudo-harmonic function on a surface M^2. For a real valued continuous function $V : M^2 \to \mathbb{R}$ to be a conjugate pseudo-harmonic function of U on M^2 it is necessary and sufficient that V is open on level sets of U.

Keywords: *a pseudo-harmonic function, a conjugate, a surface, an interior transformation*

Let M^2 be a surface, i.e. a 2-dimensional and separable manifold, $U : M^2 \to \mathbb{R}$ be a real-valued function on M^2. Denote also by

$$D = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$$

the open unit disk in the plane.

Definition 1 (see [1,2]). *A function U is called* pseudo-harmonic *in a point $p \in M^2$ if there exist a neighbourhood N of p on M^2 and a homeomorphism $T : D \to N$ such that $T(0, 0) = p$ and a function*

$$u = U \circ T : D \to \mathbb{R}^2$$

is harmonic and not identically constant.

A neighbourhood N is called simple neighbourhood of p.

We can even choose N and T from previous definition to comply with the equality

$$u(z) = U \circ T(z) = \operatorname{Re} z^n + U(p)\,, \quad z = x + iy \in D\,,$$

for a certain $n = n(p) \in \mathbb{N}$ (see [2]).

Definition 2 (see [1,2]). *A function U is called* pseudo-harmonic on M^2 *if it is pseudo-harmonic in each point $p \in M^2$.*

Let $U : M^2 \to \mathbb{R}$ be a pseudo-harmonic function on M^2 and $V : M^2 \to \mathbb{R}$ be a real valued function.

Definition 3 (see [1]). *A function V is called a* conjugate pseudo-harmonic function of U in a point $p \in M^2$ *if there exist a neighbourhood N of p on M^2 and a homeomorphism $T : D \to N$ such that $T(0,0) = p$ and*

$$u = U \circ T : D \to \mathbb{R}^2 \quad \text{and} \quad v = V \circ T : D \to \mathbb{R}^2$$

are conjugate harmonic functions.

We can choose N and T from previous definition in such way that

$$\begin{aligned} u(z) &= U \circ T(z) = \operatorname{Re} z^n + U(p)\,, \\ v(z) &= V \circ T(z) = \operatorname{Im} z^n + V(p)\,, \quad z = x + iy \in D\,, \end{aligned}$$

for a certain $n = n(p) \in \mathbb{N}$ (see [2]).

Definition 4 (see [1]). *A function V is called a* conjugate pseudo-harmonic function of U on M^2 *if it is a conjugate pseudo-harmonic function of U in every $p \in M^2$.*

Definition 5. *Let U and V be continuous real valued functions on a surface M^2. We say that V* is open on level sets of U *if for every $c \in U(M^2)$ a mapping*

$$V|_{U^{-1}(c)} : U^{-1}(c) \to \mathbb{R}$$

is open on the space $U^{-1}(c)$ in the topology induced from M^2.

Theorem 1. *Let U be a pseudo-harmonic function on M^2. For a real valued continuous function $V : M^2 \to \mathbb{R}$ to be a conjugate pseudo-harmonic function of U on M^2 it is necessary and sufficient that V is open on level sets of U.*

Let us remind following definition.

Definition 6 (see [3]). *A mapping $G : M_1^2 \to M_2^2$ of a surface M_1^2 to a surface M_2^2 is called* interior *if it complies with conditions:*

1) *G is open, i. e. an image of any open subset of M_1^2 is open in M_2^2;*
2) *for every $p \in M_2^2$ its full preimage $G^{-1}(p)$ does not contain any nondegenerate continuum (closed connected subset of M_1^2).*

In order to prove theorem 1 we need following

Lemma 1. *Let U be a pseudo-harmonic function on M^2. Let a real valued continuous function V be open on level sets of U.*

Then the mapping $F : M^2 \to \mathbb{C}$,

$$F(p) = U(p) + iV(p)\,, \quad p \in M^2$$

is interior.

First we will verify one auxiliary statement. Denote $I = [0, 1]$, $\mathring{I} = (0, 1) = I \setminus \{0, 1\}$.

Proposition 1. *In the condition of Lemma 1 the following statement holds true.*

Let $\gamma : I \to M^2$ be a simple continuous curve and $\gamma(I) \subseteq U^{-1}(c)$ for a certain $c \in \mathbb{R}$. If the set $\gamma(\mathring{I})$ is open in $U^{-1}(c)$ in the topology induced from M^2, then the function $V \circ \gamma : I \to \mathbb{R}$ is strictly monotone.

Proof. Suppose that contrary to the statement of Proposition the equality $V \circ \gamma(\tau_1) = V \circ \gamma(\tau_2)$ is valid for certain $\tau_1, \tau_2 \in I$, $\tau_1 < \tau_2$.

Since the function $V \circ \gamma$ is continuous and a set $[\tau_1, \tau_2]$ is compact, then following values

$$d_1 = \min_{t \in [\tau_1, \tau_2]} V \circ \gamma(t)\,,$$

$$d_2 = \max_{t \in [\tau_1, \tau_2]} V \circ \gamma(t)\,,$$

are well defined. Let us fix $s_1, s_2 \in [\tau_1, \tau_2]$ such that $d_i = V \circ \gamma(s_i)$, $i = 1, 2$.

We designate $W = (\tau_1, \tau_2)$. It is obviously the open subset of $\mathring{I}$.

Let us consider first the case $d_1 = d_2$. It is clear that

$$[\tau_1, \tau_2] \subseteq (V \circ \gamma)^{-1}(d_1)$$

in this case. So the open subset $\gamma(W)$ of the level set $U^{-1}(c)$ is mapped by V onto a one-point set $\{d_1\}$ which is not open in $\mathbb{R}$ and V is not open on level sets of U.

Assume now that $d_1 \neq d_2$. Since $V \circ \gamma(\tau_1) = V \circ \gamma(\tau_2)$ due to our previous supposition, then either s_1 or s_2 is contained in W.

Let $s_1 \in W$ (the case $s_2 \in W$ is considered similarly). Then $V \circ \gamma(W) \subseteq [d_1, +\infty)$ and the open subset $\gamma(W)$ of the level set $U^{-1}(c)$ can not be mapped by V to an open subset of $\mathbb{R}$ since its image containes the frontier point $d_1 = V \circ \gamma(s_1)$. So, in this case V is not open on level sets of U.

The contradiction obtained shows that our initial supposition is false and the function $V \circ \gamma$ is strictly monotone on I. $\square$

Proof of Lemma 1. Let $p \in M^2$ and Q be an open neighbourhood of p.

We are going to show that the set $F(Q)$ containes a neigbourhood of $F(p)$. At the same time we shall show that p is an isolated point of a level set $F^{-1}(F(p))$.

Without loss of generality we can assume that $U(p) = V(p) = 0$.

Let N be a simple neighbourhood of p and $T : D \to N$ be a homeomorphism such that for a certain $n \in \mathbb{N}$ the following equality holds true $u(z) = U \circ T(z) = \operatorname{Re} z^n$, $z \in D$ (see Definition 1 and the subsequent remark). It is clear that without losing generality we can regard that N is small enough to be contained in Q.

Observe that for an arbitrary level set Γ of U an intersection $\Gamma \cap T(D) = \Gamma \cap N$ is open in Γ. Consequently, since T is homeomorphism then a mapping $v = V \circ T : D \to \mathbb{R}$ is open on level sets of $u = U \circ T : D \to \mathbb{R}$ (see Definition 5).

Let us consider two possibilities.

Case 1. Zero is a regular point of the smooth function $u = U \circ T$, i. e. $n = 1$ and $u(z) = \operatorname{Re} z$, $z \in D$.

In this case

$$u^{-1}(u(0)) = u^{-1}(U(p)) = T^{-1}(U^{-1}(U(p))) = \{0\} \times (-1,1).$$

According to Proposition 1 the function v is strictly monotone on every segment which is contained in this interval, so it is strictly monotone on $\{0\} \times (-1,1)$. Consequently, for points $z_1 = 0 - i/2$ and $z_2 = 0+i/2$ the following inequality holds true $v(z_1)\cdot v(z_2) < 0$.

Let us note that from previous it follows that V is monotone on the arc $\beta = T(\{0\} \times (-1,1)) = U^{-1}(U(p)) \cap N$. And since $F^{-1}(F(p)) \cap N \subset \beta$ then $F^{-1}(F(p)) \cap N = \{p\}$ and p is an isolated point of its level set $F^{-1}(F(p))$.

Let $d_1 = v(z_1) < 0$ and $d_2 = v(z_2) > 0$ (The case $d_1 > 0$ and $d_2 < 0$ is considered similarly). Denote

$$\varepsilon = \frac{1}{2} \min(|d_1|, |d_2|) > 0\,.$$

Function v is continuous, so there exists $\delta > 0$ such that following implications are fulfilled

$$|z - z_1| < \delta \quad \Rightarrow \quad |v(z) - d_1| < \varepsilon\,,$$
$$|z - z_2| < \delta \quad \Rightarrow \quad |v(z) - d_2| < \varepsilon\,.$$

Let us examine a neighbourhood $W = (-\delta, \delta) \times (-1/2, 1/2)$ of 0, which is depicted on Figure 13. It can be easily seen that for every $x \in (-\delta, \delta)$ following relations are valid

$$u(x + iy) = x\,, \quad y \in (-\varepsilon, \varepsilon)\,,$$
$$v(x - i/2) < v(z_1) + \varepsilon < -2\varepsilon + \varepsilon = -\varepsilon\,,$$
$$v(x + i/2) > v(z_2) - \varepsilon > 2\varepsilon - \varepsilon = \varepsilon\,.$$

From two last lines and from the continuity of v on a segment $\{x\} \times [-1/2, 1/2]$ it follows that $v(\{x\} \times [-1/2, 1/2]) \supseteq (-\varepsilon, \varepsilon)$. Therefore

$$F \circ T(\{x\} \times [-1/2, 1/2]) \supseteq \{x\} \times (-\varepsilon, \varepsilon)\,, \quad x \in (-\delta, \delta)\,.$$

Since $T(W) \subseteq N \subseteq Q$ by the choise of N, then

$$0 = F(p) \in (-\delta, \delta) \times (-\varepsilon, \varepsilon) \subseteq F \circ T(W) \subseteq F(Q)\,.$$

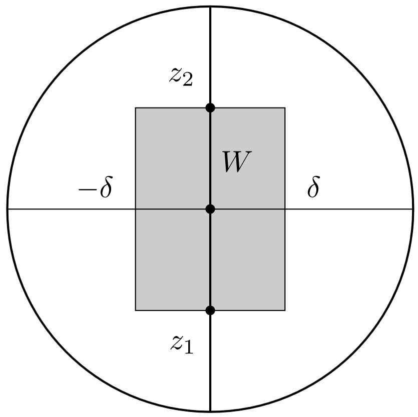

FIGURE 13

Case 2. Zero is a saddle point of $u = U \circ T$, i. e. $u(z) = \operatorname{Re} z^n$, $z \in D$ for a certain $n > 1$.

In this case

$$u^{-1}(u(0)) = T^{-1}(U^{-1}(U(p))) = \{0\} \cup \bigcup_{k=0}^{2n-1} \gamma_k \,,$$

where $\gamma_k = \{z \in D \mid z = a \cdot \exp(\pi i(k - 1/2)/n),\ a \in (0,1)\}$, $k = 1, \dots, 2n-1$.

As above, applying Proposition 1 we conclude that function

$$v = V \circ T$$

is strictly monotone on each arc γ_k, $k = 1, \dots, 2n-1$. Since v is continuous and 0 is a boundary point for each γ_k, then

$$v(z) \neq v(0)$$

for all $z \in \bigcup_k \gamma_k$. Therefore, $0 = (F \circ T)^{-1}(F \circ T(0))$ and $F^{-1}(F(p)) \cap N = \{p\}$, i. e. p is the isolated point if its level set $F^{-1}(F(p)$.

Let us designate by

$$R_k = \left\{ z \in D \;\middle|\; z = ae^{i\varphi},\ a \in [0,1),\ \varphi \in \left[\tfrac{\pi(k-1/2)}{2}, \tfrac{\pi(k+1/2)}{2} \right] \right\}, \quad k = 0, \dots, 2n-1$$

sectors on which disk D is divided by the level set $u^{-1}(u(0))$.

We also denote

$$D_l = \{ z \in D \mid \operatorname{Re} z \leq 0 \},$$
$$D_r = \{ z \in D \mid \operatorname{Re} z \geq 0 \}.$$

Consider map $\Phi : D \to D$ given by the formula $\Phi(z) = z^n$, $z \in D$. It is easy to see that for every $k \in \{0, \dots, 2n-1\}$ depending on its parity sector R_k is mapped homeomorphically by Φ either onto D_l or onto D_r. Let a mapping $\Phi_k : R_k \to D_r$ is given by relation

$$\Phi_k = \begin{cases} \Phi|_{R_k}, & \text{if } k = 2m\,, \\ \operatorname{Inv} \circ \Phi|_{R_k}, & \text{if } k = 2m+1\,, \end{cases} \qquad k = 0, \dots, 2n-1\,,$$

where $\operatorname{Inv} : D \to D$ is defined by formula $\operatorname{Inv}(z) = -z$, $z \in D$. Evidently, all Φ_k are homeomorphisms.

We consider now inverse mappings $\varphi_k = \Phi_k^{-1} : D_r \to D$, $k = 0, \dots, 2n-1$. By construction all of these mappings are embeddings. Moreover, it is easy to see that

$$u_k(z) = u \circ \varphi_k(z) = \begin{cases} \operatorname{Re} z\,, & \text{when } k = 2m\,, \\ -\operatorname{Re} z\,, & \text{when } k = 2m+1\,. \end{cases}$$

Let us fix $k \in \{0, \dots, 2n-1\}$. It is clear that φ_k homeomorphically maps a domain

$$\mathring{D}_r = \{ z \in D \mid \operatorname{Re} z > 0 \}$$

onto a domain

$$\mathring{R}_k = \left\{ z \in D \;\middle|\; z = ae^{i\varphi},\ a \in (0,1),\ \varphi \in \left(\tfrac{\pi(k-1/2)}{2}, \tfrac{\pi(k+1/2)}{2} \right) \right\},$$

so with the help of argument similar to the observation preceding to case 1 we conclude that the mapping $\mathring{v}_k = v \circ \varphi_k|_{\mathring{D}_r} : \mathring{D}_r \to \mathbb{R}$

is open on level sets of the function $\mathring{u}_k = u \circ \varphi_k|_{\mathring{D}_r} : \mathring{D}_r \to \mathbb{R}$. As above, applying Proposition 1 we conclude that function $\mathring{v}_k$ is strictly monotone on each arc

$$\alpha_c = \mathring{u}_k^{-1}(\mathring{u}_k(c+0i)) = \{z \in \mathring{D}_r \mid \operatorname{Re} z = c\}, \quad c \in (0,1).$$

We already know that the function v is strictly monotone on the arcs γ_k and γ_s, where $s \equiv k+1 \pmod{2n}$. Therefore the function $v_k = v \circ \varphi_k : D_r \to \mathbb{R}$ is strictly monotone on the arcs

$$\alpha_- = \varphi_k^{-1}(\gamma_k) = \{z \in D_r \mid \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z < 0\},$$
$$\alpha_+ = \varphi_k^{-1}(\gamma_s) = \{z \in D_r \mid \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z > 0\}.$$

Let us verify that v_k is strictly monotone on the arc

$$\alpha_0 = \alpha_- \cup \{0\} \cup \alpha_+ = u_k^{-1}(u_k(0)) = \{z \in D_r \mid \operatorname{Re} z = 0\}.$$

Since $v_k(0) = v(0) = V(p) = 0$ according to our initial assumptions and 0 is the boundary point both for α_- and α_+, then v_k is of fixed sign on each of these two arcs.

So we have two possibilities:

- either v_k has the same sign on α_- and α_+, then $v_k|_{\alpha_0}$ has a local extremum in 0;
- or v_k has different signs on α_- and α_+, then v_k is strictly monotone on α_0.

Suppose that v_k has the same sign on α_- and α_+.

We will assume that v_k is negative both on α_- and α_+. The case when v_k is positive on α_- and α_+ is considered similarly.

Denote $z_1 = 0 - i/2 \in \alpha_-$, $z_2 = 0 + i/2 \in \alpha_+$. Let

$$\hat{\varepsilon} = \tfrac{1}{2}\min(|v_k(z_1)|, |v_k(z_2)|) > 0.$$

From the continuity of v_k it follows that there exists $\hat{\delta} > 0$ to comply with the following implications

$$\begin{aligned} |z - z_1| < \hat{\delta} &\Rightarrow |v_k(z) - v_k(z_1)| < \hat{\varepsilon}, \\ |z - z_2| < \hat{\delta} &\Rightarrow |v_k(z) - v_k(z_2)| < \hat{\varepsilon}, \\ |z| = |z - 0| < \hat{\delta} &\Rightarrow |v_k(z) - v_k(0)| = |v_k(z)| < \hat{\varepsilon}. \end{aligned} \tag{1}$$

Let $c \in (0, \hat{\delta})$. Then the point $w_0 = c + i0$ is situated on the curve α_c between points $w_1 = c - i/2$ and $w_2 = c + i/2$. It follows from (1) that $v_k(w_1) < -\hat{\varepsilon}$, $v_k(w_2) < -\hat{\varepsilon}$ and $v_k(w_0) \in (-\hat{\varepsilon}, 0)$. But these three correlations can not hold true simultaneously since v_k is strictly monotone on α_c as we already know.

The contradiction obtained shows us that v_k has different signs on α_- and α_+. So, v_k is strictly monotone on α_0.

Now, repeating argument from case 1 we find such $\varepsilon_k > 0$ and $\delta_k > 0$ that the set

$$\hat{W}_k = [0, \delta_k) \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

meets the relations

$$\begin{aligned} F \circ T \circ \varphi_k(\hat{W}_k) &\supseteq [0, \delta_k) \times (-\varepsilon_k, \varepsilon_k)\,, \quad \text{if } k = 2m\,, \\ F \circ T \circ \varphi_k(\hat{W}_k) &\supseteq (-\delta_k, 0] \times (-\varepsilon_k, \varepsilon_k)\,, \quad \text{if } k = 2m + 1\,. \end{aligned} \tag{2}$$

Let us denote $W_k = \varphi_k(\hat{W}_k)$,

$$W = \bigcup_{k=0}^{2n-1} W_k\,, \quad \delta = \min_{k=0,\dots,2n-1} \delta_k > 0\,, \quad \varepsilon = \min_{k=0,\dots,2n-1} \varepsilon_k > 0\,.$$

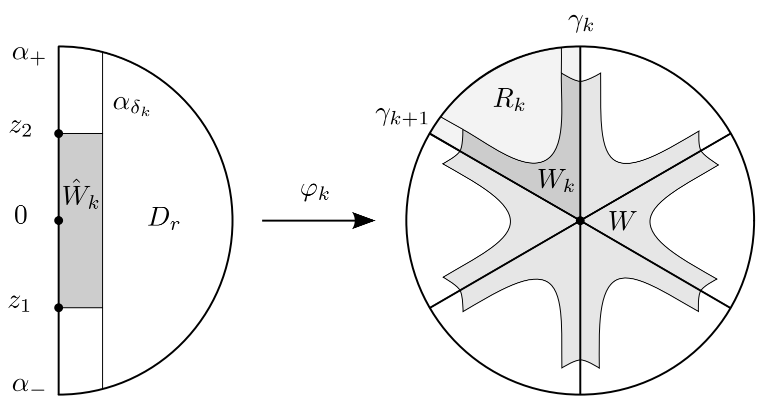

Figure 14

It is easy to show that W is an open neighbourhood of 0 in D. From (2) and from our initial assumptions it follows that

$$F(Q) \supseteq F(N) \supseteq F \circ T(W) \supseteq (-\delta, \delta) \times (-\varepsilon, \varepsilon)\,.$$

So, we have proved that for an arbitrary point $p \in M^2$ and its open neighbourhood Q a set $F(Q)$ contains a neigbourhood of $F(p)$. Hence the mapping $F : M^2 \to \mathbb{C}$ is open.

At the same time we have shown that an arbitrary $p \in M^2$ is an isolated point of its level set $F^{-1}(F(p))$. It is easy to see now that any level set $F^{-1}(F(p))$ can not contain a nondegenerate continuum.

Consequently, the map F is interior. □

Proof of Theorem 1. Necessity. Let $U, V : M^2 \to \mathbb{R}$ be conjugate pseudoharmonic functions on M^2 (see Definitions 3 and 4).

Obviously, V is continuous on M^2. Suppose that contrary to the statement of Theorem there exists such $c \in \mathbb{R}$ that V is not open on the level set $\Gamma_c = U^{-1}(c) \subset M^2$, i. e. a map $V_c = V|_{\Gamma_c} : \Gamma_c \to \mathbb{R}$ is not open on Γ_c in the topology induced from M^2.

Let us verify that V_c has a local extremum in some $p \in \Gamma_c$.

Note that the space Γ_c is locally arcwise connected, i. e. for every point $a \in \Gamma_c$ and its open neighbourhood Q there exists a neighbourhood $\hat{Q} \subseteq Q$ of a such that every two points $b_1, b_2 \in \hat{Q}$ can be connected by a continuous curve in Q. This is a straightforward corollary of the remark subsequent to Definition 1.

Since the map V_c is not open by our supposition, then there exists an open subset O of Γ_c such that its image $R = V_c(O)$ is not open in $\mathbb{R}$. Therefore there is a point $d \in R \setminus \operatorname{Int} R$. Fix $p \in V_c^{-1}(d) \cap O$.

Let us show that p is a point of local extremum of V_c. Fix a neighbourhood $\hat{O} \subseteq O$ of p such that every two points $b_1, b_2 \in \hat{O}$ can be connected by a continuous curve $\beta_{b_1,b_2} : I \to \Gamma_c$ which meets relations $\beta(0) = b_1$, $\beta(1) = b_2$ and $\beta(I) \subseteq O$. It is clear that an image of a path-connected set under a continuous mapping is path-connected, therefore following inclusions are valid

$$\big(V_c(b_1), V_c(b_2)\big) \subset V_c(I) \quad \text{if} \quad V_c(b_1) < V_c(b_2)\,,$$
$$\big(V_c(b_2), V_c(b_1)\big) \subset V_c(I) \quad \text{if} \quad V_c(b_2) > V_c(b_1)\,.$$

Evidently, p is not an interior point of $V_c(\hat{O})$ since it is not the interior point of $V_c(O)$ by construction and $V_c(\hat{O}) \subseteq V_c(O)$. Then there does not exist a pair of points $b_1, b_2 \in \hat{O}$ such that

$$V_c(b_1) < V_c(p) < V_c(b_2)$$

and either $V(b) \leq V(p)$ for all $b \in \hat{O}$ or $V(b) \geq V(p)$ for all $b \in \hat{O}$, i. e. p is the point of local extremum of V_c.

Now, since V is the conjugate pseudo-harmonic function of U in the point p (see Definition 3), we can take by definition a neighbourhood N of p in M^2 and a homeomorphism $T : D \to N$ such that a map $f : D \to \mathbb{C}$

$$f(z) = u(z) + iv(z)\,, \quad z \in D$$

is holomorphic on D. Here

$$u = U \circ T : D \to \mathbb{R}$$

and

$$v = V \circ T : D \to \mathbb{R}.$$

It is clear that without loss of generality we can choose N so small that either $V(b) = V_c(b) \leq V_c(p) = V(p)$ for every $b \in N \cap \Gamma_c$ or $V(b) \geq V(p)$ for all $b \in N \cap \Gamma_c$.

Let for definiteness p is the local maximum of V_c and

$$V(b) \leq V(p)$$

for every $b \in N \cap \Gamma_c$. The case when p is the local minimum of V_c is considered similarly.

On one hand it follows from what we said above that

$$\big(\{U(p)\} \times (V(p), +\infty)\big) \cap f(D) = \varnothing$$

since $u^{-1}(U(p)) = T^{-1}(\Gamma_c \cap N)$ and $v(z) = V(T(z)) \leq V(p)$ for all $z \in T^{-1}(\Gamma_c \cap N)$ by construction. Therefore a point

$$U(p) + iV(p) = f(T^{-1}(p))$$

is not the interior point of a set $f(D)$.

On the other hand it is known that the holomorphic map f is open, so the point $f(T^{-1}(p))$ must be the interior point of the domain $f(D)$.

The contradiction obtained shows that our initial assumption is false and V is open on level sets of U.

Sufficiency. Let U be a pseudo-harmonic function on M^2 and a continuous function $V : M^2 \to \mathbb{R}$ be open on level sets of U.

From Lemma 1 it follows that the mapping $F : M^2 \to \mathbb{C}$, $F(p) = U(p) + iV(p)$, $p \in M^2$ is interior.

Let $p \in M^2$ and N is a simple neighbourhood of p in M^2. Then there exists a homeomorphism $T : D \to N$. It is straightforward that for the open set N a mapping $F_N = F|_N : N \to \mathbb{C}$ is interior and its composition $F_N \circ T = F \circ T : D \to \mathbb{C}$ with the homeomorfism T is also an interior mapping.

Now from Stoilov theorem it follows that there exists a complex structure on D such that the mapping $F \circ T$ is holomorphic in this complex structure (see [3]). But from the uniformization theorem (see [4]) it follows that a simple-connected domain has a unique complex structure. So the mapping $F \circ T$ is holomorphic on D in the standard complex structure. Thus the functions

$$u = \mathrm{Re}(F \circ T) = U \circ T$$

and

$$v = \mathrm{Im}(F \circ T) = V \circ T$$

are conjugate harmonic functions on D. Consequently, V is a conjugate pseudo-harmonic function of U in the point p.

From arbitrariness in the choise of $p \in M^2$ it follows that V is a conjugate pseudo-harmonic function of U on M^2. $\square$

Corollary 1. *Let $U, V : M^2 \to \mathbb{R}$ be conjugate pseudoharmonic functions on M^2.*

Then there exists a complex structure on M^2 with respect to which U and V are conjugate harmonic functions on M^2.

Proof. This statement follows from Theorem 1, Lemma 1 and the Stoilov theorem which says that there exists a complex structure on M^2 such that the interior mapping $F(p) = U(p) + iV(p)$, $p \in M^2$ is holomorphic in this complex structure (see [3]). □

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