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On P-numbers of quadratic forms

In this paper we introduce P-numbers of quadratic forms over $\mathbb{R}$ and study their properties.

In this paper, by a quadratic form we mean a quadratic form over the field of real numbers $\mathbb{R}$

$$f(z) = f(z_1, \dots, z_n) = \sum_{i=1}^{n} f_i z_i^2 + \sum_{i<j} f_{ij} z_i z_j.$$

The set of all such form is denoted by $\mathcal{R}$, and the set of all $f(z) \in \mathcal{R}$ with $f_1, \dots, f_n = 1$ is denoted by $\mathcal{R}_0$.

Let $f(z) \in \mathcal{R}_0$ and $s \in \{1, \dots, n\}$. We introduce the notion of *the s-deformation of $f(z)$* as follows:

$$f^{(s)}(z, a) = f^{(s)}(z_1, \dots, z_n, a) = a z_s^2 + \sum_{i \neq s} z_i^2 + \sum_{i<j} f_{ij} z_i z_j,$$

where a is a parameter. Denote by $F_+^{(s)}$ the set of all $b \in \mathbb{R}$ such that the form $f^{(s)}(z, b)$ is positive definite, and put

$$F_-^{(s)} = \mathbb{R} \setminus F_+^{(s)}.$$

In other words, $b \in F_-^{(s)}$ iff there exists a nonzero vector

$$r = (r_1, \dots, r_n) \in \mathbb{R}^n$$

such that $f^{(s)}(r_1, \dots, r_n, b) \leq 0$. Further, put

$$m_f^{(s)} = \sup \mathrm{F}_-^{(s)} \in \mathbb{R} \cup \infty$$

(since $x \in F_-^{(s)}$ implies $y \in F_-^{(s)}$ for any $y < x$, this supremum is a limit point). We call $m_f^{(s)}$ *the s-th P-number of $f(z)$.*

Proposition 1. *Let $f(z_1, \dots, z_n) \in \mathcal{R}_0$. Then*

1) $m_f^{(s)} \geq 0$;

2) $m_f^{(s)} = \infty$ if the form

$$f_{-s}(z_1, \dots, z_{s-1}, z_{s+1}, \dots, z_n) = f(z_1, \dots, z_{s-1}, 0, z_{s+1}, \dots, z_n)$$

is not positive definite.

Both these assertions follow easily from the definitions.

Theorem 1. *Let $f(z_1, \dots, z_n) \in \mathcal{R}_0$ and let $m_f^{(s)} \neq \infty$. Then*

1) $m_f^{(s)} \in F_-^{(s)}$, and consequently $m_f^{(s)}$ is the greatest number of $F_-^{(s)}$.

2) the form $f^{(s)}(z, m_f^{(s)})$ is non-negative definite;

Proof. 1) We may assume, without loss of generality, that $s = n$. Consider the matrix $S(a)$ of the quadratic form $f^{(n)}(z, a)$:

$$S(a) = \frac{1}{2} \begin{pmatrix} 2 & f_{12} & \dots & f_{1,n-1} & f_{1n} \\ f_{12} & 2 & \dots & f_{2,n-1} & f_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ f_{1,n-1} & f_{2,n-1} & \dots & 2 & f_{n-1,n} \\ f_{1n} & f_{2n} & \dots & f_{n-1,n} & 2a \end{pmatrix}.$$

Demote by $\Delta_k, k = 1, \dots, n-1$, the principal $k \times k$ minor of $S(a)$ and by Δ_{in} the $(n-1) \times (n-1)$ minor of $S(a)$ which is obtained from $S(a)$ by deleting ith arrow and nth column. The determinant of $S(a)$ is denoted by $\Delta(a)$. Then by the well-known formula,

$$\Delta(a) = 1/2[(-1)^{n+1} f_{1n}\Delta_{1n} + (-1)^{n+2} f_{2n}\Delta_{2n} + \cdots \\ \cdots + (-1)^{2n-1} f_{n-1,n}\Delta_{n-1,1n}] + a\Delta_{n-1},$$

whence

$$\Delta(a) = a\Delta_{n-1} + N \qquad (*)$$

where $N = 1/2[(-1)^{n+1} f_{1n}\Delta_{1n} + (-1)^{n+2} f_{2n}\Delta_{2n} + \cdots + (-1)^{2n-1} f_{n-1,n}\Delta_{n-1,1n}]$.

By assertion 2) of Proposition 1 the form $f_{-n}(z_1, \ldots, z_{n-1})$ is positive definite (since $m_f^{(n)} \neq \infty$). From Silvestr's criterion of positive definiteness of quadratic forms it follows that

$$\Delta_1 > 0, \ldots, \Delta_{n-1} > 0\,.$$

Further, from this criterion it follows that $f(z, a)$ is positive definite if $\Delta(a) > 0$, and is not positive definite if $\Delta(a) \leq 0$. Consequently (see $(*)$)

$$\begin{aligned} F_-^{(n)} &= \{b \in \mathbb{R} \mid \Delta(b) \leq 0\} \\ &= \{b \in \mathbb{R} \mid b\Delta_{n-1} \leq -N\} \\ &= \{b \in \mathbb{R} \mid b \leq -N/\Delta_{n-1}\}. \end{aligned}$$

So $m_f^{(n)} = -N/\Delta_{n-1} \in F_-^{(n)}$, as claimed.

2) **The first proof.** Suppose that $f^{(s)}(z, m_f^{(s)})$ is not nonnegative definite. Then there is a vector $r = (r_1, \ldots, r_n) \in \mathbb{R}^n$ such that $f^{(s)}(r, m_f^{(s)}) = \alpha < 0$. Fix $0 < \varepsilon < -\alpha$. By continuity of $f(z, a)$, there exist $\delta_i > 0$ for $i = 1, \ldots, n$ and $\delta > 0$ such that

$$|f^{(s)}(r_1 + \mu_1, \ldots, r_n + \mu_n, m_f^{(s)} + \mu) - f^{(s)}(r_1, \ldots, r_n, m_f^{(s)})| < \varepsilon$$

whenever $|\mu_i| < \delta_i$ for $i = 1, \ldots, n$ and $|\mu| < \delta$. Put $\mu_i = 0$ for $i = 1, \ldots, n$ and fix $0 < \mu_0 < \delta$. Then

$$|f^{(s)}(r_1, \ldots, r_n, m_f^{(s)} + \mu_0) - \alpha| < \varepsilon.$$

It follows that $f^{(s)}(r_1, \ldots, r_n, m_f^{(s)} + \mu_0) - \alpha < \varepsilon$, whence

$$f^{(s)}(r_1, \ldots, r_n, m_f^{(s)} + \mu_0) < \varepsilon + \alpha < 0.$$

So $m_f^{(s)} + \mu_0 \in F_-^{(s)}$, a contradiction to the definition of $m_f^{(s)}$.

The second proof. Let $s = n$. It follows from the proof of assertion 1) (of this theorem) that $\delta(m_f^{(n)}) = 0$. Since

$$\Delta_1 > 0, \quad \ldots, \quad \Delta_{n-1} > 0,$$

the form $f^{(n)}(z, m_f^{(n)})$ is non-negative definite (see, for example, [1, P.322]). $\square$

References

[1] *V. V. Voevodin* Linear algebra. Moskow: Nauka, 1980, 400p. (in Russian).