Збірник праць
Ін-ту математики НАН України
2009, т.6, №2, 288-302

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Feedback Equivalence of 1-dimensional Control Systems of the 1-st Order

The problem of local feedback equivalence for 1-dimensional control systems of the 1-st order is considered. The algebra of differential invariants and criteria for the feedback equivalence for regular control systems are found.

Keywords: *differential invariants, invariant differentiation*

1. Introduction

In this paper we study the problem of local feedback equivalence for 1-dimensional control systems of 1-st order.

As in paper ([8]) we use the method of differential invariants. To this end we consider control systems as underdetermined ordinary differential equations. This gives a representation of feedback transformations as a special type of Lie transformations, and we study and find differential invariants of these representation.

Remark also that from the EDS point of view the case of control systems considered here is equivalent to the case of second order systems considered in ([8]), but from ODE point of view they have different algebras of feedback differential invariants.

To find a structure of the algebra of feedback differential invariants we first find 3 feedback invariant derivations. Then the differential invariants algebra is generated by two basic differential invariants J and K of orders 2 and 3 respectively and by all their invariant derivations.

This description allows us to find invariants for the formal feedback equivalence problem.

To get a local feedback equivalence we introduce a notion of *regular* control system and connect with such a system a 3-dimensional submanifold Σ in $\mathbb{R}^{14}$.

The main result of the paper states that two regular control systems are locally feedback equivalent if and only if the corresponding 3-dimensional submanifolds Σ coincide.

2. Representation of Feedback Pseudogroup

Let

$$\dot{x} = F(x, u, \dot{u}), \tag{1}$$

be an autonomous 1-dimensional control system of the 1-st order.

Here the function $x = x(t)$ describes a dynamic of the state of the system, and $u = u(t)$ is a scalar control parameter.

We shall consider this system as an undetermined ordinary differential equation of the first order on sections of 2-dimensional bundle $\pi : \mathbb{R}^3 \to \mathbb{R}$, where $\pi : (x, u, t) \longmapsto t$.

Let $\mathcal{E} \subset J^1(\pi)$ be the corresponding submanifold. In the canonical jet coordinates $(t, x, u, x_1, u_1, \dots)$ this submanifold is given by the equation:

$$x_1 = F(x, u, u_1).$$

It is known (see, for example, [6]) that Lie transformations in jet bundles $J^k(\pi)$ for 2-dimensional bundle π are prolongations of point transformations, that is, prolongations of diffeomorphisms of the total space of the bundle π.

We shall restrict ourselves by point transformations which are automorphisms of the bundle π.

Moreover, if these transformations preserve the class of systems (1) then they should have the form

$$\Phi : (x, u, t) \to (X(x), U(x, u), t). \tag{2}$$

Diffeomorphisms of form (2) is called *feedback transformations.* The corresponding infinitesimal version of this notion is a *feedback vector field,* i.e. a plane vector field of the form

$$X_{a,b} = a\,(x)\,\partial_x + b\,(x,u)\,\partial_u.$$

The feedback transformations in a natural way act on the control systems of type (1):

$$\mathcal{E} \longmapsto \Phi^{(1)}\,(\mathcal{E})\,,$$

where $\Phi^{(1)} : J^1\,(\pi) \to J^1\,(\pi)$ is the first prolongation of the point transformation Φ.

Passing to functions F, defining the systems, we get the following action on these functions:$\widehat{\Phi} : F \longmapsto G$, where the function G is a solution of the equation

$$X_x\; G = F\,(X, U, U_x G + U_u u_1)\,. \tag{3}$$

The infinitesimal version of this action leads us to the following representation $X_{a,b} \longmapsto \widehat{X_{a,b}}$ of feedback vector fields:

$$\widehat{X_{a,b}} \;= a\;\partial_x + b\;\partial_u + (u_1 b_u + f\;b_x)\,\partial_{u_1} + a_x\; f\;\partial_f. \tag{4}$$

In this formula $\widehat{X_{a,b}}$ is a vector field on the 4-dimensional space $\mathbb{R}^4$ with coordinates (u, u, u_1, f), and this field corresponds to the above action in the following sense.

Each control system (1) determines a 3-dimensional submanifold $L_F \subset \mathbb{R}^4$, the graph of F :

$$L_F = \{f = F\,(x, u, u_1)\}\,.$$

Let A_t be the 1-parameter group of shifts along vector field $X_{a,b}$ and let $B_t : \mathbb{R}^4 \to \mathbb{R}^4$ be the corresponding 1-parameter group of shifts along $\widehat{X_{a,b}}$, then these two actions related as follows

$$L_{\widehat{A_t}(F)} = B_t\,(L_F)\,.$$

In other words, if we consider an 1-dimensional bundle

$$\kappa : \mathbb{R}^4 \to \mathbb{R}^3,$$

where $\kappa((u, u, u_1, f)) = (u, u, u_1)$, then formula (4) defines the representation $X \longmapsto \widehat{X}$ of the Lie algebra of feedback vector fields into the Lie algebra of Lie vector fields on $J^0(\kappa)$, and the action of Lie vector fields $\widehat{X}$ on sections of bundle κ corresponds to the action of feedback vector fields on right hand sides of (1)

3. Feedback Differential Invariants

By a *feedback differential invariant* of order $\leq k$ we understand a function $I \in C^\infty\left(J^k\kappa\right)$ on the space of k-jets $J^k(\kappa)$, which is invariant under of the prolonged action of feedback transformations.

Namely,

$$\widehat{X_{a,b}}^{(k)}(I) = 0,$$

for all feedback vector fields $X_{a,b}$.

In what follows we shall omit subscript of order of jet spaces, and say that a function I on the space of infinite jets $I \in C^\infty(J^\infty\kappa)$ is a feedback differential invariant if

$$\widehat{X_{a,b}}^{(\cdot)}(I) = 0,$$

where $\widehat{X_{a,b}}^{(\cdot)}$ is the prolongation of the vector field $X_{a,b}$ in the space of infinite jets $J^\infty\kappa$.

In a similar way one defines a *feedback invariant derivations* as combinations of total derivatives

$$\nabla = A\frac{d}{dx} + B\frac{d}{du} + C\frac{d}{du_1},$$
$$A, B, C \in C^\infty(J^\infty\kappa),$$

which are invariant with respect to prolongations of feedback transformations, that is,

$$[\widehat{X_{a,b}}^{(\cdot)}, \nabla] = 0$$

for all feedback vector fields $X_{a,b}$.

Remark that for these derivations functions $\nabla(I)$ are differential invariants (of order, as a rule, higher then order of I) for any feedback differential invariant I. This observation allows us to construct new differential invariants from known ones only by the differentiations.

Recall the construction of the Tresse derivations in our case. Let $J_1, J_2, J_3 \in C^\infty\left(J^k\kappa\right)$ be three feedback differential invariants, and let

$$\widehat{d}J_i = \frac{dJ_i}{dx}dx + \frac{dJ_i}{du}du + \frac{dJ_i}{du_1}du_1$$

be their total derivatives.

Assume that we are in a domain $\mathcal{D}$ in $J^k\kappa$, where

$$\widehat{d}J_1 \wedge \widehat{d}J_2 \wedge \widehat{d}J_3 \neq 0.$$

Then, for any function $V \in C^\infty\left(J^l\kappa\right)$ over domain $\mathcal{D}$, one has decomposition

$$\widehat{d}V = \lambda_1\widehat{d}J_1 + \lambda_2\widehat{d}J_2 + \lambda_3\widehat{d}J_3.$$

Coefficients λ_1, λ_2 and λ_3 of this decomposition are called the *Tresse derivatives* of V and are denoted by

$$\lambda_i = \frac{DV}{DJ_i}.$$

The remarkable property of these derivatives is the fact that they are feedback differential invariants (of higher, as a rule, order then V) each time when V is a feedback differential invariant.

In other words, the Tresse derivatives

$$\frac{D}{DJ_1}, \frac{D}{DJ_2} \text{ and } \frac{D}{DJ_3}$$

are feedback invariant derivations.

4. Dimensions of Orbits

First of all, we remark that the submanifold $\{f = 0\}$ is a singular orbit for the feedback action in the space of 0-jets $J^0\kappa$. The generating function of the feedback vector field $\widehat{X_{a,b}}$ has the form:

$$\phi_{a,b} = a_x f - a f_x - b f_u - \left(u_1 b_u + f b_x\right) f_z,$$

and the formula for prolongations of vector fields ([6]) shows that in the space of 1-jets $J^1\kappa$, in addition, we have one more singular orbit $\{f_{u_1} = 0\}$. In similar way, we have one more singular orbit $\{f_{u_1u_1} = 0\}$ in the space of 2-jets. There are no more additional singular orbits in the spaces of k-jets, when $k \geq 3$.

We say that a point $x_k \in J^k\kappa$ is *regular*, if $f \neq 0, f_{u_1} \neq 0, f_{u_1u_1} \neq 0$ at this point.

In what follows we shall consider orbits of regular points only.

It is easy to see, that the $k-$th prolongation of the feedback vector field $\widehat{X_{a,b}}$ depends on $(k+1)$-jet of function $a\,(x)$ and $(k+1)$-jet of function $b\,(x,u)$.

Denote by V_i^k and W_{ij}^k the components of the decomposition

$$\widehat{X_{a,b}}^{(k)} = \sum_{0\leq i\leq k+1} a^{(i)}\,(x)\,V_i^k + \sum_{0\leq i+j\leq k+1} \frac{\partial^{i+j}b}{\partial x^i \partial u^j} W_{ij}^k.$$

Then, by the construction, the vector fields $V_i^k, 0 \leq i \leq k+1$, and $W_{ij}^k, 0 \leq i+j \leq k+1$, generate a completely integrable distribution on the space of k-jets, integral manifolds of which are orbits of the feedback action in $J^k\kappa$.

Straightforward computations show that there are no non trivial feedback differential invariants of the 1-st order.

Let $\mathcal{O}_{k+1}$ be a feedback orbit in $J^{k+1}\kappa$, then the projection $\mathcal{O}_k = \kappa_{k+1,k}\,(\mathcal{O}_{k+1}) \subset J^k\kappa$ is an orbit too, and to determine dimensions of the orbits one should find dimensions of the bundles: $\kappa_{k+1,k} : \mathcal{O}_{k+1} \to \mathcal{O}_k$. To do this we should find conditions on functions a and b under which $\widehat{X_{a,b}}^{(k)} = 0$ at a point $x_k \in J^k\kappa$.

Assume that $\widehat{X_{a,b}}^{(k-1)} = 0$ at the point $x_{k-1} \in J^{k-1}\kappa$. Then the vector field $\widehat{X_{a,b}}^{(k)}$ is a $\kappa_{k,k-1}$-vertical over this point.

Components

$$\frac{d^k\phi}{dx^i du^j}\frac{\partial}{\partial f_{\sigma_{ij}}}$$

of this vector field, where $\sigma_{ij} = (\underbrace{x, \dots, x}_{i\text{-times}}, \underbrace{u, \dots, u}_{j\text{-times}}), i + j = k$, and components

$$\frac{d^k \phi}{dx^i du^j du_1} \frac{\partial}{\partial f_{\tau_{ij}}},$$

where $\tau_{ij} = (\underbrace{x, \dots, x}_{i\text{-times}}, \underbrace{u, \dots, u}_{j\text{-times}}), i + j = k - 1$ depend on

$$\frac{\partial^{k+1} b}{\partial x^i \partial u^j},$$

and

$$\frac{d^{k+1} a}{dx^{k+1}}$$

respectively.

All others components

$$\frac{d^k \phi}{dx^r du^s du_1^t} \frac{\partial}{\partial f_\sigma}$$

are expressed in terms of k-jet of $b(x, u)$ and k-jet of function $a(x)$.

It shows that the bundles: $\kappa_{k,k-1} : \mathcal{O}_k \to \mathcal{O}_{k-1}$ are $(k+3)$-dimensional, when $k > 1$.

Feedback orbits in the space of 2-jets can be found by direct integration of 12-dimensional completely integrable distribution generating by the vector fields $V_i^1, 0 \le i \le 3$, and $W_{ij}^1, 0 \le i + j \le 2$. Summarizing, we get the following result.

Theorem 1. (1) *The first non-trivial differential invariants of feedback transformations appear in order* 2 *and they are functions of the basic invariant*

$$J = \frac{f^2 \ f_{u_1 u_1}}{(u_1 f_{u_1} - f) \ f_{u_1}^2}.$$

(2) *There are*

$$\frac{k(k+1)}{2} - 2$$

independent differential invariants of pure order k.

(3) *Dimension of the algebra of differential feedback invariants of order $k \geq 2$, is equal to*

$$\frac{k^3}{6} + \frac{k^2}{2} - \frac{5k}{3} + 1.$$

(4) *Dimension of the regular feedback orbits in the space of k-jets, $k \geq 2$, is equal to*

$$\frac{(k+1)^2}{2} + \frac{23k}{3} + \frac{5}{2}.$$

5. Invariant Derivations

We'll need the following result which allows us to compute invariant derivations.

Assume that an infinitesimal Lie pseudogroup $\mathfrak{g}$ is represented in the Lie algebra of contact vector fields on the manifold of 1-jets $J^1(\mathbb{R}^n)$.

Moreover, we will identify elements $\mathfrak{g}$ with the corresponding contact vector fields , i.e. we assume that elements of $\mathfrak{g}$ have the form X_f (see [6]), where f is the generating function.

Lemma 1. *Let $x_1, \ldots, x_n$ be coordinates in $\mathbb{R}^n$, and let*

$$(x_1, \ldots, x_n, u, p_1, \ldots, p_n)$$

be the corresponding canonical coordinates in the 1-jet space $J^1(\mathbb{R}^n)$. Then a derivation

$$\nabla = \sum_{i=1}^{n} A_i \frac{d}{dx_i}$$

is $\mathfrak{g}$-invariant if and only if functions $A_i \in C^\infty(J^\infty \mathbb{R}^n)$, $j = 1, \ldots, n$, are solutions of the following PDE system:

$$X_f(A_i) + \sum_{j=1}^{n} \frac{d}{dx_j}\left(\frac{\partial f}{\partial p_i}\right) A_j = 0, \tag{5}$$

for all $i = 1, \ldots, n$, and $X_f \in \mathfrak{g}$.

Proof. We have, [6]:

$$X_f^{\bullet} = \mathbf{E}_f - \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \frac{d}{dx_i},$$

where

$$\mathbf{E}_f = \sum_{\sigma} \frac{d^{|\sigma|} f}{dx^{\sigma}} \frac{\partial}{\partial p_\sigma}$$

is the evolutionary derivation, σ is a multi index and $\{p_\sigma\}$ are the canonical coordinates in $J^\infty \mathbb{R}^n$.

Using the fact that evolutionary derivations commute with the total ones and the relation

$$[\nabla, X_f^{\bullet}] = 0,$$

we get

$$\begin{aligned} 0 &= \left[\sum_{j=1}^{n} A_j \frac{d}{dx_j}, \mathbf{E}_f - \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \frac{d}{dx_i} \right] \\ &= -\sum_j \mathbf{E}_f \left(A_j\right) \frac{d}{dx_j} + \sum_{i,j} \left(-A_j \frac{d}{dx_j} \left(\frac{\partial f}{\partial p_i} \right) \frac{d}{dx_i} + \frac{\partial f}{\partial p_i} \frac{dA_j}{dx_i} \frac{d}{dx_j} \right) \\ &= -\sum_s \left(X_f^{\bullet} \left(A_s\right) + \sum_j A_j \frac{d}{dx_j} \left(\frac{\partial f}{\partial p_s} \right) \right) \frac{d}{dx_s}. \end{aligned}$$

□

In our case we expect three linear independent feedback invariant derivations. To solve PDE system (5) we first assume that the unknown functions are functions on the 1-jet space $J^1\mathbb{R}^3$. Then collect terms in (5) with $a, a^{'}, a''$ and $b, b_x, b_u, b_{xx}, b_{xu}$ and $b_{u\ u}$ we get the system of 8 differential equations for 3 unknown functions. Solving the system we found two independent invariant derivations. The last one we get in a similar way by assuming that the unknown functions are functions on the 2-jet space $J^2\mathbb{R}^3$.

Finally, we have 3 feedback invariant derivations:

$$\begin{aligned}
\nabla_1 &= \frac{u_1 f_{u_1} - f}{f_{u_1}} \frac{d}{du} + \frac{f - u_1 f_{u_1}}{f_{u_1}^2} f_u \frac{d}{du_1}, \\
\nabla_2 &= \frac{f}{f_{u_1}} \frac{d}{du_1}, \\
\nabla_3 &= f\ \frac{d}{dx} + \frac{f}{f_{u_1}} \frac{d}{du} + \\
&\quad \left(\frac{f_x f_{u_1} + f_u - z f_{u\ u_1} - f_{x u_1}}{f_{u_1 u_1}} + \frac{u_1 f_{u_1} - f}{f_{u_1}^2} f_u \right) \frac{d}{du_1}.
\end{aligned}$$

These derivations obey the following commutation relations

$$\begin{aligned}
[\nabla_2, \nabla_1] &= J\ \nabla_1 \\
[\nabla_3, \nabla_1] &= K\ \nabla_2 \\
[\nabla_3, \nabla_2] &= -\nabla_3 + J\ \nabla_1 + L\ \nabla_2
\end{aligned}$$

where K and L are some differential invariants of the 3rd order (see below).

6. Differential Invariants of the 3-rd Order

Theorem 1 shows that there are four independent differential invariants of the 3-rd order. We get three of them simply by invariant differentiations:

$$\nabla_1 (J), \nabla_2 (J), \nabla_3 (J).$$

The symbols of these invariants contain:

- symbol of $\nabla_2 (J)$ depends on $f_{u_1 u_1 u_1}$,
- symbol of $\nabla_1 (J)$ depends on $f_{u_1 u_1 u_1}$ and $f_{u u_1 u_1}$,
- symbol of $\nabla_3 (J)$ depends on $f_{u_1 u_1 u_1}$,$f_{u u_1 u_1}$ and $f_{x u_1 u_1}$.

It shows that these differential invariants are independent.

The similar observation shows that the differential invariant L, which appears in the commutation relations, is a function of

$J, \nabla_1(J), \nabla_2(J), \nabla_3(J)$, and the differential invariant K is the forth independent invariant. It has the following form:

$$K = -u_1 f_{xu} + 2u_1 \frac{f_u^2}{f f_{u_1}} - 2\frac{f_u^2}{{f_{u_1}}^2} + \\ + \frac{f_{uu} u_1 - 2 f_u f_x + f f_{xu}}{f_{u_1}} - u_1 \frac{(f_{uu} u_1 - 2 f_u f_x)}{f} + \\ + \frac{c_1}{f_{u_1} {f_{u_1 u_1}}^2} + \frac{c_2}{f {f_{u_1 u_1}}^2} + \frac{c_3}{{f_{u_1 u_1}}^2} + \frac{c_4}{f f_{u_1 u_1}} + \frac{c_5}{f_{u_1} f_{u_1 u_1}} + \frac{c_6}{f_{u_1 u_1}},$$

where

$$c_1 = -f f_u f_{x u_1} f_{u_1 u_1 u_1} - u_1 f_u f_{u u_1} f_{u_1 u_1 u_1} + f_u^2 f_{u_1 u_1 u_1},$$

$$c_2 = u_1 \left(f_u f_{u_1} f_{u u_1 u_1} - f_u^2 f_{u_1 u_1 u_1} - f_x f_u f_{u_1} f_{u_1 u_1 u_1} + f_x {f_{u_1}}^2 f_{u u_1 u_1} \right) \\ + {u_1}^2 f_{u u_1} \left(-f_{u_1} f_{u u_1 u_1} + f_u f_{u_1 u_1 u_1} \right),$$

$$c_3 = f f_{x u_1} f_{u u_1 u_1} + f_x f_u f_{u_1 u_1 u_1} - f_u f_{u u_1 u_1} - f_x f_{u_1} f_{u u_1 u_1} \\ + u_1 \left(f_u f_{x u_1} f_{u_1 u_1 u_1} - f_{u_1} f_{x u_1} f_{u u_1 u_1} + f_{u u_1} f_{u u_1 u_1} \right),$$

$$c_4 = -u_1 \left(2 f_{u_1} f_x f_{u u_1} - f_{u_1} f_u f_{x u_1} + f_u f_{u u_1} + f_{u_1} f_{uu} + {f_{u_1}}^2 f_{xu} \right) \\ + {u_1}^2 \left(f_{u_1} f_{u u u_1} - f_u f_{u u_1 u_1} + {f_{u u_1}}^2 \right),$$

$$c_5 = f f_u f_{x u_1 u_1} - f f_{x u_1} f_{u u_1} + f_u f_{u u_1} + u_1 \left(f_u f_{u u_1 u_1} - {f_{u u_1}}^2 \right),$$

$$c_6 = f_{uu} - f_u f_{x u_1} + 2 f_x f_{u u_1} + f_{u_1} f_{xu} - f f_{x u u_1} \\ + u_1 \left(f_{u_1} f_{x u u_1} - f_{u u u_1} + f_{x u_1} f_{u u_1} - f_u f_{x u_1 u_1} \right).$$

7. Algebra of Feedback Differential Invariants

By *regular orbits* we mean feedback orbits of regular points.

Counting the dimensions of regular feedback orbits shows that the following result is valid.

Theorem 2. *Algebra of feedback differential invariants in a neighborhood of a regular orbit is generated by differential invariant J of the 2-nd order, differential invariant K of the 3-rd order and all their invariant derivatives.*

8. The Feedback Equivalence Problem

Consider two control systems given by functions F and G. Then, to establish feedback equivalence, we should solve the differential equation

$$F\left(X, U, U_x G\left(x, u, u_1\right) + U_u u_1\right) - X_x\ G\left(x, u, u_1\right) = 0 \tag{6}$$

with respect to unknown functions $X\left(x\right)$ and $U\left(x, u\right)$.

Let us denote the left hand side of (6) by H. Then assuming the general position one can find functions X, X_x, U, U_x, U_u from the equations

$$H = H_{u_1} = H_{u_1}^{(2)} = H_{u_1}^{(3)} = H_{u_1}^{(4)} = 0.$$

Remark, that the above general conditions are feedback invariant, depends on finite jet of the system and holds in a dense open domain of the jet space. Therefore, it holds in regular points.

Assume that we get

$$\begin{aligned} &U = A\left(x, u, u_1\right), U_x = B\left(x, u, u_1\right), \\ &U_u = C\left(x, u, u_1\right), X = D\left(x, u, u_1\right), \\ &X' = E\left(x, u, u_1\right) \end{aligned}$$

Then the conditions

$$\begin{aligned} &A_{u_1} = B_{u_1} = C_{u_1} = D_{u_1} = E_{u_1} = 0, \\ &D_u = E_u = 0 \end{aligned}$$

and

$$B = A_x, C = A_u, E = D_x$$

show that if (6) has a formal solution at each point (x, u, u_1) in a domain then this equation has a local smooth solution.

On the other hand if system F at a point $p = (x^0, u^0, u_1^0)$ and system G at a point $\widetilde{p} = (\widetilde{x}^0, \widetilde{u}^0, \widetilde{u}_1^0)$ has the same differential invariants then, by the definition, there is a formal feedback transformation which send the infinite jet of F at the point p to the infinite jet of G at the point $\widetilde{p}$.

Keeping in mind these observations and results of theorem 2 we consider the space $\mathbb{R}^3$with coordinates (x, u, u_1) and the space $\mathbb{R}^{14}$ with coordinates $(j, j_1, j_2, j_3, j_{11}, j_{12}, j_{13}, j_{22}, j_{23}, j_{33} k, k_1, k_2, k_3)$.

Then any control system, given by function $F(x, u, u_1)$, defines a map

$$\sigma_F : \mathbb{R}^3 \to \mathbb{R}^{14},$$

by

$$\begin{aligned} j &= J^F, k = K^F, \\ j_i &= (\nabla_i(J))^F, k_i = (\nabla_i(K))^F, \\ j_{ij} &= (\nabla_i \nabla_j(J))^F, \end{aligned}$$

where $i, j = 1, 2, 3$, and the subscript F means that the differential invariants are evaluated due to the system.

Let

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^3$$

be a feedback transformation.

Then from the definition of the feedback differential invariants it follows that

$$\sigma_F \circ \Phi = \sigma_{\widehat{\Phi}(F)}.$$

Therefore, the geometrical image

$$\Sigma_F = Im(\sigma_F) \subset \mathbb{R}^{14}$$

does depend on the feedback equivalence class of F only.

We say that a system F is *regular* in a domain $D \subset \mathbb{R}^3$ if

(1) 4-jets of F belong to regular orbits,
(2) $\sigma_F(D)$ is a smooth 3-dimensional submanifold in $\mathbb{R}^{14}$, and
(3) three of five functions j, j_1, j_2, j_3, k are coordinates on Σ_F.

Assume, for example, that functions j_1, j_2, j_3 are coordinates on Σ_F. The following lemma gives a relation between the Tresse derivatives and invariant differentiations $\nabla_1, \nabla_2, \nabla_3$.

Lemma 2. *Let*

$$\frac{D}{DJ_1}, \frac{D}{DJ_2}, \frac{D}{DJ_3}$$

be the Tresse derivatives with respect to differential invariants $J_i = \nabla_i(J)$.

Then the following decomposition

$$\nabla_i = \sum_j R_{ij} \frac{D}{DJ_j} \tag{7}$$

with feedback differential invariants R_{ij} *of order* ≤ 4 *is valid.*

Proof. Applying both parts of (7) to invariant J_k we get

$$\nabla_i(J_k) = R_{ik}$$

which is a feedback differential invariant of order ≤ 4. □

Theorem 3. *Two regular systems* F *and* G *are locally feedback equivalent if and only if*

$$\Sigma_F = \Sigma_G. \tag{8}$$

Proof. Let us show that the condition 8 implies a local feedback equivalence.

Assume that

$$J^F = j^F(J_1, J_2, J_3), J_{ij}^F = j_{ij}^F(J_1, J_2, J_3),$$
$$K^F = k^F(J_1, J_2, J_3), K_i^F = k_i^F(J_1, J_2, J_3)$$

on Σ_F, and

$$J^G = j^G(J_1, J_2, J_3), J_{ij}^G = j_{ij}^G(J_1, J_2, J_3),$$
$$K^G = k^G(J_1, J_2, J_3), K_i^G = k_i^G(J_1, J_2, J_3)$$

on Σ_G.

Then condition 8 shows that $j^F = j^G, j_{ij}^F = j_{ij}^G, k_i^F = k_i^G$ and $k^F = k^G$.

Moreover,as we have seen the invariant derivations $\nabla_1, \nabla_2, \nabla_3$ are linear combinations of the Tresse derivatives with coefficients which are feedback differential invariants of order ≤ 4.

In other words, the above functions $j^F, k^F, j_{ij}^F, k_i^F$ and their partial derivatives in j_1, j_2, j_3 determine the restrictions of all differential invariants.

Therefore, condition 8 equalize restrictions of differential invariants not only to order ≤ 4 but in all orders, and provides formal and therefore local feedback equivalence between F and G. □

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