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Stability and Sensitivity Analysis of Digital Filters under Finite Word Length Effects via Normal Form Transformation

Main objective of this paper is to present the general solution of the first order matrix difference system $X(n+1) = AX(n)B + C(n)U(n)D(n)$, and then study the stability and sensitivity analysis of the digital filters via eigenvalue sensitivity and normal form transformations.

Предложено общее решение разностной системы матриц первого порядка $X(n+1) = AX(n)B + C(n)U(n)D(n)$. Исследована стабильность и чувствительность цифровых фильтров посредством анализа чувствительности собственного значения и преобразования нормальной формы.

K e y w o r d s: digital filters, difference equations, exponentiation of a matrix, stability and sensitivity analysis.

1. Introduction. Digital filters are widely used in many branches of science and engineering. More specifically, in communication systems and in control of linear and nonlinear systems the theory of digital filters is of immense importance due to discrete nature of the signals. Further, the study of discretization methods for differential equations has also increased the scope of the theory and applications of difference equations. In recent years, the investigation of the theory of difference equations has assumed greater significance as a well-deserved independent discipline due to various applications in signal processing and control systems. The control system theory in fact first gained considerable maturity in the discipline of engineering and has been successfully applied in a variety of branches of engineering, particularly receiving great impetus from aerospace engineering [1]. Most of the analog signals cannot be exactly implemented by the digital schemes, since the undesirable finite world length effect will occur during A/D process and / or in the registers of the CPU. Further, in some applications, the signals are faint and result in a degenerate accuracy of the digitalization. For example, in space makers the signals captured from a human body are discrete and hence finite word length (FWL) effects are very essential. The finite word length effects have been an important issue in hardware implementation and are widely discussed in literature [2, 3].

A novel approach is presented in [2] to analyze and minimize fixed point arithmetic errors for digital filter implementation based on eigenvalue sensitivity analysis. Many sensitive methods proposed for finding the optimal state-space realization are effective to deal with all linear systems under the FWL effects, especially with the digital filter and digital control systems [4]. In the year 2003 and 2004 in [3] an analytically algebraic method is proposed for solving an optimal transformation to achieve the minimal sensitivity subject to the FWL effects for digital control systems. For more information in this area of research we refer to [3].

Our paper is organized as follows. Section 2 presents criteria for constructing the general solution of the homogeneous system $X(n+1)=AX(n)B$ in terms of two fundamental matrix solutions of $X(n+1)=AX(n)$ and $X(n+1)=B^{*}X(n)$. The general solution of the first order matrix system

$$X(n+1)=AX(n)B+C(n)U(n)D(n),\ X(0)=X_0, \tag{1}$$

state equation

$$R(n)=EX(n)+FU(n), \tag{2}$$

output equation, where A and B are $(k\times k)$ non-singular constant matrices and $X(n)\in R^{k\times k}$ (or $C^{k\times k}$) are the components defined on $N_{n0}^{\pm}=\{n_0, n_{0\pm1}, n_{0\pm2}, \dots$ $\dots, n_{0\pm k}, \dots\}$, where $k\in N^{+}$ and $n_0\in N$, N being the set of integers. $U(n)$ is the control function and is of the order $(s\times k)$, $C(n)$ being $k\times s$ matrix, and the output signal R is $(r\times k)$. Standard terminology is that the difference system (1) is said to be time invariant, if the coefficient matrices A and B do not vary with time.

In section 3, we present a set of sufficient conditions for the stability analysis of the system (1) and then show through examples that the system $X(n+1)=$ $=AX(n)$ need not to be stable and the matrix B can be chosen so that the first order difference system $X(n+1)=AX(n)B$ be stable. Section 4 is concerned with sensitivity analysis of the linear system and eigenvalue sensitivity minimization and the optimally similar transformation. Section 5 is concerned with eigenvalue sensitivity and normal form transformation.

2. Preliminaries. In this section, we shall be concerned with the general solution of the first order matrix difference system

$$X(n+1)=AX(n)B+C(n)U(n)D(n),\ X(0)=X_0 \tag{3}$$

in terms of the two fundamental matrix solutions of the system $X(n+1)=AX(n)$ and $X(n+1)=B^{*}X(n)$ and then present a technique to compute A^n. For a linear system of the first order difference system, the computation of A^n is the analogous computation of e^{At} in the first order system of ordinary differential equations. From (3) we have

$$X(0)=X_0,$$
$$X(1)=AX_0B+C(0)U(0)D(0),$$
$$X(2)=AX_1B+C(1)U(1)D(1)=$$
$$=A^2X_0B^2+AC(0)U(0)D(0)B+C(1)U(1)D(1),$$

. . . .

$$X(n)=A^nX_0B^n+\sum_{j=0}^{n-1}A^{n-j-1}C(j+1)U(j+1)D(j+1)B^{n-j-1}.$$

Furthermore, $X(n)$ can be computed with the following

$$X(n)=\Phi(n,n_0)X_0\psi^*(n,n_0)+$$
$$+\sum_{j=0}^{n-1}\Phi(n_0,j+1)C(j+1)U(j+1)D(j+1)\psi^*(n_0,j+1)$$

and, if $\overline{X}(n)$ is a particular solution of (3), then any solution of (3) is given by $X(n)=\Phi(n,n_0)X_0\psi^*(n,n_0)+\overline{X}(n)$, where $\Phi(n,n_0)=A^{n-n_0}$ and $\psi^*(n,n_0)=$ $=B^{n-n_0}$. It is a well-known fact that the exponentiation of the matrix A defined by

$$e^{At}=I+(At)+\frac{(At)^2}{2!}+...+\frac{(At)^n}{n!}+...$$

is a solution of $y'=Ay$ (y being an n-vector) and any solution of $y'=Ay, y(0)=y_0$ is given by $e^{At}y_0$. The next theorem is useful for the stability analysis of digital filters under the FWL effects via normal-form transformation.

Theorem 1. Let A be a constant non-singular $(k\times k)$ matrix with characteristic polynomial $|\lambda I-A|=\lambda_k+C_{k-1}\lambda^{k-1}+\ ...\ +C_1\lambda\ +C_0=(\lambda-\lambda_1)(\lambda-\lambda_2)...(\lambda-\lambda_k)$. Then

$$A^n=x_1(n)I+x_2(n)A+x_3(n)A^2+...+x_k(n)A^{k-1},$$

where $x_1, x_2, ..., x_k$ are the k-linearly independent solutions of the k^{th} order scalar differential equation $X(n+k)+C_{k-1}X(n+k-1)+...+C_1X(n+1)+C_0X(n)=0$ satisfying the initial conditions

$$\begin{aligned}&x_1(0)=1,\ x_2(0)=0,\ ...,\ x_k(0)=0,\\&x_2(1)=0,\ x_2(1)=1,\ ...,\ x_k(1)=0,\\&.\ .\\&x_1(n-1)=0,\ x_2(n-1)=0,\ ...,\ x_k(n-1)=1.\end{aligned}\qquad(4)$$

P r o o f. Let A be a constant non-singular $(k\times k)$ matrix and let $G(\lambda)= =|\lambda I - A| = \lambda_k + C_{k-1}\lambda^{k-1} + ... + C_1\lambda + C_0 = 0$ be its characteristic polynomial. Define $\phi(n) = x_1(n)I + x_2(n)A + x_3(n)A^2 + ... + x_k(n)A^{k-1}$, where x_1, x_2, ..., x_k are the k-linearly independent solutions of the k^{th} order scalar difference equation $X(n+k) + C_{k-1}X(n+k-1) + ... + C_1X(n+1) + C_0X(n) = 0$ satisfying the initial conditions (4). Then

$$\phi(n+k) + C_{k-1}\phi(n+k-1) + ... + C_1\phi(n+1) + C_0\phi(n) =$$
$$= [x_1(n+k) + C_{k-1}x_1(n+k-1) + ... + C_1x_1(n+1) + C_0x_1(n)] +$$
$$+ [x_2(n+k) + C_{k-1}x_2(n+k-1) + ... + C_1x_2(n+1) + C_0x_2(n)]A +$$
$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots +$$
$$+ [x_k(n+k) + C_{k-1}x_k(n+k-1) + ... + C_1x_k(n+1) + C_0x_k(n)]A^{k-1} =$$
$$= 0.I + 0.A + ... + 0.0A^{k-1} = 0.$$

Thus, $\phi(n+k) + C_{k-1}\phi(n+k-1) + ... + C_1\phi(n+1) + C_0\phi(n) = 0$ for all $n \varepsilon\, N_0^+$ and

$$\phi(0) = x_1(0)I + x_2(0)A + x_3(0)A^2 + ... + x_k(0)A^{k-1} = I,$$
$$\phi(1) = x_1(1)I + x_2(1)A + x_3(1)A^2 + ... + x_k(1)A^{k-1} = A,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\phi(k-1) = x_1(k-1)I + x_2(k-1)A + x_3(k-1)A^2 + ... + x_k(k-1)A^{k-1} = A^{k-1}.$$

Clearly $\phi(n+k) = x_1(n)I + x_2(n)A + x_3(n)A^2 + ... + x_k(n)A^{k-1}$ satisfies the initial value problem $\phi(n+k) = C_{k-1}\phi(n+k-1) + ... + C_1\phi(n+1) + C_0\phi(n) = 0$ and $\phi(0) = I$, $\phi(1) = A$, $\phi(2) = A^2$, ..., $\phi(k-1) = A^{k-1}$.

Thus, it follows from the uniqueness of initial value problems that $\phi(n)$ is the unique solution of (3) and $A^n = x_1(n)I + x_2(n)A + x_3(n)A^2 + ... + x_k(n)A^{k-1}$. To illustrate the importance of the above theorem, we consider the following example.

Consider the matrix A given by

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 3 & 1 \\ -3 & 1 & 4 \end{bmatrix}.$$

The characteristic polynomial $G(\lambda)$ is given by $G(\lambda) = |\lambda I - A| = (\lambda - 2)^2(\lambda - 3) = 0$. Therefore, the eigenvalues of the matrix are given by 2, 2, 3. Clearly $A^n =$

$=x_1(n)I+x_2(n)A-x_3(n)A^2$. Let $X(n)=C_1(2)^n+C_2 n(2)^n+C_3(3)^n$. To determine the constants C_1, C_2 and C_3 we have

$$x_1(0)=1;\ x_2(0)=0;\ x_3(0)=1,$$
$$x_1(1)=0;\ x_2(1)=1;\ x_3(1)=0,$$
$$x_1(2)=0;\ x_2(2)=0;\ x_3(2)=1.$$

Now, the three linearly independent solutions $X_1(n)$, $X_2(n)$ and $X_3(n)$ are given by

$$x_1(n)=-3(2)^n-3(2)^n+4(3)^n,$$
$$x_2(n)=4(2)^n+2.5n(2)^n-4(3)^n,$$
$$x_3(n)=-(2)^n-0.5n(2)^n+(3)^n.$$

Therefore,

$$A^n=x_1(n)I+x_2(n)A+x_3(n)A^2=$$
$$=\begin{bmatrix}(2)^{n+1}-(3)^n-n(2)^{n-1} & n(2)^{n-1} & -(2)^n+(3)^n\\ (2)^n-(3)^n-n(2)^{n-1} & (n+2)(2)^{n-1} & -(2)^n+(3)^n\\ (2)^{n+1}-2(3)^n-n(2)^{n-1} & n(2)^{n-1} & -(2)^n+2(3)^n\end{bmatrix}.$$

Now consider the homogeneous system $X(n+1)=AX(n)B$, $X(0)=I$, where A is the matrix given above and

$$B=\begin{bmatrix}\frac{1}{4} & 0 & 0\\ 0 & \frac{1}{4} & 0\\ 0 & 0 & \frac{1}{4}\end{bmatrix},$$

we have

$$B^n=\begin{bmatrix}2^{-2n} & 0 & 0\\ 0 & 2^{-2n} & 0\\ 0 & 0 & 2^{-2n}\end{bmatrix},$$

and hence $X(n)=A^n\times(0)B^n$ is a solution of the system $X(n+1)=AX(n)B$.

3. Stability analysis. Stability is a very important concept in signal / image processing. A small perturbation in the initial data effects a substantial deviation in the image processing of the signal, then such a system is not acceptable even approximately. In general, stability in a system means that small changes in the system inputs or in initial conditions or in the system parameters do not effect a substantial change in the system behaviour. Most of the working systems are designed in such a way that they are perfectly stable [5] (stability means ability of the system to come back to the original state after a small perturbation). We need to investigate in this section conditions under which the homogeneous system is stable and asymptotically stable homogeneous system.

One of the most important requirements in the performance of control systems is stability. This is true for continuous-data systems as well as digital control systems.

Example 1. Consider the difference system

$$X(n+1)=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}X(n)=AX(n).$$

The eigenvalues of the matrix A are given by $\lambda_1=\lambda_2=-1$. The two linearly independent solutions are given by

$$X_1(n)=(1-n)(-1)^n,$$

$$X_2(n)=-n(-1)^n.$$

Therefore a solution matrix $\varphi(n)$ is given by

$$\varphi(n)=X_1(n)I+X_2(n)A=A^n=\begin{bmatrix}(1-n) & -n\\ n & (1+n)\end{bmatrix}(-1)^n.$$

Example 2. Consider the difference system

$$X(n+1)=AX(n)=\begin{bmatrix}0 & 1\\ -2 & 3\end{bmatrix}X(n).$$

The eigenvalues of the matrix A are $\lambda_1=1$ and $\lambda_2=2$, and hence the system is unstable in fact

$$A^n=\begin{bmatrix}2(1)^n-(2)^n & -(1)^n+(2)^n\\ 2(1)^n-2(2)^n & -(1)^n+2(2)^n\end{bmatrix}=\begin{bmatrix}2-(2)^n & -(1)^n+(2)^n\\ 2-2(2)^n & -(1)^n+2(2)^n\end{bmatrix}.$$

As $n\to\infty$, the eigenvalues of A^n do not lie inside the unit circle and hence the system is unstable. On the other hand consider

$$X(n+1)=AX(n)B=\begin{bmatrix}0 & 1\\ -2 & 3\end{bmatrix}X(n)\begin{bmatrix}-\frac{1}{2^2} & 0\\ 0 & -\frac{1}{2^2}\end{bmatrix}$$

with $X(0) = I$, where I is the unit matrix. Then $X(n) = A^n \times (0)\, B^n$ implies

$$X(n)=\begin{bmatrix} 2(1)^n-(2)^n & -(1)^n+(2)^n \\ 2(1)^n-2(2)^n & -(1)^n+2(2)^n \end{bmatrix}\begin{bmatrix} -2^{-2n} & 0 \\ 0 & -2^{-2n} \end{bmatrix}.$$

One can easily see that $\lim_{n\to\infty} X(n)=0$, and hence the system is stable and in fact asymptotically stable.

Theorem 2. Let $\phi(n)$ and $\psi(n)$ be fundamental matrix solutions of $T(n+1)= =AT(n)$ and $T(n+1)=B^* T(n)$ satisfying $\phi(n_0)=\psi(n_0)=I$. Then the system $T(n+1)=AT(n)B$ is stable if, and only if there exists an $M>0$ such that $\|\phi(n)\|\le M$ and $\|\psi(n)\|\le M$ for all $n\ge n_0$. In addition if $\|\phi(n)\|\to 0$ as $n\to\infty$ or $\|\psi(n)\|\to 0$ as $n\to\infty$, then the system $T(n+1)=AT(n)B$ is asymptotically stable.

P r o o f. First, we note that $\phi(n)=A^n$ and $\psi(n)=B^n$. Suppose there exists an $M>0$ such that $\|\phi(n)\|\le M$ and $\|\psi(n)\|\le M$ for all $n\ge n_0$. Then any solution of $T(n+1)=AT(n)B$ is of the form $T(n)=\phi(n)T(0)\psi(n)$. Let $\varepsilon>0$, then $\|T(0)\|\le$ $\le \dfrac{\varepsilon}{M^2}$ implies $\|T(n)\|<\varepsilon$ for all $n\ge n_0$. Hence the system is stable. Conversely, suppose the system is stable. Therefore, for a given $\varepsilon>0$, there exists a $\delta>0$ such that $\|T(0)\|\le\delta$ implies $\|T(n)\|\le\varepsilon$ for all $n\ge n_0$. This implies $\|\phi(n)T(0)\psi(n)\|\le\varepsilon$ for all $n\ge n_0$. A simple argument shows that $\|\phi(n)T(0)\psi(n)\|\le m_{ij}^2\delta<\varepsilon\left(\delta<\dfrac{\varepsilon}{\sum m_{il}^2}\right)$. This is true from all $i, j = 1, 2, ..., n$. Hence $\|T(n)\|\le\dfrac{2n\varepsilon}{\delta}=M$. Hence, the proof of the theorem is complete.

Further, if $\|\phi(n)\|\to 0$ as $n\to\infty$, and $\|\psi(n)\|\le M$ or $\|\phi(n)\|\le M$ and $\|\psi(n)\|\to 0$, then $\|T(n)\|\to 0$ as $n\to\infty$. Thus in both cases the system is asymptotically stable.

4. Sensitivity analysis. In the section, we present sensitivity analysis of the signal processing of the first order matrix difference system.

$$X(n+1)=AX(n)B+C(n)U(n). \tag{5}$$

The main reason for discussing sensitivity analysis is two-fold.

i) The mathematical models of several process are effected by measurement errors and for these reasons it is important to study the set of all solutions of admissible perturbations of the parameters.

ii) If a backward numerically stable algorithm is implemented to solve the general first order matrix difference system, then the computed solution will be exact solution of a slightly perturbed system. In fact, the sensitivity analysis gives a perturbation bound for the solution as a function of the perturbation in the signal processing data and also gives the estimate for the actual error in the computed solution.

Suppose the coefficient matrices in (5) are subject to small perturbation, say

$$A \to A^{+} = A + \delta A;\ B \to B^{+} = B + \delta B;\ C \to C^{+} = C + \delta C;\ U \to U^{+} = U + \delta U.$$

Now, the corresponding perturbed system is given by

$$(X+\delta X)(n+1) = (A+\delta A)(X+\delta X)(n)(B+\delta B) +$$
$$+ (C+\delta C)(n)(U+\delta U)(n)(D+\delta D)(n) =$$
$$= AX(n)B + C(n)U(n) + [A\delta X + \delta AX(n) + \delta A\delta X(n)](B+\delta B) +$$
$$+ C(n)\delta U(n)D(n) + \delta C(n)\delta U(n) =$$
$$= X(n+1) + (A+\delta A)\delta X(n)(B+\delta B) + C(n)\delta U(n) + \delta C(n)U(n).$$

Thus

$$\delta X(n+1) = (A+\delta A)\delta X(n)(B+\delta B) + C(n)\delta U(n) + \delta C(n)U(n) +$$
$$+ \delta C(n)\delta U(n) = (A+\delta A)\delta X(n)(B+\delta B) + C(n)\delta U(n) + \delta C(n)U(n).$$

Since the perturbations are small, we neglect the term $\delta C(n)\delta U(n)$

$$\delta X(n+1) = (A+\delta A)\delta X(n)(B+\delta B) + (C(n) + \delta C(n)\delta U(n)) + F(n, U(n)) =$$
$$= (A^{+}\delta X(n)B^{+} + C^{+}\delta U(n) + F(n, U(n)),$$

$$\delta X(n) = \phi(n, n_0)\delta X_0 \Psi^{*}(n, n_0) + \sum_{j=0}^{n-1} \phi(n_0, j+1)C^{+}(j)\delta U(j) +$$
$$+ \phi^{*}(n_0, j+1) + \sum_{j=0}^{\infty} F(j, U(j)).$$

If $\|\delta A\| = \varepsilon_1$, $\|\delta B\| = \varepsilon$ and F satisfies Lipschitz condition, then we have

$$\delta X(n) = L(\delta X)(n) = \phi(n, n_0)\delta X \psi^{*}(n, n_0) +$$
$$+ \sum_{j=0}^{n-1} \phi(n_0, j+1)C(j)\delta U(j)\psi^{*}(n_0, j+1) + \sum_{j=0}^{\infty} F(j, U(j)).$$

Let $\mu = \operatorname{Sup}_n \{\phi(n, n_0)\psi^{*}(n, n_0)\}$ and $\eta = \operatorname{Sup}_n \{\text{cond}(\phi(n, n_0))\,\text{cond}(\psi^{*}(n, n_0))\}$.

Then

$$\|L(\delta X)(n)\| \le \mu\|C\| + \eta\|C(j)\delta U(j)\| + \|F(j, U(j))\|.$$

Now, for any $\delta X(n)$ and $\delta Y(n)$, we have

$$\|L(\delta X)(n) - L(\delta Y)(n)\| \le k\|\delta X(n) - \delta Y(n)\|.$$

Thus L is a contraction map and hence by the Banach fixed point theorem the operator L has a unique fixed point and this fixed point is the solution of the perturbed control system.

5. Eigenvalue sensitivity and normal form transformation. In [3] the question is considered of eigenvalue sensitivity and normal form transformation to the difference system

$$x(n+1) = Ax(n) + Bu(n),$$

$$y(n) = Cx(n) + Du(n),$$

where $x(n)$, $u(n)$ and $y(n)$ are the state, input and output vectors, respectively. The above system is known as control of linear multivariate system.

In this section, we consider the more general systems (1) and (2) and analyze the sensitivity and normal form transformation by using eigenvalues and eigenvectors of the system (1). The main purpose in this section is to reduce the FWL effects via the sensitivity minimization approach, since the eigenvalues of an ideal digital filter may be perturbed under the FWL effects resulting in poor performances or even instability for actual system implementation. Further, sensitivity minimization becomes one of the effective schemes against the FWL effects of a state space realization.

It may be noted that if λ and $\overline{\lambda}$ are the eigenvalues of the matrix A and the perturbed matrix A^{+} and if μ and $\overline{\mu}$ are the corresponding eigenvalues of the matrix B and B^{+}, then (assuming that the perturbations are small) the system $X(n+1) = AX(n)B$ is said to be stable, if $|\lambda - \overline{\lambda}| < \varepsilon$ and $|\mu - \overline{\mu}| < \varepsilon$ for all sufficiently small $\varepsilon > 0$.

Theorem 3. Suppose the system matrices A and B in (5) are diagonalizable and let λ_k and μ_k be the k^{th} eigenvalues of A and B, respectively. Then

$$\frac{\delta|\lambda_k|}{\delta A} \cdot \frac{\delta|\mu_k|}{\delta B} = \frac{\operatorname{Re}(\lambda_k \mu_k u_k y_k v_k^H x_k^H)}{|\lambda_k \mu_k|},$$

where x_k and y_k are the k^{th} non-zero right and left eigenvectors of A and u_k, v_k are the non-zero right and left eigenvectors of B and v_k^H , x_k^H are the Hermetian transpose vectors of v_k and x_k, respectively.

In all practical purposes, all eigenvalues of a digital filter will be perturbed by the FWL effect and hence the sensitivity measure is considered as follows:

$$\Gamma = \left\| \sum_{k=1}^{n} \frac{\partial |\lambda_k|}{\partial A} \right\|_2, \quad \Gamma_1 = \left\| \sum_{k=1}^{n} \frac{\partial |\mu_k|}{\partial B} \right\|_2$$

and $\Gamma \le \|y\|_2 \|x\|_2$, $\Gamma_1 \le \|u\|_2 \|v\|_2$ where x and y are the sets of right and left eigenvalues of A, and U, and V are the sets of right and left eigenvectors of B.

For convenience, we define the supreme of Γ as $\overline{\Gamma}$ and supremem Γ_1 as $\overline{\Gamma}_1$. To tackle, the FWL effects of a digital filter and to find the corresponding optimal state-space realization can be specifically formulated as a minimization problem as $\Gamma_{\min} = \min_T \overline{\Gamma}$ and $\Gamma_{1\min} \le \min_{T_1} \overline{\Gamma}_1$, where T is the similarity transformation of A, and T_1 is the similarity transformation of B. If some optimality is reached by similarity transformations, then

$$A(\text{opt}) = T_{(\text{opt})}^{-1} A T_{(\text{opt})}, \quad D(\text{opt}) = T_{(\text{opt})}^{-1} D,$$

$$B(\text{opt}) = T_{1(\text{opt})}^{-1} B T_{1(\text{opt})}, \quad E(\text{opt}) = T_{(\text{opt})}^{-1} E,$$

$$C(\text{opt}) = T_{(\text{opt})}^{-1} C, \quad F(\text{opt}) = T_{(\text{opt})}^{-1} F$$

and the reciprocal right and left eigenvectors can also be expressed as

$$X(\text{opt}) = T_{(\text{opt})}^{-1} X \text{ and } Y_{(\text{opt})} = T_{(\text{opt})}^{1} y_1,$$

$$U(\text{opt}) = T_{(\text{opt})}^{-1} U \text{ and } V_{(\text{opt})} = T_{(\text{opt})}^{1} v.$$

Theorem 4. Assume X has norms with full rank, U has norms with full rank. Then the following hold

$$\|y\|_2 \|x\|_2 \le \left\| x^H y \right\|_2, \quad \|v\|_2 \|u\|_2 \le \left\| U^H V \right\|_2.$$

Theorem 5. Let the optimal similarity transformation for minimizing the $\overline{\Gamma}$ be expressed as $T_{(\text{opt})}$ and for $\overline{\Gamma}_1$ be expressed as $T_{1(\text{opt})}$. Then

$$T_{(\text{opt})} = (X\, \overline{X}^{-H})^{V_2} \alpha, \quad T_{1(\text{opt})} = (U\, \overline{U}^{-H})^{y_2} \beta,$$

where α and β are real orthogonal vectors. It may be noted that $X\overline{X}^{-H} = I$, and $U\overline{U}^{-H} = I$.

Theorem 6. If the system matrices A and B of the filter system described by (5) are transformed to $A_{(\text{opt})}$ and $B_{(\text{opt})}$, respectively, by using $T_{(\text{opt})}$ and $T_{1(\text{opt})}$, the optimal realizations of $A_{(\text{opt})}$ and $B_{(\text{opt})}$, respectively, are the normal form realizations, that is

$$\|A_{(\text{opt})}\|_2 = \rho(A_{(\text{opt})}), \quad \|B_{(\text{opt})}\|_2 = \rho(B_{(\text{opt})}),$$

where ρ denotes the spectral radius (the absolute value of all eigenvalues).

P r o o f.

$$\|A_{(\text{opt})}\|_2 = \sqrt{\lambda \max(A_{(\text{opt})}^H), A_{(\text{opt})}},$$

$$\|B_{(\text{opt})}\|_2 = \sqrt{\mu \max(B_{(\text{opt})}^H), B_{(\text{opt})}},$$

therefore

$$\|A_{(\mathrm{opt})}\|_2 = \max_k |\lambda_k| = \rho(A_{(\mathrm{opt})}),$$
$$\|B_{(\mathrm{opt})}\|_2 = \max_k |\mu_k| = \rho(B_{(\mathrm{opt})}).$$

Запропоновано загальний розв'язок різницевої системи матриць першого порядку $X(n+1) = AX(n)B + C(n)U(n)D(n)$. Досліджено стабільність і чутливість цифрових фільтрів за допомогою аналізу чутливості власного значення та перетворення нормальної форми.

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