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Fluctuation Analysis in a Queue with (L,N)-Policy and Secondary Maintenance. Discrete Time Parameter Process[2]

This paper generalizes numerous classes of queues with vacationing servers. In our model, a server not only leaves the system, but he services packets of jobs at a secondary facility up until the total number of single jobs exceeds a specific threshold. The strategy of server processes is represented for different state of queue. We use various techniques (including fluctuation analysis) to deliver explicit formulas for the queueing process with discrete time parameters. We also utilize some game-theoretic principles (namely sequential games) to efficiently construct our model.

Обобщены многочисленные классы очередей с простаивающими серверами. В предлагаемой модели сервер не просто выходит из системы, а обрабатывает пакеты задач в фоновом режиме до тех пор, пока общее число одиночных заданий не превысит определенный порог. Представлена стратегия работы сервера при различных состояниях очереди. Использованы различные методики получения явных формул (включая флуктуационный анализ) для процесса массового обслуживания с дискретным временем, а также некоторые подходы теории игр (последовательные игры) для эффективного конструирования модели.

K e y w o r d s: queueing, game theory, random walk analysis, fluctuation theory, marked point process, multiple vacations.

1. Introduction. In this article we consider a complex queueing model, in which a single server leaves an exhausted system to run a secondary work. Unlike traditional vacations queues, in our system the server works on real units and not until he is done with some minimum of them, does he return to the system again. In a nutshell the server enters a secondary facility (SF) with an unlimited quantity of jobs placed in packets of random sizes. The server begins processing them one by one and when the total number of jobs he processed reached a minimum of to L he is allowed return to the primary facility (PF).

Now, since the jobs are placed in packets, the server is not permitted to break any packet when he completes processing L jobs. Consequently, there will

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be an excess of L by the time his obligation at the SF is fulfilled. When the server returns to the PF thereafter, he is expected to see at least N customers waiting for service. If the line is shorter than that, the server returns to the SF and processes exactly one packet followed by its checking with the queue. If it still less than N, then a next packet will be processed and so on. By the time the queue in the PF accumulates to N or more units, the server upon the completion of a packet, returns to the PF, now to begin a busy period.

Consequently, the server vacates during two consecutive phases. By the end of the second phase (if it will occur), the server will be done with a number of jobs at the SF($\geq L$), and the number of customers waiting at the PF will be greater than or equal to N. Thus there will be three reference values: the number of processed jobs at the SF, the number of accumulated customers at the PF upon the end of server's absence, and the time of his absence. Using Kendall's symbolic, we will call our system the $M^X/G/S_L - P_N/1/\infty$ queue (with secondary and primary facilities and the associated exit thresholds L and N). Naturally, we will refer to this model as a queue with (L, N)-policy.

Our article will primarily focus on the evaluation of their joint distribution in the form of a transform using fluctuation analysis and noncooperative games. Once this transform is known to us, we proceed with the embedded queueing process upon departures. We however, postpone the continuous time parameter process (that requires time sensitive analysis) to a follow-up paper considering a limited space and our desire to focus on fluctuations.

As already mentioned, our model differs from common queues with vacationing server(s) (cf. [1]) in the sense that our server performs a background activity during his vacations expressed in terms of processed jobs. The protocol of the associated process will come to its full realization in our forthcoming article on continuous time parameter process with some optimization. We still calculate the number of fully processed jobs during a prolonged service cycle by using fluctuation analysis and a game-theoretic approach developed in a number of papers by the second author.

The game aspect of this model is integrated during the first two phases of an extended idle period of the server, during which not only does he work at a secondary facility, but he is commuting between the two facilities continuing servicing secondary jobs on the packet by packet bases. Consequently, the first phase of server's vacation is modeled by a game of two players A and B who attack each other at random times with strikes of random magnitudes. Each player can sustain only a certain amount of losses specified by two control levels L and N.

In earlier work by the second author [2], the game ended when one of the players made the other player's losses exceed an associated threshold of tolerance. In the present case, the first phase of the game lasts up until player A suffers losses in excess L of regardless of how player B has been doing. Upon the end of the this phase, if besides player A who was severely damaged, player B's losses also exceeded N, then the game is over. Otherwise, the game will continue until player B's losses will finally exceed N. During this period of time, called phase II, player A continues sustaining further damages. When the game is over two phases later, it is not clear who really won the game, but in this case it is not an issue and all we need is to know the exit time of the game and the total quantity of damages to each player. (By the way, should we worry about who is the winner and who is the loser, we can always figure it out using a reasonable criterion.)

As it is easy to guess, player A represents the secondary facility and player B — the primary facility and the associated losses are the amount of jobs fully processed at the SF and the number of customers accumulated at the PF. A somewhat similar approach was previously introduced in [3] by Al-Matar and Dshalalow, only [3], the server was waiting during the second phase instead of commuting back and forth between the SF and PF. Other related literature is on sequential games [4] and fluctuation analysis applied to finance [5], physics [6], and queueing [7].

The paper is laid out as follows. Sections 2 and 3 are preliminaries with formalism of our model and a general theorem from fluctuation analysis. Sections 4 and 5 deal with two phases of a model under the assumption that service times of jobs are exponentially distributed and the sizes of associated packs are geometric. Sections 6 and 7 consider two special cases with ordinary input and no N-policy, respectively. Section 8 concludes with Kendall's-like formula for the discrete time parameter process. In a follow-up paper we will treat the continuous time parameter queueing process using time sensitive analysis and semi-regenerative techniques, and obtain key performance measures such as the number of switchovers, potential of processed jobs, and the mean buffer load.

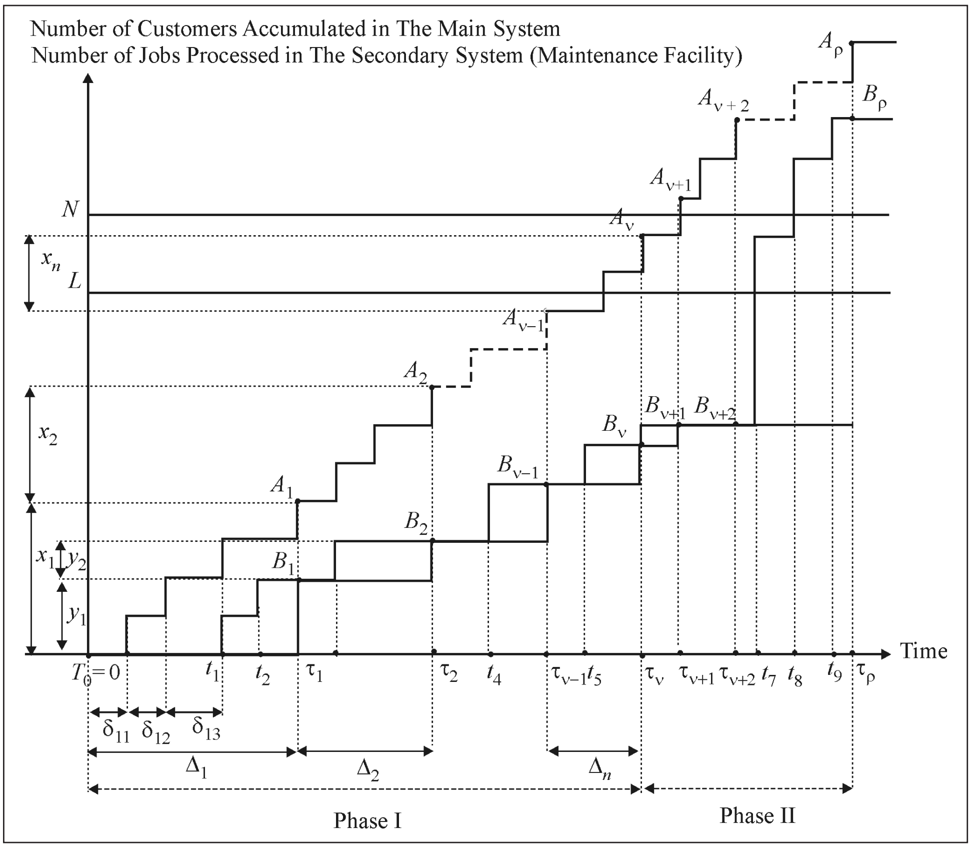
2. Formalism of the model. When the queue is exhausted, the server departs from the system and begins serving secondary jobs at the SF. We assume that upon its arrival at SF, there are unlimited quantities of jobs in packed in groups of random sizes. The ith job requires a random time of processing. We assume that their processing times are iid r.v.'s from an equivalence class $[\delta]$. Suppose the kth packet contains X_k jobs requiring service time Δ_k. Hence the time τ_n needed to process n packets is

$$\tau_n = \Delta_1 + \ldots + \Delta_n = \delta_{11} + \ldots + \delta_{X_1 1} + \ldots + \delta_{1n} + \ldots + \delta_{X_n n}, \tag{1}$$

where δ_{ij} is the processing time of the ith job from the jth packet. As previously mentioned,

$$\delta_{ij} \in [\delta], \ i, j = 1, 2, \ldots \tag{2}$$

and they are iid.



The server continues processing the packets of jobs up until it is done with L of them. Since server's work on any packet cannot be interrupted, it will continue processing the packet, even if the total number of jobs rendered is L . Consequently, the server's SF is finished if it completes a minimum of L jobs and not more than $L+X_{n+1}-1$ such that $X_1+...+X_n<L$ within the condition that only integer number of packets may be processed. After this, the server is called back to the system.

Upon its return, the server may or may not find waiting customers. In the former case, the server starts processing them immediately. Otherwise, it again resumes its work at the SF on the packet-by-packet basis. In other words, after it is done with a packet it checks if the queue in the PF is replenished with at least one customer, and if this is the case it unconditionally returns to the system. Otherwise, it keeps on working on the next pack and so on, up until the queue becomes positive.

We may say that the server enters the second phase of work at the SF after the total quantity of processed jobs hits L. The second phase resembles somewhat the system with multiple vacations, with the processing time of a packet being the time of a vacation segment.

Once the queue becomes positive for the first time upon server's completion of a packet, the server enters phase III during which it resumes its work at the PF and this phase is referred to the busy period.

We proceed with some more formalism. The total number of jobs processed with k packages is $A_k = \sum_{i=1}^{k} X_i$. The cumulative time needed to complete A_k jobs is $\tau_k = \sum_{i=1}^{k} \Delta_i$, where $\Delta_i = \sum_{j=1}^{X_i} \delta_{ij}$. Now, we introduce the so-called exit index I

$$\nu := \inf\{n : A_n \ge L\}. \tag{3}$$

Associated with ν are the following key r.v.'s (see Figure):

A_ν is the number of jobs (in excess of L) by the end of phase I;

B_ν is the number of customers in the buffer by the end of phase I;

τ_ν is the time of server's return at the end of phase I (exit time I).

We define B_ν as follows. Let y_{11} be the number of arriving customers in the interval $[T_0, T_0 + \delta_{11}]$, where $T_0 = 0$. We now consider all pertinent events from the trace σ-algebra $\mathcal{F} \cap \{Q_0 = 0\}$ from probability space $(\Omega, \mathcal{F}, P)$. Thus,

$$B_\nu = \underbrace{y_{11} + \ldots + y_{X_1,1}}_{Y_1} + \ldots + \underbrace{y_{1\nu} + \ldots + y_{X_\nu,\nu}}_{Y_\nu},$$

where in particular, $y_{X_1,1}$ is the number of arriving customers in interval $(\delta_{11} + \ldots + \delta_{X_1-1,1}, \delta_{11} + \ldots + \delta_{X_1 1}]$; Y_1 is the number of arriving customers in interval $[0, \Delta_1]$; Y_i is the number of arriving customers in interval $(\tau_{i-1}, \tau_i]$, $i = 1, 2, \ldots$. Let $\rho := \inf\{r : B_\nu > 0\}$ referred to as the exit index II. Then,

$\widetilde{A}_\rho$ is the number of jobs done at the SF by the end of phase II;

$\widetilde{B}_\rho$ is the number of customers in the buffer by the end of phase II;

$\widetilde{\tau}_\rho$ is the exit time II (from phase II).

In the beginning we will be concerned with the first exit functional

$$\Phi_\nu(u, z, \theta) = E u^{A_\nu} z^{B_\nu} e^{-\theta \tau_\nu}$$

and using this, we will find the second exit functional

$$\Phi_\rho(u, z, \theta) = E u^{\widetilde{A}_\rho} z^{\widetilde{B}_\rho} e^{-\theta \widetilde{\tau}_\rho}.$$

3. The random walk analysis of the system during phase I. As noticed in the Introduction, the first phase is specified by the trivariate «interdependent» process on a probability space $(\Omega, \mathcal{F}, P)$:

$$(\mathcal{A}, \mathcal{B}, \mathcal{T}) := \sum_{i=0}^{\infty} (X_i, Y_i) \varepsilon_{\tau_i} \tag{4}$$

(ε_α is the unit mass at α) is a bivariate marked point process with position dependent marking. We can also regards $(\mathcal{A}, \mathcal{B}, \mathcal{T})$ as a trivariate generalized random walk process on a three-dimensional random lattice. According to (3), τ_ν is the hitting time or exit time of the random walk from the set $[0, L) \times \mathbb{N} \times \mathbb{R}_+$.

The random walk $(\mathcal{A}, \mathcal{B}, \mathcal{T})$ can be specified by the following conditions adapted to our model:

(*i*) since the server departs from the system when the buffer is completely depleted, we have $B_0 = 0$;

(*ii*) without loss of generality, we consider the system on the first service cycle, which starts at $T_0 = 0$ and ends at $T_1 = \tilde{\tau}_\rho + \sigma_1$, where $\tilde{\tau}_\rho$ is the end of the second phase (if any) and σ_1 is the first service duration at the PF. Hence the server finds itself at the SF at $T_0 = \tau_0 = 0$;

(*iii*) the random walk $(\mathcal{A}, \mathcal{B}, \mathcal{T})$ corresponds to phase I and it is pertinent to the paths of the entire process on the trace σ-algebra $\mathcal{F} \cap \{Q_0 = 0\}$;

(*iv*) at time T_0, the number of processed jobs $A_0 = 0$.

Accordingly, the initial functional γ_0 is defined as

$$\gamma_0(u, z, \theta) = E[u^{A_0} z^{B_0} e^{-\theta\tau_0}] = 1. \tag{5}$$

From the construction of our model, $(\mathcal{A}, \mathcal{B}, \mathcal{T})$ is a bivariate marked renewal process with position dependent marking. The random vectors (X_1, Y_1, Δ_1), (X_2, Y_2, Δ_2), … are independent and identically distributed as a generic random vector (X, Y, Δ), so that (X_i, Y_i, Δ_i)'s belong to the equivalent class $[(X, Y, \Delta)]$. We are interested in the joint functional of (X, Y, Δ).

First we assume the knowledge of $\delta(\theta) = Ee^{-\theta\delta}$, $\hat{\delta} = E\delta < \infty$ (see (2)). Furthermore, suppose $G(u) = Eu^X$ with $\hat{X} = EX$, be a given pgf (probability generating function) of the r.v. X. Then, $\gamma(u, z, \theta) = E[u^X z^Y e^{-\theta\Delta}]$ can be expressed in terms of $\delta(\theta)$ and $G(u)$ as follows:

$$\gamma(u, z, \theta) = E[u^X z^Y e^{-\theta\Delta}] = E[u^{X_1} z^{Y_1} e^{-\theta\Delta_1}] =$$

$$= E[u^{X_1} E[z^{y_{11} + \ldots + y_{X_1,1}} e^{-\theta(\delta_{11} + \ldots + \delta_{1,X_1})} | X_1]] =$$

$$= E[E[u^{X_1} (Ez^{y_{11}} e^{-\theta\delta_{11}})^{X_1} | X_1] = E[u^{X_1} \delta(\theta + \lambda - \lambda a(z))^{X_1}] =$$

$$= G[u\delta(\theta + \lambda - \lambda a(z))]. \tag{6}$$

Introduce a transformation

$$D_p\{f(p)\}(x) := \sum_{p=0}^{\infty} x^p f(p)(1-x), \ \| x \| < 1, \tag{7}$$

where f is an integrable function defined on set $\mathbb{N}_0 = \{0, 1, \dots\}$. The inverse operator below can restore f, if we apply it for all k : $\mathcal{D}_x^k (D_p\{f(p)\}(x)) = f(k)$, $k = 0, 1, \dots$, where the inverse $\mathcal{D}^k$ is

$$k \mapsto \mathcal{D}_x^k \varphi(x, y) = \begin{cases} \lim\limits_{x\to 0} \dfrac{1}{k!} \dfrac{\partial^k}{\partial x^k} \left[\dfrac{1}{1-x} \varphi(x, y) \right], & k \geq 0, \\ 0, & k < 0 \end{cases} \tag{8}$$

if applied to a function $\varphi(x, y)$ analytic at zero in the first variable. Now, we will use the general result [8] for the extended functional.

Theorem 1. Let $(\mathcal{A}, \mathcal{B}, \mathcal{T})$ given in (4) be a bivariate marked point process with position dependent marking which starts at $\tau_0 = 0$ with $A_0 = B_0 = 0$ as the initial conditions, and the entire process on the trace σ-algebra $\mathcal{F} \cap \{Q_0 = 0\}$ ends at τ_ν when its active component $\mathcal{A}$ crosses the threshold L. Then the joint functional $\Phi_\nu(u, z, \theta)$ satisfies the following formula:

$$\Phi_\nu(u, z, \theta) = E[u^{A_\nu} z^{B_\nu} e^{-\theta\tau_\nu}] =$$

$$= \mathcal{D}_x^{L-1} \left\{ \gamma_0(u, z, \theta) - \gamma_0(ux, z, \theta) + \frac{\gamma_0(ux, z, \theta)}{1-\gamma(ux, z, \theta)} [\gamma(u, z, \theta) - \gamma(ux, z, \theta)] \right\},$$

where $\mathcal{D}$ (the inverse operator of (7)) is defined in (8).

Considering the initial functional $\gamma_0(u, z, \theta) = E[u^{A_0} z^{B_0} e^{-\theta\tau_0}] = 1$ of (5) we arrive at

$$\Phi_\nu(u, z, \theta) := E[u^{A_\nu} z^{B_\nu} e^{-\theta\tau_\nu}] =$$

$$= 1 - 1 + \mathcal{D}_x^{L-1} \left\{ \frac{1}{1-\gamma(ux, z, \theta)} [\gamma(u, z, \theta) - \gamma(ux, z, \theta)] \right\} =$$

$$= \mathcal{D}_x^{L-1} \left\{ \frac{\gamma(u, z, \theta) - \gamma(ux, z, \theta)}{1-\gamma(ux, z, \theta)} \right\} = 1 - [1 - \gamma(u, z, \theta)] \mathcal{D}_x^{L-1} \left\{ \frac{1}{1-\gamma(ux, z, \theta)} \right\}$$

followed by Theorem with the implementation of (6).

Theorem 2. In light of Theorem 1 and under the specification in (6), the joint functional $\Phi_\nu(u, z, \theta) = E[u^{A_\nu} z^{B_\nu} e^{-\theta\tau_\nu}]$ satisfies the following formula:

$$\Phi_\nu(u, z, \theta) := E[u^{A_\nu} z^{B_\nu} e^{-\theta\tau_\nu}] =$$

$$= 1 - [1 - G[u\delta(\theta + \lambda - \lambda a(z))]] \mathcal{D}_x^{L-1} \left\{ \frac{1}{1 - G[xu\delta(\theta + \lambda - \lambda a(z))]} \right\}. \tag{9}$$

4. Special case of phase I. The number of processed jobs. We will consider the following special case:

(i) δ is exponential with parameter $d > 0$, i.e.

$$\delta(\theta) = Ee^{-\theta\delta} = \frac{d}{d+\theta}; \tag{10}$$

(ii) The number of jobs in a packet is geometric with parameter p, i.e.

$$G(u) = Eu^X = \frac{pu}{1-qu}. \tag{11}$$

We will use assumptions (i-ii) in light of formula (8) for $\Phi_\nu(u,z,\theta)$. Firstly, from (6) we have:

$$\gamma(u,z,\theta) = G[u\delta(\theta+\lambda-\lambda a(z))] =$$

$$= \frac{pu\delta(\theta+\lambda-\lambda a(z))}{1-qu\delta(\theta+\lambda-\lambda a(z))} = \frac{pdu}{d+\theta+\lambda-\lambda a(z)-qdu}. \tag{12}$$

Then, from (12),

$$1-G[u\delta(\theta+\lambda-\lambda a(z))] = \frac{d+\theta+\lambda-\lambda a(z)-du}{d+\theta+\lambda-\lambda a(z)-qdu}, \tag{13}$$

and finally, from (13),

$$\frac{1}{1-G[u\delta(\theta+\lambda-\lambda a(z))]} = q + p\frac{1}{1-\dfrac{d}{d+\theta+\lambda-\lambda a(z)}u}. \tag{14}$$

From (8) and (14),

$$\mathcal{D}_x^{L-1}\left\{\frac{1}{1-G[xu\delta(\theta+\lambda-\lambda a(z))]}\right\} = \mathcal{D}_x^{L-1}\left\{q + p\frac{1}{1-\dfrac{d}{d+\theta+\lambda-\lambda a(z)}xu}\right\}.$$

Using the property of $\mathcal{D}$ as a linear operator having fixed points at all constant functions, and that

$$\mathcal{D}_x^k\left\{\frac{1}{1-bx}\right\} = \frac{1-b^{k+1}}{1-b}$$

(see Section 9, (40)) we have

$$\mathcal{D}_x^{L-1}\left\{q + p\frac{1}{1-\dfrac{d}{d+\theta+\lambda-\lambda a(z)}xu}\right\} = q + p\frac{1-\left(\dfrac{du}{d+\theta+\lambda-\lambda a(z)}\right)^L}{1-\dfrac{d}{d+\theta+\lambda-\lambda a(z)}u}. \tag{15}$$

Now, returning to formula (8), along with (13) and (15), we have

$$\Phi_\nu(u,z,\theta)=1-\frac{d+\theta+\lambda-\lambda a(z)-ud}{d+\theta+\lambda-\lambda a(z)-udq}\left[q+p\frac{1-\left(\frac{du}{d+\theta+\lambda-\lambda a(z)}\right)^L}{1-\frac{d}{d+\theta+\lambda-\lambda a(z)}u}\right]=$$

$$=\frac{p(du)^L}{(d+\theta+\lambda-\lambda a(z)-dqu)(d+\theta+\lambda-\lambda a(z))^{L-1}}. \tag{16}$$

For the sequel, we need the marginal functionals of $\Phi_\nu(u,z,\theta)=E[u^{A_\nu}z^{B_\nu}e^{-\theta\tau_\nu}]$ under assumptions ($i-ii$). We start with the amount of jobs A_ν processed by the end of phase I. From (16) we have

$$\Phi_\nu(u,1,\theta)=Eu^{A_\nu}=\frac{p(du)^L}{(d+\theta+\lambda-\lambda a(1)-dqu)(d+\theta+\lambda-\lambda a(1))^{L-1}}$$

which simplifies to

$$Eu^{A_\nu}=\frac{pu}{1-qu}u^{L-1}.$$

This compact expression consists of two parts: $\frac{pu}{1-qu}$ and u^{L-1}. The first factor is the pgf of the geometrically distributed packet size, while the second factor is the pgf of a constant packet of size $L-1$. It stands for reason to interpret the result as follows. After the total number of processed jobs reached $L-1$, the server takes one more packet of random size to complete the first phase.

The expected number of the processed jobs in phase I will therefore be $Eu^{A_\nu}=\frac{1}{p}+L-1.$

5. The random walk analysis of the system during phase II. When the server completes its first phase at the SF, it returns to the system and begins servicing all customers one by one if there are N or more customers lined up and waiting. Otherwise it keeps on commuting to the SF working on one-by-one packed basis up until on one of its returns from the SF he finds N or more customers. Since we do not alter our assumptions on server's policy at the SF during phase II, we continue to operate with the same functional

$$\gamma(u,z,\theta)=E[u^X z^Y e^{-\theta\Delta}], \tag{17}$$

where X stands for a packet size, Δ — the time needed to process all jobs included in that packet, and Y is the number of customers entering the queue during time Δ. (Recall that $[X, Y, \Delta]$ is the equivalence class of all such random vectors).

In this case, phase II with the initial status of the system at the beginning of phase II is the exit status of the system from phase I. Namely,

$$\widetilde{A}_0 = \widetilde{X}_0 = A_\nu, \tag{18}$$

$$\widetilde{B}_0 = \widetilde{Y}_0 = B_\nu, \tag{19}$$

and

$$\widetilde{\tau}_0 = \widetilde{\Delta}_0 = \tau_\nu. \tag{20}$$

Altogether, we have the new delayed random walk process

$$(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{T}}) := \sum_{i=0}^{\infty} (\widetilde{X}_i, \widetilde{Y}_i)\, \varepsilon_{\widetilde{\tau}_i}, \tag{21}$$

with the initial conditions specified by (18) — (20) and increments $(\widetilde{X}_i, \widetilde{Y}_i, \widetilde{\Delta}_i) \in$ $\in [X, Y, \Delta]$, $i = 1, 2, \ldots$ distributed as those during phase I starting with $i = 1, 2, \ldots$ and with the associated functional (17). As far as the initial functional

$$E[u^{\widetilde{X}_0} z^{\widetilde{Y}_0} e^{-\theta \widetilde{\Delta}_0}] =: \Gamma_0(u, z, \theta),$$

we equate it to $\Phi_\nu(u, z, \theta)$ of (9) or its more particular version (16). The random walk $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{T}})$ of (21) will now be terminated once $\widetilde{Y}_0 + \ldots + \widetilde{Y}_n$ exceeds $N - 1$ for some n to occur at some $\widetilde{\tau}_n$. Again we define the second exit index $\rho = \min\{n : \widetilde{B}_n = \widetilde{Y}_0 + \ldots + \widetilde{Y}_n \geq N\}$. The functional of our further interest will be

$$\Psi_\rho(u, z, \theta) := E[u^{\widetilde{A}_\rho} z^{\widetilde{B}_\rho} e^{-\theta \widetilde{\tau}_\rho}], \tag{22}$$

where $\Psi_\rho(u, z, \theta)$ satisfies the formula (cf. [8])

$$\Psi_\rho(u, z, \theta) = \Gamma_0(u, z, \theta) - [1 - \Gamma(u, z, \theta)]\, \mathcal{D}_y^{N-1} \left\{ \frac{\Gamma_0(u, yz, \theta)}{1 - \Gamma(u, yz, \theta)} \right\}, \tag{23}$$

where as already mentioned

$$\Gamma_0(u, z, \theta) = \Phi_\nu(u, z, \theta) \tag{24}$$

and

$$\Gamma(u, z, \theta) = \gamma(u, z, \theta) = G[u\delta(\theta + \lambda - \lambda a(z))]. \tag{25}$$

6. Special case I. The ordinary Poisson input. This model will be treated under the assumptions made in (10), (11) and in addition, we assume that input is ordinary. In other words, we set $a(z) = z$. For consistency, we list again all three assumptions:

(*i*) $\delta(\theta) = Ee^{-\theta\delta} = \dfrac{d}{d + \theta}$;

$(ii)\; G(u) = Eu^X = \dfrac{pu}{1-qu}$;

$(iii)\; a(z) = z.$

First we will use assumptions (*i- ii*) in light of formula (16):

$$\Gamma_0(u,z,\theta) = \Phi_\nu(u,z,\theta) = \frac{p(du)^L}{(d+\theta+\lambda-\lambda a(z)-dqu)(d+\theta+\lambda-\lambda a(z))^{L-1}},$$

$$\Gamma(u,z,\theta) = \gamma(u,z,\theta) = \frac{pdu}{d+\theta+\lambda-\lambda a(z)-dqu}.$$

Furthermore,

$$1-\Gamma(u,z,\theta) = \frac{d+\theta+\lambda-\lambda a(z)-du}{d+\theta+\lambda-\lambda a(z)-dqu},$$

$$\frac{\Gamma_0(u,z,\theta)}{1-\Gamma(u,z,\theta)} = \frac{p(du)^L}{(d+\theta+\lambda-\lambda a(z)-du)(d+\theta+\lambda-\lambda a(z))^{L-1}}. \tag{26}$$

To continue with the main part of (22), namely

$$\mathcal{D}_y^{N-1}\left\{\frac{\Gamma_0(u,yz,\theta)}{1-\Gamma(u,yz,\theta)}\right\}$$

(with (26)), we now add our third assumption that $a(z) = z$. Then (23) — (25) will read

$$\Psi_\rho(u,z,\theta) = \Gamma_0(u,z,\theta) - [1-\Gamma(u,z,\theta)]\,\mathcal{D}_y^{N-1}\left\{\frac{\Gamma_0(u,yz,\theta)}{1-\Gamma(u,yz,\theta)}\right\} =$$

$$= \frac{p(du)^L}{(d+\theta+\lambda-\lambda z-dqu)(d+\theta+\lambda-\lambda z)^{L-1}} -$$

$$-\frac{d+\theta+\lambda-\lambda z-du}{d+\theta+\lambda-\lambda z-dqu}\,\mathcal{D}_y^{N-1}\left\{\frac{p(du)^L}{(d+\theta+\lambda-\lambda zy-du)(d+\theta+\lambda-\lambda zy)^{L-1}}\right\}. \tag{27}$$

Formula (27) will be modified to tame the use of operator $\mathcal{D}^k$:

$$\Psi_\rho(u,z,\theta) = \frac{p(du)^L}{(d+\theta+\lambda-\lambda z-dqu)(d+\theta+\lambda-\lambda z)^{L-1}} - \frac{d+\theta+\lambda-\lambda z-du}{d+\theta+\lambda-\lambda z-dqu}\times$$

$$\times \mathcal{D}_y^{N-1}\left\{ \frac{p(du)^L}{(d+\theta+\lambda-du)\left(1-\frac{\lambda zy}{d+\theta+\lambda-du}\right)} \frac{1}{(d+\theta+\lambda)^{L-1}\left(1-\frac{\lambda zy}{d+\theta+\lambda}\right)^{L-1}} \right\}. \quad (28)$$

Denote

$$\Psi(y) := \frac{p(du)^L}{(d+\theta+\lambda-du)\left(1-\frac{\lambda zy}{d+\theta+\lambda-du}\right)} \frac{1}{(d+\theta+\lambda)^{L-1}\left(1-\frac{\lambda zy}{d+\theta+\lambda}\right)^{L-1}}.$$

To find $\mathcal{D}_y^{N-1}\{\Psi(y)\}$ we will use formulas (40) and (41) from section 9 for the inverse operator $\mathcal{D}^k$.

Case 1: $L = 1$

$$\Psi(y) = \frac{pdu}{(d+\theta+\lambda-du)\left(1-\frac{\lambda zy}{d+\theta+\lambda-du}\right)},$$

$$\mathcal{D}_y^{N-1}\{\Psi(y)\} = \frac{pdu}{(d+\theta+\lambda-du)} \frac{1-\left(\frac{\lambda z}{d+\theta+\lambda-du}\right)^N}{1-\frac{\lambda z}{d+\theta+\lambda-du}} =$$

$$= \frac{pdu}{d+\theta+\lambda-\lambda z-du}\left[1-\left(\frac{\lambda z}{d+\theta+\lambda-du}\right)^N\right].$$

Returning to (28) in light of formula (40) from section 9 we have

$$\Psi_\rho(u,z,\theta) = \frac{pdu}{d+\theta+\lambda-\lambda z-dqu} - \frac{d+\theta+\lambda-\lambda z-du}{d+\theta+\lambda-\lambda z-dqu} \times$$

$$\times\left(\frac{pdu}{d+\theta+\lambda-\lambda z-du}\left[1-\left(\frac{\lambda z}{d+\theta+\lambda-du}\right)^N\right]\right) =$$

$$= \frac{pdu}{d+\theta+\lambda-\lambda z-dqu} + \left(\frac{\lambda z}{d+\theta+\lambda-du}\right)^N. \quad (29)$$

Case 2: $L \geq 2$

$$\Psi(y) := \frac{p(du)^L}{(d+\theta+\lambda-du)\left(1-\frac{\lambda zy}{d+\theta+\lambda-du}\right)} \frac{1}{(d+\theta+\lambda)^{L-1}\left(1-\frac{\lambda zy}{d+\theta+\lambda}\right)^{L-1}},$$

$$\mathcal{D}_y^{N-1}\{\Psi(y)\}=\frac{p(du)^L}{(d+\theta+\lambda-du)(d+\theta+\lambda)^{L-1}}\frac{1}{1-\dfrac{\lambda z}{d+\theta+\lambda-du}},$$

$$\sum_{j=0}^{N-1}\binom{L+j-2}{j}\left(\left(\frac{\lambda z}{d+\theta+\lambda}\right)^j-\left(\frac{\lambda z}{d+\theta+\lambda-du}\right)^N\left(\frac{d+\theta+\lambda-du}{d+\theta+\lambda}\right)^j\right)=$$

$$=\frac{p(du)^L}{(d+\theta+\lambda-\lambda z-du)(d+\theta+\lambda)^{L-1}},$$

$$\sum_{j=0}^{N-1}\binom{L+j-2}{j}\left(\left(\frac{\lambda z}{d+\theta+\lambda}\right)^j-\left(\frac{\lambda z}{d+\theta+\lambda-du}\right)^N\left(\frac{d+\theta+\lambda-du}{d+\theta+\lambda}\right)^j\right).$$

Returning to (28) in light of formula (41) from the Section 9 we have

$$\Psi_\rho(u,z,\theta)=E[u^{\widetilde{A}_\rho}z^{\widetilde{B}_\rho}e^{-\theta\widetilde{\tau}_\rho}]=\frac{p(du)^L}{(d+\theta+\lambda-\lambda z-dqu)(d+\theta+\lambda-\lambda z)^{L-1}}-$$

$$-\frac{d+\theta+\lambda-\lambda z-du}{d+\theta+\lambda-\lambda z-dqu}\left[\frac{p(du)^L}{(d+\theta+\lambda-\lambda z-du)(d+\theta+\lambda)^{L-1}}\times\right.$$

$$\left.\times\sum_{j=0}^{N-1}\binom{L+j-2}{j}\left(\left(\frac{\lambda z}{d+\theta+\lambda}\right)^j-\left(\frac{\lambda z}{d+\theta+\lambda-du}\right)^N\left(\frac{d+\theta+\lambda-du}{d+\theta+\lambda}\right)^j\right)\right]=$$

$$=\frac{p(du)^L}{d+\theta+\lambda-\lambda z-dqu}\left[\frac{1}{(d+\theta+\lambda-\lambda z)^{L-1}}-\frac{1}{(d+\theta+\lambda)^{L-1}}\times\right.$$

$$\left.\times\sum_{j=0}^{N-1}\binom{L+j-2}{j}\left(\left(\frac{\lambda z}{d+\theta+\lambda}\right)^j-\left(\frac{\lambda z}{d+\theta+\lambda-du}\right)^N\left(\frac{d+\theta+\lambda-du}{d+\theta+\lambda}\right)^j\right)\right].\qquad(30)$$

Theorem 3 . In the $M^X/G/S_L-P_N/1/\infty$ queue (with secondary and primary facilities and the associated exit thresholds L and N) under the assumptions (*i-iii*), the joint functional $\Psi_\rho(u,z,\theta)=E[u^{\widetilde{A}_\rho}z^{\widetilde{B}_\rho}e^{-\theta\widetilde{\tau}_\rho}]$ of the exit time $\widetilde{\tau}_\rho$ from phase II, the number of jobs $\widetilde{A}_\rho$, and the queue length $\widetilde{B}_\rho$ at time $\widetilde{\tau}_\rho$ satisfies formulas (29) and (30).

7. Special case II. No N-policy. In this section, we are going to investigate another special case of Ψ_ρ under the assumption that $N=1$. We however, retain the generality of the input stream. Here again we list all three assumptions pertinent for this section:

(i) $\delta(\theta)=Ee^{-\theta\delta}=\dfrac{d}{d+\theta}$;

(*ii*) $G(u)=Eu^X=\dfrac{pu}{1-qu}$;

(*iii*) $N=1$.

From (23)—(25) we have

$$\Psi_\rho(u,z,\theta)=E[u^{\widetilde{A}_\rho}z^{\widetilde{B}_\rho}e^{-\theta\widetilde{\tau}_\rho}]=\Gamma_0(u,z,\theta)-[1-\Gamma(u,z,\theta)]\left\{\frac{\Gamma_0(u,0,\theta)}{1-\Gamma(u,0,\theta)}\right\}=$$

$$=\frac{p(d+\theta+\lambda-\lambda a(z))\dfrac{(du)^L}{(d+\theta+\lambda-\lambda a(z))^L}}{d+\theta+\lambda-\lambda a(z)-duq}-$$

$$-\frac{d+\theta+\lambda-\lambda a(z)-du}{d+\theta+\lambda-\lambda a(z)-duq}\frac{p(d+\theta+\lambda)\left(\dfrac{du}{d+\theta+\lambda}\right)^L}{d+\theta+\lambda-duq}\left(q+p\frac{1}{1-\dfrac{du}{d+\theta+\lambda}}\right). \tag{31}$$

Theorem 4. In the $M^X/G/S_L-P_N/1/\infty$ queue (with secondary and primary facilities and the associated exit thresholds L and N) under the assumptions (*i-iii*), the joint functional $\Psi_\rho(u,z,\theta)=E[u^{\widetilde{A}_\rho}z^{\widetilde{B}_\rho}e^{-\theta\widetilde{\tau}_\rho}]$ of the exit time $\widetilde{\tau}_\rho$ from phase II, the number of jobs $\widetilde{A}_\rho$, and the queue length $\widetilde{B}_\rho$ at time $\widetilde{\tau}_\rho$ satisfies formula (31).

Remark. If in this special case we assume that L is a random variable with a pgf $H(z)=Ez^L$, then we can interpret (31) as the conditional expectation

$$\Psi_\rho(u,z,\theta;L)=E[u^{\widetilde{A}_\rho}z^{\widetilde{B}_\rho}e^{-\theta\widetilde{\tau}_\rho}|L].$$

Consequently, the functional $\Psi_\rho(u,z,\theta)$ will be «recovered» from

$$E[\Psi_\rho(u,z,\theta;L)=E[E[u^{\widetilde{A}_\rho}z^{\widetilde{B}_\rho}e^{-\theta\widetilde{\tau}_\rho}|L]]=$$

$$=\frac{p(d+\theta+\lambda-\lambda a(z))H\left(\dfrac{du}{d+\theta+\lambda-\lambda a(z)}\right)}{d+\theta+\lambda-\lambda a(z)-duq}-$$

$$-\frac{d+\theta+\lambda-\lambda a(z)-du}{d+\theta+\lambda-\lambda a(z)-duq}\frac{p(d+\theta+\lambda)H\left(\dfrac{du}{d+\theta+\lambda}\right)}{d+\theta+\lambda-duq}\left(q+p\frac{1}{1-\dfrac{du}{d+\theta+\lambda}}\right).$$

8. The embedded queueing process. In this section we turn to the queueing process, actually regarding all previous sections as preliminaries. We start with its formal description:

(i) the input to the system is bulk Poisson $\mathcal{I} = \sum_{i=1}^{\infty} U_i \varepsilon_{t_i}$ with position independent marking, $a(z) := Ez^{U_i}$, $a := EU_i, < \infty$, $i \geq 1$, and intensity λ of its point process $\sum_{i=1}^{\infty} \varepsilon_{t_i}$;

(ii) service is general and independent including independence of the input, i.e. service times $\sigma_1, \sigma_2, \ldots$ are iid r. v. with a common LST $\beta(\theta) = Ee^{-\sigma_1}$, $\mathrm{Re}(\theta) \geq 0$ and $b := E\sigma_1 < \infty$;

(iii) $Q(t)$ is the right continuous queueing process (the number of customers at time $t \geq 0$);

(iv) T_0, T_1, T_2, ... are successive departures of the individually processed units;

(v) $\{Q_n := Q(T_n); n = 0, 1, \ldots\}$ is the embedded process upon departures. Since our system is of $M^X/G/1$ type, the queueing process $\{Q(t)\}$ is semi-regenerative relative to the sequence $\{T_n\}$ of stopping times and the associated embedded process $\{Q_n\}$ is a Markov chain;

(vi) the transition probability matrix $P = (p_{ij})$ of $\{Q_n\}$ is a delta-2 matrix (cf. [3]) similar to that of $M^X/G/1/\infty$ queue, with only zero row different, with no impact on common the necessary and sufficient ergodicity condition $\rho := \lambda ab < 1$. The zero row contains the transition probabilities over the two-phase secondary with waiting and service period (zero-service cycle), so that the queue length upon the end of such cycle is $\Psi_\rho(z, \theta) := E[z^{\tilde{B}_\rho} e^{-\theta \tilde{\tau}_\rho}]$, and its pgf $P_0(z)$, is readily seen to be: $P_0(z) := E[z^{Q_1} | Q_0 = 0] = \beta(\lambda - \lambda a(z))\, \alpha(z)\, z^{-1}$, where $\alpha(z) := Ez^{\tilde{B}_\rho}$ with $\alpha(z)$ being of one of the two types treated in Sections 6 and 7;

(vii) consequently, if Kendall's (or Pollaczek-Khinchine) formula in the $M^X/G/1/\infty$ system is

$$P(z) = p_0 \beta(\lambda - \lambda a(z)) \frac{a(z) - 1}{z - \beta(\lambda - \lambda a(z))},$$

(cf. [9]) all we need to do is to replace $a(z)$ with $\alpha(z) := Ez^{\tilde{B}_\rho}$.

Under the special case of the ordinary input (Section 6), the joint functional of (29) will be used to find the marginal functional for $\alpha(z)$. For the case of $L = 1$,

$$\alpha(z) = \Psi_\rho(1, z, 0) = \frac{pd}{\lambda - \lambda z + pd} + z^N \tag{32}$$

and for the case of $L \geq 2$, we will use the joint functional formula in (30):

$$\alpha(z) = \Psi_\rho(1, z, 0) = \frac{pd^L}{(pd + \lambda - \lambda z)(d + \lambda - \lambda z)^{L-1}} - \frac{pd}{pd + \lambda - \lambda z}\left(\frac{d}{d+\lambda}\right)^{L-1} \times$$

$$\times \sum_{j=0}^{N-1} \binom{L+j-2}{j} \left(\frac{\lambda}{d+\lambda}\right)^j (z^j + z^N). \tag{33}$$

From Section 7 for the case of $N = 1$, we will get the marginal functional from the joint functional (31),

$$\alpha(z) = \Psi_\rho(1, z, 0) = \frac{pd^L}{(pd + \lambda - \lambda a(z))(d + \lambda - \lambda a(z))^{L-1}} -$$

$$- \frac{pd(1 - a(z))}{pd + \lambda - \lambda a(z)}\left(\frac{d}{d+\lambda}\right)^{L-1} \tag{34}$$

to get

$$P(z) = p_0 \beta(\lambda - \lambda a(z) \frac{\alpha(z) - 1}{z - \beta(\lambda - \lambda a(z))}.$$

Here $P(z)$ is the pgf of the embedded queueing process in equilibrium. Also, in Kendall's formula, $p_0 = (1-\rho)/a$. So, we also replace a with $\alpha = E\tilde{B}_\rho$. For the case of $L = 1$ when $a(z) = z$, we will continue from formula (32):

$$\alpha = \lim_{z \to 1} \frac{\partial \alpha(z)}{\partial z} = \frac{\lambda}{pd} + N \tag{35}$$

and for the case of $L \geq 2$, we will use formula in (33);

$$\alpha = \frac{\lambda}{d}\left(\frac{1}{p} + L - 1\right) + \left(\frac{d}{d+\lambda}\right)^{L-1} \sum_{j=0}^{N-1} \binom{L+j-2}{j} \left(\frac{\lambda}{d+\lambda}\right)^j (N - j). \tag{36}$$

For the case of $N = 1$ in Section 7, we will use formula (34):

$$\alpha = a\frac{\lambda}{d}\left(\frac{1}{p} + L - 1\right) + a\left(\frac{d}{d+\lambda}\right)^{L-1}. \tag{37}$$

We can summarize the above as

Theorem 5. The embedded queueing process $\{Q_n\}$ in the $M^X/G/1/\infty$ type queue with a two-phase is ergodic if and only if $\rho<1$. Under this condition the pgf of the invariant probability measure $\mathbf{p}=(p_0, p_1, ...)$ of the transition probability matrix P satisfies the Kendall like formula

$$P(z)=\frac{1-\rho}{\alpha}\beta(\lambda-\lambda a(z))\frac{\alpha(z)-1}{z-\beta(\lambda-\lambda a(z))},$$

where $\alpha(z)$ and α, under the assumptions of (*i-iii*) of Sections 6 and 7, satisfy formulas (32)—(34) and (35)—(37).

9. Properties of operator $\mathcal{D}^k$. ***Proposition 1.*** Let g be an analytic function at zero. Then, it holds true that

$$\mathcal{D}_x^k(x^j g(x))=\mathcal{D}_x^{k-j}(g(x)). \qquad (38)$$

P r o o f. Indeed, first we use the Leibnitz formula

$$\frac{d^k}{dx^k}(F(x)G(x))=\sum_{s=0}^{k}\binom{k}{s}(F(x))^{(s)}G^{(k-s)}(x)$$

and set $F(x)=x^j$ and $G(x)=\dfrac{g(x)}{1-x}$. Hence when applying $\mathcal{D}^k$ we have

$$\mathcal{D}_x^k(x^j g(x))=\frac{1}{k!}\sum_{s=0}^{k}\binom{k}{s}\frac{d^s}{dx^s}(x^j)\bigg|_{x=0}(k-s)!\mathcal{D}_x^{k-1}(g(x)).$$

□

Let $b(x)=\sum_{i=0}^{\infty}b_i x^i$. Then, using Proposition 1 (38) with $g(x)=1$ for all x we have

$$\mathcal{D}_x^k(b(x))=\sum_{i=0}^{k}b_i. \qquad (39)$$

Now using Taylor series expansion of $\dfrac{1}{1-bx}$ and formula (39) we have

$$\mathcal{D}_x^k\left\{\frac{1}{1-bx}\right\}=\begin{cases}\dfrac{1-b^{k+1}}{1-b}, & b\neq 1,\\ k+1, & b=1.\end{cases} \qquad (40)$$

Proposition 2. Generalization of (40). For any real number a and for a positive integer n, it holds true that

$$\mathcal{D}_x^k\left\{\frac{1}{(1-ax)^n}\right\}=\begin{cases}\sum_{j=0}^{k}\binom{n+j-1}{j}a^j, & \text{except for } a=n=1,\\ k+1, & a=n=1.\end{cases}$$

P r o o f. Using the binomial expansion for an integer α (not necessarily positive) we have

$$(1+z)^\alpha=1+\alpha z+\frac{\alpha(\alpha-1)z^2}{2!}+\ldots+\frac{\alpha\ldots(\alpha-j+1)z^j}{j!}+\ldots,$$

which converges in the open unit ball $B(0, 1)$. Thus,

$$(1-ax)^{-n}=1-n(-ax)+\frac{(-n)(-n-1)(-ax)^2}{2!}+\ldots$$

$$\ldots+(-1)^j\frac{(-n)\ldots(-n-j+1)(ax)^j}{j!}+\ldots.$$

In summary,

$$\left(\frac{1}{1-ax}\right)^n=\sum_{j=0}^{\infty}\binom{n+j-1}{j}(ax)^j.$$

The statement follows after we use formula (39).

Proposition 3. For two real numbers a and b it holds true that

$$\mathcal{D}_x^k\left\{\frac{1}{1-bx}\frac{1}{(1-ax)^n}\right\}=\begin{cases}\frac{1}{1-b}\sum_{j=0}^{k}\binom{n+j-1}{j}\left(a^j-b^{k+1}\left(\frac{a}{b}\right)^j\right), & b\neq 1,\\ =\sum_{j=0}^{k}\binom{n+j-1}{j}a^j(k-j+1), & b=1.\end{cases} \tag{41}$$

P r o o f. Expanding $\frac{1}{1-bx}$ in Taylor series we have

$$\mathcal{D}_x^k\left\{\frac{f(x)}{1-bx}\right\}=\mathcal{D}_x^k\left\{\sum_{i=0}^{\infty}b^ix^if(x)\right\}$$

interchanging the operator and the series, and then using Proposition 1

$$\mathcal{D}_x^k\left\{\frac{f(x)}{1-bx}\right\}=\sum_{i=0}^{k} b^i\,\mathcal{D}_x^{k-i}\{f(x)\}$$

with $f(x)=\left(\dfrac{1}{1-ax}\right)^n$

$$\mathcal{D}_x^k\left\{\frac{f(x)}{1-bx}\right\}=\sum_{i=0}^{k} b^i\;\mathcal{D}_x^{k-i}\left\{\left(\frac{1}{1-ax}\right)^n\right\}.$$

From Proposition 2

$$\mathcal{D}_x^k\left\{\frac{f(x)}{1-bx}\right\}=\sum_{i=0}^{k} b^i\sum_{j=0}^{k-i}\binom{n+j-1}{j}a^j$$

interchanging the sums

$$\mathcal{D}_x^k\left\{\frac{f(x)}{1-bx}\right\}=\sum_{j=0}^{k}\binom{n+j-1}{j}a^j\sum_{i=0}^{k-j} b^i=\sum_{j=0}^{k}\binom{n+j-1}{j}a^j\,\frac{1-b^{k-j+1}}{1-b}=$$

$$=\frac{1}{1-b}\sum_{j=0}^{k}\binom{n+j-1}{j}\left(a^j-b^{k+1}\left(\frac{a}{b}\right)^j\right),\;\; b\neq 1.$$

For $b=1$, it equals

$$\mathcal{D}_x^k\left\{\frac{f(x)}{1-x}\right\}=\sum_{j=0}^{k}\binom{n+j-1}{j}a^j(k-j+1).$$

Узагальнено багаточисельні класи черг серверів що простоюють. В запропонованій моделі сервер не просто виходить з системи, а обробляє пакети задач у фоновому режимі до тих пір, поки загальне число поодиноких завдань не перевищить означений поріг. Представлено стратегію роботи сервера в умовах різного стану черги. Використано різні методики отримання явних формул (включаючи флуктуаційний аналіз) для процесу масового обслуговування з дискретним часом, а також деякі підходи теорії гри (послідовні ігри) для ефективного конструювання моделі.

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