# THE ANALYTIC PROPERTIES OF THE S-MATRIX FOR ARBITRARY INTERACTIONS WHICH PASS EXTERNALLY INTO THE CENTRIFUGAL AND RAPIDLY DECREASING POTENTIALS[1]

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The analytic structure of non-relativistic unitary and non-unitary $S$-matrices is reviewed for the cases of arbitrary interactions (and may be, with the unspecified equations of motion) inside a sphere of radius $r \leq a$ which pass outside it (at $r > a$) into the centrifugal and decreasing (exponentially, by the Yukawa law, or more rapidly) potentials on the base of the author's papers from 1961 till 2006. The one-channel case and special examples of many-channel cases are considered. Some kinds of the symmetry conditions are imposed. The Schrödinger equation for $r > a$ for the particle motion and the condition of completeness of the corresponding wave functions are assumed. Finally, a scientific program of the future research is presented as a clear continuation and an extension of the obtained results.

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## 1. Introduction

Only a rather small number of papers is concentrated on the study of the analytic properties of the $S$-matrix with the minimal number of assumptions about the interaction properties on small distances (practically nothing, except for very general physical and mathematical principles, such as certain symmetry properties, causality or the condition of the completeness of the wave functions in the external interaction range, and the possibility of the $S$-matrix analytic continuation in the complex plane of kinetic energies or wave numbers). This approach ascends to the old idea of Heisenberg [1] (see also [2–5] and precedent references therein) of the unique fundamental quantity ($S$-matrix) which will be sufficient for the predictions of many observable quantities basing only on the general physical and mathematical principles.

We now outline the main results of [5] for the unitary $S$-matrix, since they will be the initial base of the further reviewed results of the author's papers [6–12]. Namely in [5], the analytical expression for the function $S_l(k)$ had been obtained. It defines the relation between the amplitudes of ingoing and outgoing $l$-waves for the elastic scattering of non-relativistic particles without spin (with $l = 0$) for an arbitrary interaction localized inside the sphere of radius $a$, starting from the unitary condition

$$S_l(k)S_l^*(k^*) = 1, \quad (1)$$

the symmetry condition

$$S_l(k)S_l(-k) = 1 \quad (2)$$

or

$$S_l^*(k)S_l(-k^*) = 1, \quad (3)$$

and the particular "causality" condition (*if the ingoing wave packet is normalized so that if it represents one particle at $t = -\infty$, then the total probability to find the particle at any successive time moment (for instance, $t = 0$) outside the interaction sphere cannot be more than 1).* Strictly speaking, this condition is not the causality but the conservation of the total probability. In [13], it was shown that it does directly follow from the orthogonality of the eigenfunctions of a self-adjoint operator describing the motion and the interaction of colliding particles.

Then the existence of the analytic continuation of $S_l(k)$ into the complex plane of $k$ and the condition of quadratic integrability of the weight functions of wave packets had been also assumed. This, in turn, ensured the uniform convergence (in the range $r > a$) of the integrals over the momentum in the Fourier-transforms of wave packets. Finally, the following expression for $S_0(k)$ was obtained (which had been named then by the Bargmann representation for the Bargmann potentials):

$$S_0(k) = \exp(-2ik\alpha) \prod_\lambda \frac{k_\lambda - k}{k_\lambda + k} \prod_s \frac{(k_s - k)(k_s^* + k)}{(k_s^* - k)(k_s + k)}. \quad (4)$$

[1] This review is dedicated to the memory of my first supervisor Yu.V. Tsekhmistrenko who had become a victim of the severe administrative persecution by the political motives in the 1970s

Here, $\alpha \leq a$, $k_\lambda$ are zeros on the imaginary axis (which are simple on the lower semiaxis), $k_s$ are the zeros in the upper half-plane $D^+$, and the products $\prod\limits_\lambda$ and $\prod\limits_s$ converge on the real $k$ axis. In [14], it was shown that zeros $k_\lambda$ on the lower and upper imaginary semiaxes and zeros $k_s$ correspond to bound, virtual (anti-bound), and resonance states, respectively.

If the interaction is described by a local central potential $V(r)$ independent of $k$, and the conditions

$$\int\limits_0^\infty drr^n\,|V(r)| < \infty, \quad n = 1, 2, \tag{5}$$

and

$$V(r) \equiv 0 \quad \text{for} \quad r > a, \tag{6}$$

are fulfilled, expression (4) is valid also for arbitrary values of $l$ with $\alpha = a$, and the product over $\lambda$ contains a finite number of poles on the upper imaginary semiaxis. But if only condition (5) is fulfilled, then expression (4) is, generally speaking, invalid, and one does often use the expression

$$S_l(k) = \frac{f_{l-}(k)}{f_{l+}(k)}, \tag{7}$$

where the Jost functions $f_{0\pm}(k) = f_{0\pm}(k, 0)$ for $l = 0$ and

$$f_{l\pm}(k) = \frac{k^l \exp(\pm il\pi/2)}{(2l-1)!!} \lim_{r\to 0} r^l f_{l\pm}(k, r)$$

for $l > 0$, $f_{l\pm}(k, r)$ is the Jost solution of the radial Schrödinger equation or the integral equation equivalent to it,

$$f_{l\pm}(k, r) = \pm i \exp(\pm il\pi/2) krh_l^{(1,2)}(kr) -$$

$$-\frac{2\mu}{\hbar^2 k}\int\limits_0^\infty dr' g_l(k; r, r') V(r') f_{l\pm}(k, r'), \tag{8}$$

with the boundary condition

$$\lim_{r\to\infty} f_{l\pm}(k, r)\exp(\mu ikr) = 1, \tag{9}$$

where

$$g_l(k; r, r') = \frac{ikrr'}{2}[h_l^{(1)}(kr')h_l^{(2)}(kr) - h_l^{(1)}(kr)h_l^{(2)}(kr')],$$

and $h_l^{(1,2)}(kr) = j_l(kr) \pm in_l(kr)$ are the Hankel spherical functions of the first and second kinds, respectively ($j_l(kr)$ and $n_l(kr)$ are the Bessel and Neumann spherical functions, respectively). Under such conditions, the function $S_l(k)$ can have, in addition to the singularities described by (4), additional singularities, corresponding to the singularities of $f_{l\pm}(k, r)$.

The author's papers [6–12] give the results of that approach, published gradually during 1961–2006 (mainly in the Russia and Ukraine). In the final section of this paper, I present the scientific program which highlights the remaining problems and the gradually revealed perspective, unexpected previously, of how a rigorous mathematical method or approach can help to clarify quite specific and sometimes paradoxical physical phenomena.

## 2. Properties of the Non-unitary One-channel $S$-matrix for Arbitrary Interactions Externally Passing into the Centrifugal Barrier and a Potential Which Is Decreasing More Rapidly Than Any Exponential Function

Now, following [7], we consider a generalized case where the interaction is arbitrary and the equation of motion inside the sphere of radius $a$ is unspecified as before but, at $r > a$, contains the centrifugal barrier $h^2 l(l+1)/r^2$ and a potential $V(r)$. Moreover, not only the scattering but also the partial particle absorption or generation are observed. For convenience, we introduce a new interaction characteristics, a complex "interaction constant" $\gamma$. We agree conventionally that its real part Re$\gamma$ will characterize that interaction part which causes by itself the scattering only, without the particle absorption or generation. We also agree to set up the negative (positive) value of Im$\gamma$ in correspondence with that interaction part, whose absence causes the absence of the particle absorption (generation). If we further connect the particle absorption and generation with the simple decrease or increase of the flux of scattered particles in comparison with the flux of bombarding particles and assume the conservation of their momenta and other characteristics, then it will be natural to impose the following conditions:

$$0 < |S_l(\gamma, k)|^2 \leq 1, \tag{10a}$$

$$1 \leq |S_l(\gamma^*, k)|^2 < \infty, \tag{10b}$$

with Im$\gamma < 0$, for real positive $k$. Since conditions (10a) and (10b) are evidently insufficient for the study of the analytic properties of $S_l(\gamma,k)$, we introduce, by generalizing (1)–(3), the new symmetry properties (typical of central interactions):

$$S_l(\gamma,k)S_l(\gamma,-k)=1, \qquad (2a)$$

$$S_l(\gamma^*,k)S_l(\gamma^*,-k)=1, \qquad (3a)$$

and the generalized "unitarity" condition

$$S_l(\gamma,k)S_l^*(\gamma^*,k^*)=1, \qquad (1a)$$

thus selecting, for any interaction with a constant $\gamma$(Im$\gamma$, 0), the "conjugate" interaction with a complex conjugate constant $\gamma^*$.

One can easily check that conditions (1a), (2a), (3a), (10a), and (10b) are automatically fulfilled in the case where the interaction can be described by a complex potential which satisfies condition (5) [14,15]. In that case, the values of $\gamma$ and $\gamma^*$ are not only conventional but also factual parameters of the potential $V(\gamma,r)=\mathrm{Re}\gamma V_1(r)+i\mathrm{Im}\gamma V_2(r)$.

Instead of the "causality" condition from [5], we use the condition of completeness for the wave functions outside the sphere of an unknown interaction, factually assuming, in this region (i.e., for $r\geq a$), the possibility to describe the colliding particles by the Schrödinger equation with a self-adjoint Hamiltonian:

$$\frac{2}{\pi}\int\limits_0^\infty k^2dkR_l^{(+)}(\gamma,k,r)R_l^{(+)*}(\gamma,k,r')+$$

$$+\sum_n R_{nl}(\gamma,k_{nl},r)R_{nl}(\gamma,k_{nl},r')=\frac{\delta(r-r')}{r^2}, \qquad (11)$$

where

$$R_l^{(+)}(\gamma,k,r)=\frac{i}{2kr}[f_{l-}(k,r)\exp(il\pi/2)-$$

$$-S_l(\gamma,k)f_{l+}(k,r)\exp(-il\pi/2)],$$

$$R_{nl}=\frac{1}{\sqrt{2\pi}}B_{nl}(\gamma,k_{nl})f_{l+}(k_{nl},k)/r,$$

the functions $f_{l\pm}(k,r)$ are the Jost solutions of Eq. (8); Im$k_{nl}>0$ and, consequently, the functions $R_{nl}$ are integrable together with their squares (at least, in the range $a\leq r<\infty$); all information on the interaction inside the sphere with radius $r<a$ is contained in the functions $S_l(\gamma,k)$ and the constants $B_{nl}(\gamma,k_{nl})$. In (11), we assume that $R_{nl}(\gamma,k_{nl},r)=R_{nl}^*(\gamma^*,k_{nl}^*,r)$.

Equation (11) represents a generalization of the completeness relation for the eigenfunctions of the most simple classes of non-Hermitian Hamiltonians [14] in the cases where all the eigenvalues $k_{nl}$ are simple (non-multiple) and are situated outside the real axis $k$. When $\gamma=\mathrm{Re}\gamma$, the functions $R_{nl}$ describe simply the bound states of the system. For the complex values of $\gamma$, they have the same boundary conditions as the bound states, and their properties for the non-singular potentials with the negative imaginary part are partially described in [16].

In order to be sure that $S_l(\gamma,k)$ can have the analytic continuation into the complex plane of $k$, one has to impose some limitations on the potential tails in the range $r>a$. In correspondence with the study of the potential scattering in [14–17], we can try, at least, to limit ourselves to the cases where there is a potential in the range $r>a$ in addition to the centrifugal barrier. This potential satisfies the condition

$$\int\limits_0^\infty drr\,|V(r)|\exp(br)<\infty \qquad (12)$$

at least with any arbitrarily small $b$.

Using the known properties

$$f_{l+}^*(k^*,r)=f_{l+}(-k,r)=f_{l-}(k,r) \qquad (13)$$

for real $k$ and relations (1a),(2a), and (3a) for $S_l\gamma,k)$, one can transform (11) into the form

$$\frac{1}{rr'}\int\limits_C dkf_{l+}(k,r)f_{l-}(k,r')-$$

$$-\frac{(-1)^l}{rr'}\int\limits_C dkS_l(\gamma,k)f_{l+}(k,r)f_{l+}(k,r')+$$

$$+\frac{1}{rr'}\sum_n(B_{nl})^2f_{l+}(k_{nl},r)f_{l+}(k_{nl},r')=\frac{2\pi\delta(r-r')}{r^2}, \qquad (14)$$

where the integration trajectory $C$ goes along the real axis $k$ from -∞ to +∞, bypassing the point $k=0$, where $f_{l\pm}$ have the pole of the $l$-th order, along a semicircle of the infinitesimal radius located in the upper half-space.

We limit ourselves to the case where $f_{l\pm}(k,r)$ behaves itself as $\exp(\pm ikr)$ in the whole complex plane at $|k|\to\infty$. Let us shift the integration contour into $D^+$ and enclose all the singularities by closed contours. By utilizing the equalities

$$\int\limits_{\Gamma+} dk f_{l+}(k,r) f_{l-}(k,r) =$$

$$= \int\limits_{\Gamma+} dk e^{ik(r-r')} = \int\limits_{-\infty}^{\infty} dk e^{ik(r-r')} = 2\pi\delta(r-r'), \quad (15)$$

$$\int\limits_{\Gamma+} dk S_l(\gamma,k) f_{l+}(k,r) f_{l+}(k,r') = \int\limits_{\Gamma+} dk S_l(\gamma,k) e^{ik(r+r')}, \quad (16)$$

after reasonings and derivations performed within an approach that was outlined in [7,10,12] (see also [9,13]), we obtain

$$S_l(\gamma,k) = \exp(-2i\alpha k)\times$$

$$\times \prod_n \frac{k_{nl}+k}{k_{nl}-k} \prod_\lambda \frac{k_\lambda - k}{k_\lambda + k} \prod_s \frac{k_s - k}{k_s + k} \prod_{s'} \frac{k_{s'} - k}{k_{s'} + k}, \quad (17a)$$

$$S_l(\gamma^*,k) = \exp(-2i\alpha k)\times$$

$$\times \prod_n \frac{k^*_{nl}+k}{k^*_{nl}-k} \prod_\lambda \frac{k^*_\lambda - k}{k^*_\lambda + k} \prod_s \frac{k^*_s - k}{k^*_s + k} \prod_{s'} \frac{k^*_{s'} - k}{k^*_{s'} + k}, \quad (17b)$$

which generalizes (4), taking conditions (1a)–(3a) into account. Here, $k_{nl}$ are the poles in the lower half-space $D^-$, $k_\lambda$ are the zeros in $D^+$, and $k_s$ and $k_{s'}$ are the zeros in the first and second quadrants, respectively. (For $\gamma = \mathrm{Re}\gamma$, the points $k_\lambda$ are the zeros $k_{nl}$ on the lower imaginary semiaxis, corresponding to bound states, and also the zeros on the upper imaginary semiaxis which define virtual (anti-bound) states and correspond to the poles situated, at least, by one between the poles $k_{nl}$ and $k_{n+1}$, following an approach that was outlined in [9,13].) Results (23a) and (23b) had been first explicitly obtained in [7].

The above-written simplified assumptions about the eigenvalues $k_{nl}$ in the completeness condition (11) factually brings to an insignificant limitation of the interaction class. The absence of values $k_{nl}$ on the real axis $k$, i.e. the absence of poles and zeros (spectral points) of $S_l(\gamma,k)$ and $S_l(\gamma^*,k)$ corresponding to them (as well as the absence of values of $k_s$ and $k_{s'}$) does simply signify the rejection of the cases of the total absorption of bombarding particles and also the rejection of the infinite increase of the new-particle birth for the physical values of $k\geq 0$. The condition of the absence of the eigenvalues $k_{nl}$ with the multiplicity of more than 1 apparently does not also bring to the essential limitation of the interaction class. Really, if one naturally assumes that a smooth change of the interaction parameter $\gamma$ brings to a smooth shift of the values $k_{nl}$, then an arbitrarily small change of the parameter $\gamma$ will brings to a certain small divergence of the various trajectories $k_{nl}(\gamma)$ from the point of their ($k_{nl}$) coincidence. In [9], it was shown (with the help of another method) that expressions (23a,b) are valid for local potentials inside $r\leq a$ with a hard (infinite) core of radius $r_0 < a$, for non-local separable potentials of the type $v(r)\ v(r')$ with $0<r,\ r'<a$, and for non-local separable potentials with a hard(infinite) core of radius $r_0 < a$. Expressions (23a,b) were generalized for local complex potentials with multiple zeros $-k_{nl}$, $k_\lambda$, and $k_s$. In the last case, relations (23a,b) will contain factors of the type $(\frac{k_{nl}+k}{k_{nl}-k})^\alpha_{nl}(\frac{k_\lambda-k}{k_\lambda+k})^\alpha_\lambda(\frac{k_s-k}{k_s+k})^\alpha_s$, where $\alpha_{nl}$, $\alpha_\lambda$, and $\alpha_s$ are the multiplicities of the zeros $-k_{nl}$, $k_\lambda$, and $k_s$, respectively.

If there are the centrifugal barrier and a potential which is decreasing more rapidly than any exponential function in the external region $r\geq a$, then results (23a) and (23b) remain valid, since the functions $f_{l\pm}(k,r)$ are analytical everywhere in this case (see, e.g., [12,14]), besides the points $k=0$ and $\infty$. In the limit $|k|\to\infty$, they tend to $(\exp \pm ikr)$.

If there are the centrifugal barrier and an exponential potential of the type $V = V_0\exp(-br)$, $V_0$, $b>0$, in the external region, where $r\geq a$, then the functions $f_{l\pm}(k,r)$ have the simple poles at the points $k=\mp i\frac{b}{2}m$ ($m=1,2,\ldots$) and, at the limit $|k|\to\infty$, they tend to $\exp(\pm ikr)$. Similar results can be obtained for the Eckart, Hulthén, and Woods–Saxon potentials [13]. Let the centrifugal barrier and a potential of the type $V = V_0 P_n(r)\exp(-br)$, where $P_n(r)$ is an $n$th-order polynomial and $b>0$, be present in the external region (with $r\geq a$). Then the function $f_{l\pm}(-k,r)$ has poles of an order not higher than $n+1$ at the points $\mu ib/2$, $\mu ib$, $\mu 3ib/2$, ..., and it is analytic at all other points of the complex plane.

Moreover, the following theorem was first proved in [6].

For $f_0(-k,r)$ to have poles of the order not higher than $(n_1+1)$ at the points $ib_1/2$, $ib_1$, $3ib_1/2$,..., not higher than $(n_2+1)$ at the points $ib_2/2$, $ib_2$, $3ib_2/2$,..., and not higher than $(n_m+1)$ at the points $ib_m/2$, $ib_m$, $3ib_m/2$,..., it is necessary and sufficient that the corresponding potential have a term $\sum_{n_m} P_{n_m}(r)\exp(-b_m r)$. Then, in [12, 14], the appropriate integral equation which allows computing $f_l(-k,r)$ from $f_0(-k,r)$, it was shown that, in this case, $f_l(-k,r)$ has the same isolated singular points as $f_0(-k,r)$. Thus, one can obtain also results (17a) and (17b) in this case where, in $\prod_m$, one must include the factors corresponding to "redundant" poles $\frac{i}{2}bm'$ $(m'=1,2,...)$ of the first order in the presence of an exponential potential tail $V=V_0\exp(-br)$ and factors of the type $\prod_{m''}(\frac{k_{m''}-k}{k_{m''}+k})^n$ corresponding to multiple "redundant" poles $\frac{i}{2}bm''$ $(m''=1,2,...)$ in the presence of a potential tail of the type $V_0\sum_n P_n(r)\exp(-br)$.

If the centrifugal barrier and a central Yukawa potential of the type $V=V_0[(br)^{-1}\exp(-br)]$, $V_0$, $b^{-1}\sim a$, are present in the external region $r\geq a$, we can study the analytic properties of $f_{l\pm}(k,r)$ and $S_l(\gamma,k)$ following [12], where it was obtained:

$$S_l(\gamma,k)=\exp(-2i\alpha k)F(k)\times$$

$$\times\prod_n\frac{k_{nl}+k}{k_{nl}-k}\prod_\lambda\frac{k_\lambda-k}{k_\lambda+k}\prod_s\frac{k_s-k}{k_s+k}\prod_{s'}\frac{k_{s'}-k}{k_{s'}+k}, \tag{18a}$$

$$S_l(\gamma^*,k)=\exp(-2i\alpha k)F(k)\times$$

$$\times\prod_n\frac{k^*_{nl}+k}{k^*_{nl}-k}\prod_\lambda\frac{k^*_\lambda-k}{k_\lambda+k}\prod_s\frac{k^*_s-k}{k^*_s+k}\prod_{s'}\frac{k^*_{s'}-k}{k^*_{s'}+k}. \tag{18b}$$

Here, $F(k)$ is a singular factor,

$$F(k)=\frac{\left[1-\frac{i\rho}{2k}\ln\left(1-\frac{2ik}{b}\right)\right]}{\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]}, \tag{19}$$

containing the logarithmic branch points at $k=\pm k_\gamma=\pm ib/2$ ($\rho=2\mu V_0/\eta^2b^3<0$, and $\mu$ is the reduced mass).

## 3. Properties of the Non-unitary $S$-Matrix for Arbitrary Non-central and Parity-violating Interactions Externally Passing into the Centrifugal Barrier and a Potential Which Is Decreasing More Rapidly Than Any Exponential Function

Following [10], we suppose that the interaction between two colliding particles is such that the $S$-matrix is diagonal, as regards the total momentum $j$, does not depend on the total-momentum projection onto an arbitrary axis, and contains both diagonal and non-diagonal elements regarding the orbital momentum $l$ with the mixed neighboring values $l,l'=j\pm\lambda$ of equal ($\lambda=1$) or opposite ($\lambda=\frac{1}{2}$) parities. Particularly, there is a mixture of values $l,l'=l\pm1$ (in the case of a tensor interaction admixture) or there is no mixture at all ($l=l'=j,\lambda=0$). There is a mixture $l,l'=j+\frac{1}{2}$ in the case of a parity-violating interaction like $\upsilon(r)\hat{\boldsymbol\sigma}\hat{\mathbf p}+\hat{\boldsymbol\sigma}\hat{\mathbf p}\upsilon(r)$, where $r$ is the relative distance between two particles, $\hat{\boldsymbol\sigma}$ is the Pauli pseudo-vector matrix, and $\hat{\mathbf p}$ is the momentum operator for the relative motion of a nucleon and a nucleus with spin 0. Of course, $l=l'=j$ and $\lambda=0$ for central interactions in all the cases.

Thus, we consider an arbitrary non-central or parity-violating interaction inside the sphere $r<a$ surrounded by the centrifugal barrier and a central potential which is decreasing more rapidly than any exponential function $V(r)$. Supposing that there is not only the scattering, but also the absorption or the creation of particles, it is natural to put, by generalizing (10a) and (10b), the following conditions for the elements $S^j_{ll'}$ of the $S$-matrix:

$$0<\sum_{l'}\left|S^j_{ll'}(\gamma,k)\right|^2\leq1, \tag{20a}$$

$$1\geq\sum_{l'}\left|S^j_{ll'}(\gamma,k)\right|^2<\infty, \tag{20b}$$

and, by generalizing (1a)–(3a), the extended "unitarity" condition

$$\sum_l S^j_{l_1l}(\gamma,k)S^{j*}_{ll_2}(\gamma^*,k^*)=\delta_{l_1l_2} \tag{21}$$

and symmetry condition

$$S^{j*}_{ll'}(\gamma^*,k^*)=(-1)^{l+l'}S^j_{ll'}(\gamma,-k) \tag{22}$$

(as regards the Im$k$ axis), and the symmetry condition of $S^j_{ll'}$ regarding the lower indices:

$$S^j_{ll'}(\gamma,k)=S^j_{l'l}(\gamma,k). \tag{23}$$

A state of the system for $r\geq a$ can be described by the wave functions

$$R^{j*}_{ll'}(\gamma,k,r)=\frac{i}{2kr}[\delta_{ll'}f_{l'-}(k,r)\exp(il'\pi/2)-$$

$$-S^j_{ll'}(\gamma,k)f_{l'+}(k,r)\exp(-il'\pi/2)] \tag{24}$$

in the continuous part of the spectrum and

$$R_l^{j(n)}(\gamma,k_{nl},r)=(2\pi)^{-1/2}B_l(\gamma,k_{nl})f_{l+}(k_{nl},r)r^{-1} \quad (25)$$

in the discrete part.

Generalizing the completeness relation (11) for an arbitrary non-central or parity-violating interaction inside the sphere $r<a$ surrounded by the centrifugal barrier and a central potential which is decreasing more rapidly than any exponential function $V(r)$, we can write

$$\frac{2}{\pi}\sum_l\int\limits_0^\infty k^2dkR_{l'l}^{j(+)}(\gamma,k,r)R_{l''l}^{j(+)*}(\gamma,k,r')+$$

$$+\sum_n R_{l'}^{j(n)}(\gamma,k_{nj,}r)R_{l''}^{j(n)}(\gamma^*,k_{nj},r')=\frac{\delta(r-r')}{r^2}\delta_{l'l''}. \quad (26)$$

Relation (26) is a generalization of the completeness condition for eigenfunctions of a class of non-Hermitian Hamiltonians [17, 18], for which all eigenvalues are simple (not multiple) and are situated outside the Re$k$ axis.

If there are the centrifugal barrier and a central potential which is decreasing more rapidly than any exponential function $V(r)$ in the external region $r\geq a$, we can study the analytic properties of $S_{ll'}^j(\gamma,k)$, following [10], where it was obtained:

$$S_{ll'}^j(\gamma,k)=A_{ll'}(\gamma,k)\exp[-i(\alpha_l+\alpha_{l'})]\prod_n\frac{1+k/k_{nl}}{1-k/k_{nl}}\times$$

$$\times\prod_{m,s,s',p,r,t,t'}\frac{(1-k/k_p)(1-k/k_r)(1-k/k_t)(1-k/k_{t'})}{(1-k/k_m)(1-k/k_s)(1-k/k_{s'})}. \quad (27)$$

Here, $A_{ll'}=\delta_{ll'}+(1-\delta_{ll'})Ck^{l_>+1}$, $C=i\mathrm{Im}C$ is a constant, and $\alpha_l=a-\beta_l\leq a$. The topology of the poles $k_{nj}$, $k_m$, $k_s$, $k_{s'}$ and the zeros $k_{nj}$, $k_p$, $k_r$, $k_t$, $k_{t'}$ was specified as follows:

In $D^-$, all the elements $S_{ll'}^j(\gamma,k)$ have the same poles $k_{nj}$ (on the semiaxis Im$k<0$), $k_s$ (in the 4-th quadrant), and $k_{s'}$ (in the 3-rd quadrant) which correspond to the zeros in $D^+$ of the function $d_j(\gamma,k)=S_{ll}^j(\gamma,k)S_{l'l'}^j(\gamma,k)-[S_{ll'}^j(\gamma,k)]^2$ defined by the equalities

$$S_{ll}^j(\gamma,-k)=S_{l'l'}^j(\gamma,k)/d_j(\gamma,k),$$

$$S_{ll'}^j(\gamma,-k)=-S_{ll'}^j(\gamma,k)/d_j(\gamma,k)$$

(with $l'\neq l$) and also the zeros $-k_{nj}$ which correspond to the poles $k_{nj}$ in $D^+$. In addition, every diagonal element $S_{ll'}^j(\gamma,k)$ can have additional poles on the semiaxis Re$k<0$ ($k_\mu$), in the 4-th quadrant ($k_\sigma$) and in the 3-rd quadrant ($k_{\sigma'}$) which correspond to the zeros $-k_\mu$, $-k_\sigma$ and $-k_{\sigma'}$ of two functions $S_{ll}^j(\gamma,k)$ and $S_{ll'}^j(\gamma,k)$ in $D^+$. Moreover, one can conclude from formulae (23) that the zeros $k_p$ (on the axis Im$k$), $k_r$ (on the axis Re$k$), $k_t$ (in the 1-st and 4-th quadrants) and $k_{t'}$ (in the 2-nd and 3-rd quadrants) of the diagonal element $S_{ll}^j(\gamma,k)$ correspond to the zeros $-k_p$, $-k_r$, $-k_t$ and $-k_{t'}$ of the second diagonal element $S_{l'l'}^j(\gamma,k)$, $l'\neq l$, and also that the zeros of the non-diagonal element $S_{ll'}^j(\gamma,k)$, $l'\neq l$, can appear only in pairs $\pm k_\pi$ (on the semiaxis Im$k$), $\pm k_\rho$ (on the semiaxis Re$k$), $\pm k_t$ (in the rest of the complex plane). Evidently, the last assertion is true for those zeros which are not general zeros of all the elements $S_{ll'}^j(\gamma,k)$.

In the case of $\gamma=\mathrm{Re}\gamma$, the zeros appear in the pairs $\pm k_r$ and $k_{s'}=-k_s^*$, $k_{t'}=-k_t^*$ because of the symmetry condition (22), and then

$$S_{ll'}^j(\mathrm{Re}\gamma,k)=A_{ll'}\exp[-i(\alpha_l+\alpha_{l'})]\prod_n\frac{1+k/k_{nl}}{1-k/k_{nl}}\times$$

$$\times\prod_{m,s,s',p,r,t,t'}\frac{(1-k/k_p)(1-k/k_r)^2(1-k/k_t)(1-k/k_t^*)}{(1-k/k_m)(1-k/k_s)(1-k/k_s^*)}. \quad (28)$$

## 4. The Properties of the Non-unitary Multi-channel S-matrix for Arbitrary Interactions Externally Passing into the Centrifugal Barrier and a Potential Which Is Decreasing More Rapidly Than Any Exponential Function

In [8, 11], we considered an idealized case where the structure of colliding particles is described by an infinite discrete set of non-degenerate energy levels $\varepsilon_n$ and the wave functions $\varphi_n(\xi)$ of bound states with zero (or "frozen") spin. Let the conditions of the orthonormalization

$$\int d\xi\varphi_m(\xi)\varphi_n(\xi)=\delta_{mn}\quad(m,n=0,1,2,\ldots) \quad (29)$$

and completeness

$$\sum_{n=0}^\infty\varphi_n(\xi)\varphi_n(\xi')=\delta(\xi-\xi') \quad (30)$$

($\xi$ is a totality of the internal particle coordinates) be fulfilled.

We assume that the central interaction between colliding particles inside the sphere of radius $a$ is arbitrary as before, but, at $r > a$, it contains the centrifugal barrier $h^2 l(l+1)/r^2$ and a central potential $V(r)$ which is decreasing more rapidly than any exponential function, and there is not only the scattering, but also the partial particle absorption or generation. We describe the state of the total system in the external region ($r \geq a$) by the function

$$R_{nl}^{(+)}(E,r,\xi)=\sum_{m=0}^{\infty}\frac{i}{2r}\sqrt{\frac{\mu}{\hbar^2 k_m}}[f_{l-}(k_n,r)\exp(il\pi/2)\delta_{mn}-$$

$$-S_l^{(mn)}(\gamma,E)f_{l+}(k_m,r)\exp(-il\pi/2)]\varphi_m(\xi) \quad (31)$$

in the region of the continuum part of the energy spectrum $E = \varepsilon_n + \frac{\hbar^2 k_n^2}{2\mu}$, i.e. for $k_n^2 \geq 0$ ($n = 0,1,2,\ldots$; $\varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \ldots$), and the functions

$$R_{\nu l}(E_{\nu l},r,\xi)=\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}B_{\nu l}^{(n)}(\gamma,k_{\nu l}^{(n)})f_{l+}(k_{\nu l}^{(n)},r)r^{-1}\varphi_n(\xi) \quad (32)$$

in the region of the discrete part of the energy spectrum $E_{\nu l} = \varepsilon_n + \frac{\hbar^2[k_{\nu l}^{(n)}]^2}{2\mu}$, i.e. for $[k_{\nu l}^{(n)}]^2 < 0$, $n = 0,1,2,\ldots$ ($\gamma$ is a complex parameter which characterizes the scattering (Re$\gamma$) and the absorption or generation (Im$\gamma$) of particles). We assume that $\sum\limits_m$ and $\sum\limits_n$ do uniformly converge.

If $\gamma = \mathrm{Re}\gamma$ and $E \geq \varepsilon_0$, the elements $S_l^{(mn)}(\gamma,E) \equiv S_l^{(mn)}(\gamma,k_0,k_1,\ldots)$ must satisfy the *conditions of unitarity*4

$$\sum_{n=0}^{N-1} S_l^{(mn)}(\gamma,E)S_l^{(nm')*}(\gamma,E) = \delta_{mm'} \quad (33)$$

with $\varepsilon_{N-1} \leq E \leq \varepsilon_N$, $0 \leq m,m' \leq N-1$, $N = 1,2,\ldots$ ($N-1$ being the number of the last open channel), which follows from the law of conservation of the particle number (see, e.g., [13,22,23]), *of the symmetry regarding the indices of open channels*

$$S_l^{(mn)T}(\gamma,E) = S_l^{(mn)}(\gamma,E) = S_l^{(nm)}(\gamma,E) \quad (34)$$

and *of the symmetry regarding the wave numbers*

$$\tilde{S}_l^{(mn)*}(\gamma,k_0,k_1,\ldots) = \tilde{S}_l^{(mn)}(\gamma,-k_0^*,-k_1^*,\ldots), \quad (35)$$

where

$$\tilde{S}_l^{(mn)} = \sqrt{\frac{k_m}{k_n}}S_l^{(mn)} \quad (36)$$

for $E \geq \varepsilon_{N-1}$ ($m,m' \leq N-1$), or, considering that

$$e^{i\pi}k_n = \begin{cases} e^{i\pi}k_n, & \text{if the } n\text{-th channel is open,} \\ k_n, & \text{if the } n\text{-th channel is closed,} \end{cases}$$

$$S_l^{(mn)*}(\gamma,k_0,k_1,\ldots) =$$

$$= S_l^{(mn)}(\gamma,-k_0,-k_1,\ldots,-k_{N-1},k_N,k_{N+1},\ldots). \quad (35a)$$

Relations (34)–(35a) follow from the principle of the reversibility of time ($T$-invariance) in the external region ($r \geq a$).

Since the values of $k_n$ ($n = 0,1,2,\ldots$), on which the elements $S_l^{(mm')}$ depend relative to (36), represent irrational (radical) functions of one independent variable (for instance, $k_m$), we will study the elements $S_l^{(mm')}$ on the Riemann surface which can be obtained, by making the cuts on the infinite number of the $k_m$-planes from points $+\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_n-\varepsilon_m)}$ till points $-\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_n-\varepsilon_m)}$, respectively, symmetrically located relative to the imaginary axis (because of the symmetry condition (36)), and connecting these planes along the cuts. Let draw these cuts in such a manner, as is shown in Figure.

The usual physical boundary conditions imposed on the asymptotics of functions (31) imply that $k_n > 0$ for $E > \varepsilon_n$ and $\mathrm{Im}k_n > 0$ for $E < \varepsilon_n$. Therefore, we choose, as the physical sheet of the Riemann surface, the $k_m$-plane, on which

$$\text{sign Re}k_n = \text{sign Re}k_m,$$

$$\text{sign Im}k_n = \text{sign Im}k_m \quad (n = 0,1,2,\ldots). \quad (37)$$

Generalizing relations (35)–(38) to the case $\gamma \neq \mathrm{Re}\gamma$ with complex values of $k_0$, $k_1$, $\ldots$, we can write

$$\sum_{n=0}^{N-1}\tilde{S}_l^{(mn)}(\gamma,k_0,k_1,\ldots)k_n\times$$

$$\times\tilde{S}_l^{(m'n)}(\gamma,-k_0,-k_1,\ldots-k_{N-1},k_N,k_{N+1},\ldots) =$$

$$= \delta_{mm'}k_m, \quad (38)$$

$0 \le m, m' \le N-1, \quad N = 1, 2, \ldots;$

$$S_l^{(mn)T}(\gamma, k_0, k_1, ...) = S_l^{(mn)}(\gamma, k_0, k_1, ...) =$$

$$= S_l^{(nm)}(\gamma, k_0, k_1, ...) \tag{39}$$

and

$$\tilde{S}_l^{(mn)*}(\gamma^*, k_0^*, k_1^*, ...) =$$

$$= \tilde{S}_l^{(mn)}(\gamma, -k_0, -k_1, ..., -k_{N-1}, k_N, k_{N+1}, ...), \tag{40}$$

$0 \le m, n \le N-1, \quad N = 1, 2, \ldots$

The completeness condition of the system's wave functions in the region $r \ge a$ can be written in the form

$$\frac{2}{\pi} \sum_{n=0}^{\infty} \int_{\varepsilon_n}^{\infty} dE R_{nl}^{(+)}(\gamma, E; r, \xi) R_{nl}^{(+)*}(\gamma, E; r', \xi') +$$

$$+ \sum_{\nu} R_{\nu l}(\gamma, E_{\nu l}; r, \xi) R_{\nu l}(\chi, , E_{\nu l}; r', \xi') =$$

$$= \frac{\delta(r - r')}{r^2} \delta(\xi - \xi'). \tag{41}$$

Then, following [8] and shifting the integration contour into $D^+$, enclosing all the singularities by closed contours, after reasoning and derivations performed within an approach that was outlined in [8], we can study the analytic properties of $S_l^{(m'm)}$, but we cannot obtain the explicit analytic expression for $S_l^{(mm)}$ $(rS_{J\Pi}^{(mn)})$ on the Riemann surface.

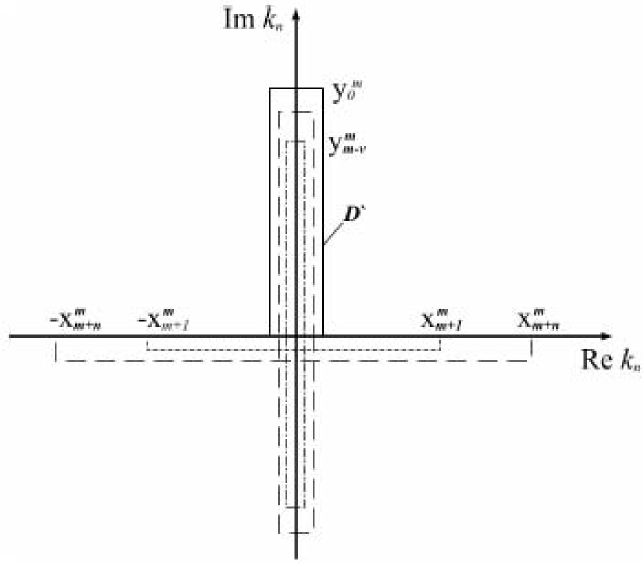
Quite similarly to the behavior of $S(\gamma, k)$ for $k \to 0$ in the one-channel case, one can obtain

$$S_l^{(mm)}(\gamma, k_0, k_1, ...) \underset{k_m \to 0}{\to} 1 + \mathrm{O}(k_m^q), \quad q \ge l+1, \tag{42}$$

$$S_l^{(m'm)}(\gamma, k_0, k_1, ...) \underset{k_m \to 0}{\to} \mathrm{O}(k_m^{q'}), \;\; m' \ne m, \;\; q' \ge l + 1/2. \tag{43}$$

In [18], the following useful simplified parametrization was obtained:

$$\hat{S}^{(\alpha)} = \hat{U}^{(\alpha)} \prod_{\nu} (1 - \frac{i\Gamma_\nu^{(\alpha)} \hat{P}_\nu^{(\alpha)}}{E - E_\nu^{(\alpha)} + i\Gamma_\nu^{(\varepsilon)}/2}) \hat{U}^{(\alpha)T},$$




Physical $k_m$-plane with the cuts between the channel thresholds $\pm x_{m+n}^m = \pm\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_{m+n} - \varepsilon_m)}$, $n = 0, 1, 2, \ldots$; $\pm y_{m-\nu}^m = \pm\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_m - \varepsilon_{m-\nu})}$, $m \ge \nu, m, \nu = 0, 1, 2, ...,$ and the contour $D'$

$$\hat{U}^{(\alpha)} \hat{U}^{(\alpha)*} = 1,$$

$$\hat{P}_\nu^{(\alpha)} = \hat{P}_\nu^{(\alpha)*} = \hat{P}_\nu^{(\alpha)2}, \mathrm{Trace} \hat{P}_\nu^{(\alpha)} = 1, \tag{44}$$

coming from the general principles of unitarity, meromorphy, and $T$-invariance of the $S$-matrix. With this, it was noted in [18] that there is a practical difficulty of the explicit consideration of $T$-invariance in the general case of the projectors $\hat{P}_\nu^{(\alpha)}$ non-symmetric and non-commuting with one another. This parametrization is the mostly convenient for overlapping and strongly overlapping resonances (see, e.g., [19,20]) and was utilized for revealing the time resonances (explosions) of compound clusters and nuclei in high-energy nuclear reactions in the range of strongly overlapping energy resonances [21]. It was shown in [22] that when the projectors $\hat{P}_\nu^{(\alpha)}$ do not depend on the values of any other resonance parameters ($E_\lambda^{(\alpha)}$ and $\Gamma_\nu^{(\varepsilon)}$), then $\hat{S}^{(\alpha)} = \hat{S}^{(\alpha)T}$. Really, in that case, one can rewrite the resonance part $\hat{S}_{\mathrm{res}}^{(\alpha)} \equiv \prod_{\nu=1}^{\Lambda^{(\alpha)}} (1 - \frac{i\Gamma_\nu^{(\alpha)} P_\nu^{(\alpha)}}{E - E_\nu^{(\alpha)} + i\Gamma_\nu^{(\alpha)}/2})$ from expression (44) for $\hat{S}^{(\alpha)}$ in the form of a sum

$$\hat{S}_{\mathrm{res}}^{(\alpha)} = 1 - i \sum_{\nu} \frac{\Gamma_\nu^{(\alpha)} P_\nu^{(\alpha)}}{E - E_\nu^{(\alpha)} + i\Gamma_\nu^{(\alpha)}/2} -$$

$$-\sum_{\nu'>\nu}\frac{\Gamma_{\nu}^{(\alpha)}\Gamma_{\nu'}^{(\alpha)}\hat{P}_{\nu}^{(\alpha)}\hat{P}_{\nu'}^{(\alpha)}}{(E-E_{\nu}^{(\alpha)}+i\Gamma_{\nu}^{(\alpha)}/2)(E-E_{\nu'}^{(\alpha)}+i\Gamma_{\nu'}^{(\alpha)}/2)}+... \tag{45}$$

which can be transformed to the Mittag–Leffler expansion

$$\hat{S}_{\rm res}^{(\alpha)}=1-i\sum_{\nu}\frac{iG_{\nu}^{(\alpha)}}{E-E_{\nu}^{(\alpha)}+i\Gamma_{\nu}^{(\alpha)}/2}, \tag{45a}$$

$$G_{\nu}^{(\alpha)}=\Gamma_{\nu}^{(\alpha)}P_{\nu}^{(\alpha)}-i\sum_{\nu'>\nu}\frac{\Gamma_{\nu}^{(\alpha)}\Gamma_{\nu'}^{(\alpha)}\hat{P}_{\nu}^{(\alpha)}\hat{P}_{\nu'}^{(\alpha)}}{E_{\nu}^{(\alpha)}-E_{\nu'}^{(\alpha)}+i(\Gamma_{\nu}^{(\alpha)}-\Gamma_{\nu'}^{(\alpha)})/2}-$$

$$-i\sum_{\nu''<\nu}\frac{\Gamma_{\nu}^{(\alpha)}\Gamma_{\nu''}^{(\alpha)}\hat{P}_{\nu}^{(\alpha)}\hat{P}_{\nu''}^{(\alpha)}}{E_{\nu}^{(\alpha)}-E_{\nu''}^{(\alpha)}+i(\Gamma_{\nu}^{(\alpha)}-\Gamma_{\nu''}^{(\alpha)})/2}+...$$

Taking (45a) and the $T$-invariance of expression (44) for $\hat{S}^{(\alpha)}$ into account, we can write

$$\hat{S}_{\rm res}^{(\alpha)}=\hat{S}_{\rm res}^{(\alpha)T}. \tag{46}$$

Then one can further rewrite (46) in the following form (see the last ref. in [28]):

$$G_{\nu}^{(\alpha)}=G_{\nu}^{(\alpha)T}\quad \nu=1,2,\ldots,\Lambda^{(\alpha)}. \tag{47}$$

Relations (47) are in general too bulky as correlations between the matrices $\hat{P}_{\nu}^{(\alpha)}$ with different $\nu$.

But if $\hat{P}_{\nu}^{(\alpha)}$ do not depend on the values of $E_{\lambda}^{(\alpha)}$ and $\Gamma_{\nu}^{(\varepsilon)}$, then the relations

$$\hat{P}_{\nu}^{(\alpha)}=\hat{P}_{\nu}^{(\alpha)T}, \tag{48}$$

$$\hat{P}_{\nu}^{(\alpha)}\hat{P}_{\nu'}^{(\alpha)}=\hat{P}_{\nu'}^{(\alpha)}\hat{P}_{\nu}^{(\alpha)},\quad \nu,\nu'=1,2,\ldots,\Lambda^{(\alpha)}. \tag{49}$$

(i.e. the matrices $\hat{P}_{\nu}^{(\alpha)}$ will be symmetric and commute with one another) are the direct consequences of (47). By the way, such a simplification (the independence of $\hat{P}_{\nu}^{(\alpha)}$ of any other resonance parameters) is justified at least when $\Lambda^{(\alpha)}$ and the number $N$ of open channels is very large. Then it follows from the properties (48) and $\hat{P}_{\nu}^{(\alpha)}=\hat{P}_{\nu}^{(\alpha)*}=\hat{P}_{\nu}^{(\alpha)2}$, Trace$\hat{P}_{\nu}^{(\alpha)}=1$ (from (44)) that the $\hat{P}_{\nu}^{(\alpha)}$ are real, i.e.

$$\hat{P}_{\nu}^{(\alpha)}=\hat{P}_{\nu}^{(\alpha)*}. \tag{50}$$

## 5. Final Remarks. Conclusions and Perspectives

The presented review contains the results of the almost complete study of the non-relativistic S-matrix analytic structure for *arbitrary* central, non-central (tensor), and parity-violating $T$-invariant interactions, linear or nonlinear, with the possible absorption and/or generation of particles inside a sphere of small radius $r\leq a$ passing in the external range ($a<r<\infty$) into a centrifugal barrier with the possible presence of decreasing (more rapidly than any exponential function, according to the exponential law, or the Yukawa-potential, *etc.*) tails for the one-channel and discrete-many-channel scatterings. This study is based on some general mathematical assumptions like the possibility of the S-matrix analytic continuation into the regions of complex values of the wave numbers or kinetic energies of particles and the completeness conditions for the external wave functions, as well as on the physical principles like the causality and some kinds of the symmetry for the $S$-matrix.

It is rather curious how the results of a research based on the well-known cognitive principle "with the least number of assumptions to obtain the most number of results of a rather general physical and mathematical character" can also help to reveal some specific physical phenomena and effects: (a) the enhancement phenomena caused by the parity violation; (b) the phenomenon of time resonances (explosions) formed from the strongly overlapping energy resonances of high-energy many-channel nuclear reactions mentioned in Section 5 (see [21]); (c) the paradoxical delay-advance phenomenon in the center-of-mass system in the range of nuclear isolated resonances distorted by the non-resonance background, and the resolution of such paradoxical phenomenon by passing to the laboratory system of reference, where it is eliminated by the correct phase analysis with introducing an additional phase parameter describing the space-time shift caused by the real motion of the decaying compound nucleus in the laboratory system (see [22–25]).

In the existing publications concerning the analytic structure of the $S$-matrix and the scattering (collision) amplitude, the topic of *dispersion relations* (certain integral relations for the scattering amplitude) attracts often a lot of attention. A great number of papers and the majority of manuals on quantum mechanics considering the range of not very high energies are dedicated to this topic (see, e.g., [5, 17, 26–28] with extensive bibliography therein). This topic is studied in details both for the known potential interactions and the microscopically unknown interactions. Several author's papers on dispersion relations considered some applications to the nuclear optical model and compound-nucleus processes [29,30].

As for the future perspectives, it is possible to propose (a) the study of the enhancement phenomena caused by violations of $T$-invariance, quite similarly to the enhancement phenomena caused by parity violations of the $S$-matrix [10], in addition to the model study in [31, 32], and (b) the study of the $S$-matrix analytic structure for arbitrary interactions which externally (in the range $r > a$) pass into a centrifugal barrier and a screened Coulomb barrier (the last one is namely the Yukawa-type potential differing from the Yukawa potential by the positive sign (repulsion instead of attraction) and by the scale.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АНАЛІТИЧНІ ВЛАСТИВОСТІ $S$-МАТРИЦІ ДЛЯ ДОВІЛЬНИХ ВЗАЄМОДІЙ, ЯКІ В ЗОВНІШНІЙ ОБЛАСТІ ПЕРЕХОДЯТЬ В ДОЦЕНТРОВИЙ ТА ШВИДКО ЗГАСАЮЧІ ПОТЕНЦИАЛИ

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Р е з ю м е

Подано огляд робіт, виконаних автором з 1961 по 2006 роки, з аналітичної структури нерелятивістської унітарної та неунітарної $S$-матриць у випадку довільних взаємодій (і, можливо, з довільними рівняннями руху) всередині сфери радіуса $r \leq a$, які в зовнішній області ($r > a$) переходять в доцентровий та швидко згасаючі (за експоненціальним чи юкавівським законами або згасаючі більш швидко) потенціали. Розглянуто одноканальний та особливі багатоканальні випадки. Накладено умови симетрії деяких типів. Використовуються рівняння Шредінгера для руху частинок в області $r > a$ та умова повноти відповідних хвильових функцій. На заключення представлено програму можливих досліджень як зрозуміле продовження і розширення одержаних результатів.