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Free Vibration of the Axially Loaded System of Two Beams Connected by a Two-Directional Viscoelastic Interlayer

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Свободные колебания системы из двух слоистых балок, соединенных внутренним вязкоупругим слоем и нагруженных осевой силой

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Предлагается аналитический метод решения задач о свободных колебаниях с затуханием слоистых балок, состоящих из двух внешних слоев, соединенных внутренним вязкоупругим слоем, который рассматривается как двунаправленное винклеровское основание. Верхний внешний слой, нагруженный осевой постоянной силой, описывается на основе модели Бернулли–Эйлера. Нижний внешний слой моделируется с помощью модели Тимошенко. Свободные колебания описываются однородной системой связанных дифференциальных уравнений в частных производных. После разделения переменных в исходной системе дифференциальных уравнений решается краевая задача. В результате получено три комплексных уравнения для определения частот и мод свободных колебаний. Задача о свободных колебаниях рассмотрена для произвольных начальных условий и различных осевых сил.

Introduction. In recent years, the Bernoulli–Euler and Timoshenko models have been applied to the solution of various mechanical and building vibration problems. The Bernoulli–Euler model has been used for the solution of the problem of vibration of sandwich beams [1–3]. The problem of free vibration of two axially loaded Bernoulli–Euler beams transversally coupled with discrete springs without damping was studied in [1]. The problem of a complex continuous dynamical system was considered in [2] with the use of the classical method and the complete theory of non-damped vibrations. Vibrations of two Bernoulli–Euler elastic beams connected by an elastic interlayer with moving loads was solved in [3].

For the first time, the influence of transverse forces and rotational inertia with the shearing coefficient in a beam was considered in [4]. Natural frequencies for continuous Timoshenko models were studied in [5], and, for discrete-continuous Timoshenko models, this problem was solved in [6].

The property of orthogonality of the complex modes of free vibration for continuous systems with damping was demonstrated in [7–11]; for discrete and discrete-continuous systems with damping, it was demonstrated in [12, 13].

The general method for the solution of problems of free vibration for complex continuous one- and two-dimensional systems with damping for various boundary conditions and different initial conditions was presented in [1]. The application of the Bernoulli–Euler and Timoshenko models to the solution of free-vibration problems for different sandwich beams with damping was considered in [2–4].

The purpose of this paper is the solution of the problem of free vibration of an axially loaded sandwich beam with damping for various axial forces. The calculations of the dynamic displacements for a two-directional interlayer are compared with similar results for a one-directional interlayer.

Statement of the Problem. The physical model of the structural system is an axially loaded sandwich beam with damping, which consists of two homogenous elastic parallel beams of equal length coupled together by a soft viscoelastic interlayer (Fig. 1). The upper external layer is simulated by the Bernoulli–Euler model and is loaded by a constant axial force P. The lower external layer is simulated by the Timoshenko model. The beams are supported at their ends. The viscoelastic interlayer has the characteristics of a homogenous continuous two-directional Winkler base [5] and is described by the Voigt–Kelvin model [14–18].

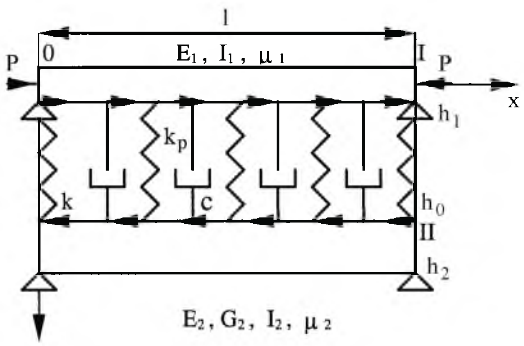

Fig. 1. Dynamical model of an axially loaded system of two beams connected by a two-parameter viscoelastic interlayer.

The mathematical model of the problem is represented by the following system of conjugate partial differential equations describing small transverse vibrations of the physical system:

$$
\begin{cases}
R_1 \dfrac{\partial^4 w_1}{\partial x^4} - P \dfrac{\partial^2 w_1}{\partial x^2} + \mu_1 \dfrac{\partial^2 w_1}{\partial t^2} - \left(\dfrac{\partial^2 w_1}{\partial x^2} \dfrac{h_1}{2} + \dfrac{d\psi_2}{dx} \dfrac{h_2}{2} \right) \dfrac{h_1}{2} k_p + \\
+ \left(k + \dfrac{\partial}{\partial t} \right) (w_1 - w_2) = 0, \\
N \left(\dfrac{\partial^2 w_2}{\partial x^2} - \dfrac{\partial \psi_2}{\partial x} \right) - \mu_2 \dfrac{\partial^2 w_2}{\partial t^2} + \left(k + c \dfrac{\partial}{\partial t} \right) (w_1 - w_2) = 0, \\
R_2 \dfrac{\partial^2 \psi_2}{\partial x^2} + N \left(\dfrac{\partial w_2}{\partial x} - \psi_2 \right) - \Xi_2 \dfrac{\partial^2 \psi_2}{\partial t^2} - \left(\dfrac{dw_1}{dx} \dfrac{h_1}{2} + \psi_2 \dfrac{h_2}{2} \right) \dfrac{h_2}{2} k_p = 0,
\end{cases}
\tag{1}
$$

where

$$R_1 = E_1 I_1, \quad R_2 = E_2 I_2, \quad N = k' G_2 F_2,$$

$$\mu_1 = \rho_1 F_1, \quad \mu_2 = \rho_2 F_2, \quad \Xi_2 = \rho_2 I_2, \quad k_p = \frac{k}{2(1+\nu_0)};$$

$w_1 = w_1(x,t)$ and $w_2 = w_2(x,t)$ are transverse deflections of beams I and II, $\psi_1 = \psi_1(x,t)$ and $\psi_2 = \psi_2(x,t)$ are the angles of rotation of the cross sections of beams I and II, E_1 and E_2 are the elastic moduli of the material for beams I and II, I_1 and I_2 are the moments of inertia of the cross sections of beams I and II, P is the axial force, F_1 and F_2 are the areas of the cross sections of beams I and II, G_2 is the Kirchhoff modulus of the material of beam II, ρ_1 and ρ_2 are the mass densities of the material of beams I and II, k' is the shearing coefficient, k is the transverse coefficient of elasticity of the interlayer, k_p is the longitudinal coefficient of elasticity of the interlayer, c is the coefficient of viscosity of the interlayer, h_1 and h_2 are the heights of beams I and II, h_0 is the height of the interlayer, and l is the length of beams I and II. The bending moment and transverse force in beam II were determined by Timoshenko [9] in the form

$$M_1 = -R_1 \frac{\partial^2 w_1}{\partial x^2}, \; Q_1 = -R_1 \frac{\partial^3 w_1}{\partial x^3}, \; M_2 = -R_2 \frac{\partial \psi_2}{\partial x}, \; Q_2 = k' G_2 F_2 \gamma_2, \tag{2}$$

where $\frac{\partial w_1}{\partial x} = \psi_1$, $\frac{\partial w_2}{\partial x} = \psi_2 + \gamma_2$, and $\gamma_2 = \gamma_2(x,t)$ is the angle of shearing in beam II.

Solution of the Boundary-Value Problem. Substituting

$$\begin{bmatrix} w_1 \\ w_2 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} W_1(x) \\ W_2(x) \\ \Psi_2(x) \end{bmatrix} \exp(i\nu t) \tag{3}$$

for w_1, w_2, and ψ_2 in the system of differential equations (1), we represent the homogenous system of conjugate ordinary differential equations describing the complex modes of vibration of the beams in the following form:

$$\begin{cases} R_1 \dfrac{d^4 W_1}{dx^4} - (P + k'_{p1}) - \mu_1 W_1 \nu^2 + (k + ic\nu)(W_1 - W_2) - k'_{p2} \dfrac{d\Psi_2}{dx} = 0, \\ N\left(\dfrac{d^2 W_2}{dx^2} - \dfrac{d\Psi_2}{dx} \right) + \mu_2 W_2 \nu^2 + (k + ic\nu)(W_1 - W_2) = 0, \\ R_2 \dfrac{d^2 \Psi_2}{dx^2} + N\left(\dfrac{dW_2}{dx} - \Psi_2 \right) + \Xi_2 \Psi_2 \nu^2 - k''_{p2} \Psi_2 - k''_{p1} \dfrac{dW_1}{dx} = 0, \end{cases} \tag{4}$$

where

$$k'_{p1} = \frac{h_1^2}{4} k_p, \quad k''_{p2} = \frac{h_2^2}{4} k_p, \quad k''_{p1} = k'_{p2} = \frac{h_1 h_2}{4} k_p \quad [3];$$

$W_1 = W_1(x)$ and $W_2 = W_2(x)$ are the complex transverse vibration modes of beams I and II, $\Psi_2 = \Psi_2(x)$ is the complex rotation mode of vibration of beam II, ν is the complex frequency of vibration of beams I and II, and t is time.

Seeking a particular solution of the system of differential equations (4) in the form

$$\begin{bmatrix} W_1 \\ W_2 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} A \\ B \\ \Theta \end{bmatrix} \exp(rx), \tag{5}$$

we obtain the following homogeneous system of linear algebraic equations:

$$\begin{cases} A[R_1 r^4 - (P + k'_{p1}) r^2 - \mu_1 \nu^2 + k + ic\nu] - B(k + ic\nu) - \Theta\, k'_{p2} r = 0, \\ A(k + ic\nu) + B(N r^2 + \mu_2 \nu^2 - k - ic\nu) - \Theta N r = 0, \\ A\, k''_{p1} r - B\, N r - \Theta(R_2 r^2 - N + \Xi_2 \nu^2 - k''_{p2}) = 0. \end{cases} \tag{6}$$

Expanding the determinant of the characteristic matrix of the system of equations (6) and equating it to zero, namely,

$$\begin{vmatrix} R_1 r^4 - (P + k''_{p1}) r^2 + n_1 & -(k + ic\nu) & -k'_{p2} r \\ (k + ic\nu) & (N r^2 - n_2) & -N r \\ -k''_{p1} r & N r & R_2 r^2 = N + \Xi_2 \nu^2 - k''_{p2} \end{vmatrix} = 0, \tag{7}$$

we obtain the characteristic equation in the form of an algebraic equation, namely,

$$r^8 + a_{11} r^6 + a_{22} r^4 + a_{33} r^2 + a_{44} = 0, \tag{8}$$

with the roots $r_j = (-1)^{j-1} i\lambda_\upsilon$, $j = (2\upsilon - 1), 2\upsilon$, $\upsilon = 1, 2, 3, 4$, where $n_1 = k + ic\nu - \mu_1 \nu^2$, $n_2 = k + ic\nu - \mu_2 \nu^2$, and a_{11}, a_{22}, a_{33}, and a_{44} are constant coefficients.

After application of the Euler formulas, the solution of the system of differential equations (4) consists of the fundamental system of solutions

$$W_1(x) = \sum_{\upsilon=1}^{4} A_\upsilon^* \sin \lambda_\upsilon x + A_\upsilon^{**} \cos \lambda_\upsilon x,$$

$$W_2(x)=\sum_{v=1}^{4} B_v^* \sin\lambda_v x + B_v^{**}\cos\lambda_v x,$$

$$\Psi_2(x)=\sum_{v=1}^{4} \Theta_v^* \cos\lambda_v x + \Theta_v^{**}\sin\lambda_v x, \tag{9}$$

where $\Psi_1(x)=\dfrac{dW_1}{dx}$, $A_v^*, A_v^{**}, B_v^*, B_v^{**}, \Theta_v^*$, and Θ_v^{**} are constants, and $\lambda_v=\alpha_v+i\beta_v$ is a parameter that describes the roots of the characteristic equation (8).

In agreement with (6), the constants in (9) satisfy the following relations:

$$a_v^*=\frac{B_v^*}{A_v^*},\quad a_v^{**}=\frac{B_v^{**}}{A_v^{**}},\quad b_v^*=\frac{\Theta_v^*}{A_v^*},\quad b_v^{**}=\frac{\Theta_v^{**}}{A_v^{**}}, \tag{10}$$

where

$$a_v^*=a_v^{**}=a_v=\frac{k''_{p1}Nr^2-(k+icv)RR_2}{(Nr)^2+NN_2RR_2},$$

$$b_v^*=b_v=\frac{1}{k'_{p2}r}\left[RR_1-(k+icv)\frac{k''_{p1}Nr^2-(k+icv)RR_2}{(Nr)^2+NN_2RR_2}\right],\quad b_v^*=-b_v^{**}, \tag{11}$$

$$RR_1=R_1r^4-(P+k'_{p1})r^2+n_1,\quad RR_2=R_2r^2-N+v^2\Xi_2=k''_{p2},$$

$$NN_2=Nr^2-n_2.$$

Substituting (10) in (9), we obtain the general solution of the system of differential equations (4) in the following form:

$$W_1(x)=\sum_{v=1}^{4} A_v^*\sin\lambda_v x + A_v^{**}\cos\lambda_v x,$$

$$\Psi_1(x)=\sum_{v=1}^{4} \lambda_v A_v^*\cos\lambda_v x - \lambda_v A_v^{**}\sin\lambda_v x,$$

$$W_2(x)=\sum_{v=1}^{4} a_v(A_v^*\sin\lambda_v x + A_v^{**}\cos\lambda_v x), \tag{12}$$

$$\Psi_2(x)=\sum_{v=1}^{4} b_v(A_v^*\cos\lambda_v x - A_v^{**}\sin\lambda_v x).$$

In order to solve the boundary-value problem, we use the following boundary conditions:

$$W_1(0)=0, \quad W_1(l)=0,$$
$$W_2(0)=0, \quad W_2(l)=0,$$
$$\frac{d\Psi_1}{dx}(0)=0, \quad \frac{d\Psi_1}{dx}(l)=0, \tag{13}$$
$$\frac{d\Psi_2}{dx}(0)=0, \quad \frac{d\Psi_2}{dx}(l)=0.$$

Substituting (12) in (13), we obtain the homogenous system of linear algebraic equations. The matrix of the system obtained has the following form:

$$\mathbf{YX}=0, \tag{14}$$

where $\mathbf{X}=[A_1^*, A_2^*, A_3^*, A_4^*, A_1^{**}, A_2^{**}, A_3^{**}, A_4^{**}]^T$ is the vector of unknowns of the system of equations and

$$\mathbf{Y}=[Y_{i*j}]_{8*8} \tag{15}$$

is the characteristic matrix of the system of equations (14).

The first four equations in (14) are represented in the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ -\lambda_1^2 & -\lambda_2^2 & -\lambda_3^2 & -\lambda_4^2 \\ -\lambda_1 b_1 & -\lambda_2 b_2 & -\lambda_3 b_3 & -\lambda_4 b_4 \end{bmatrix} \begin{bmatrix} A_1^{**} \\ A_2^{**} \\ A_3^{**} \\ A_4^{**} \end{bmatrix} = 0. \tag{16}$$

It follows from the system of equations (16) that $A_1^{**}=A_2^{**}=A_3^{**}=A_4^{**}=0.$

The other four equations in (14) give the following system of equations:

$$\begin{bmatrix} \sin\lambda_1 l & \sin\lambda_2 l & \sin\lambda_3 l & \sin\lambda_4 l \\ a_1\sin\lambda_1 l & a_2\sin\lambda_2 l & a_3\sin\lambda_3 l & a_4\sin\lambda_4 l \\ -\lambda_1^2\sin\lambda_1 l & -\lambda_2^2\sin\lambda_2 l & -\lambda_3^2\sin_3 l & -\lambda_4^2\sin\lambda_4 l \\ -\lambda_1 b_1\sin\lambda_1 l & -\lambda_2 b_2\sin\lambda_2 l & -\lambda_3 b_3\sin\lambda_3 l & -\lambda_4 b_4\sin\lambda_4 l \end{bmatrix} \begin{bmatrix} A_1^* \\ A_2^* \\ A_3^* \\ A_4^* \end{bmatrix} = 0. \tag{17}$$

A condition for the solvability of the system of equations (17) is vanishing of the characteristic determinant, i.e.,

$$\begin{bmatrix} \sin\lambda_1 l & \sin\lambda_2 l & \sin\lambda_3 l & \sin\lambda_4 l \\ a_1\sin\lambda_1 l & a_2\sin\lambda_2 l & a_3\sin\lambda_3 l & a_4\sin\lambda_4 l \\ -\lambda_1^2\sin\lambda_1 l & -\lambda_2^2\sin\lambda_2 l & -\lambda_3^2\sin\lambda_3 l & -\lambda_4^2\sin\lambda_4 l \\ -\lambda_1 b_1\sin\lambda_1 l & -\lambda_2 b_2\sin\lambda_2 l & -\lambda_3 b_3\sin\lambda_3 l & -\lambda_4 b_4\sin\lambda_4 l \end{bmatrix} = 0. \tag{18}$$

Expanding the determinant (18), we obtain the following characteristic equation:

$$\sin\lambda_1 l \sin\lambda_2 l \sin\lambda_3 l \sin\lambda_4 l = 0, \tag{19}$$

where $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$.

The characteristic equation (19) can be rewritten in the form

$$\sin\lambda l = 0, \tag{20}$$

where

$$\lambda = \alpha + i\beta \tag{21}$$

are complex numbers in the general case.

Substituting (21) in (20), we get the equation

$$\sin\alpha l \operatorname{ch}\beta l + i\cos\alpha l \operatorname{sh}\beta l = 0, \tag{22}$$

which has the following roots:

$$\alpha_s = \frac{s\pi}{l}, \quad \beta_s = 0, \quad s = 1, 2, 3, \ldots . \tag{23}$$

In view of (23), relation (21) yields the following identity:

$$\lambda_s = \alpha_s = \frac{s\pi}{l}. \tag{24}$$

Substituting $r = i\lambda_s$ in equation (8) and carrying out the corresponding transformations, we obtain the equation for frequency

$$\nu^6 + b_{11}\nu^5 + b_{22}\nu^4 + b_{33}\nu^4 + b_{55}\nu + b_{66} = 0, \tag{25}$$

from which we determine the sequence of complex natural frequencies

$$\nu_n = i\eta_n \pm \omega_n, \tag{26}$$

where $n = (3s-2), (3s-1), 3s$, and $b_{11}, b_{22}, b_{33}, b_{44}, b_{55}$, and b_{66} are constant coefficients.

Substituting equation (26) in equations (11), we obtain the following formulas for the coefficients of amplitudes:

$$a_n = -\frac{k''_{p1}N\lambda_s^2 + (k + ic\nu_n)RR_2}{(N\lambda_s)^2 - NN_2RR_2},$$

$$b_n = -\frac{1}{ik'_{p2}\lambda_s}\left[RR_1 - (k + ic\nu_n)\frac{k''_{p1}N\lambda_s^2 + (k + ic\nu_n)RR_2}{(N\lambda_s)^2 - NN_2RR_2}\right], \tag{27}$$

$$RR_1 = R_1\lambda_s^4 + (P + k'_{p1})\lambda_s^2 + n_1, \quad RR_2 = -R_2\,\lambda_s^2 - N + \nu_n^2\Xi_2 - k''_{p2},$$

$$NN_2 = -N\,\lambda_s^2 - n_2.$$

Substituting the sequences λ and a_n, b_n in (12), we get the following four sequences of modes for the free vibration of two beams:

$$\begin{aligned} W_{1n}(x) &= \sin\lambda_s x, \\ \Psi_{1n}(x) &= \lambda_s \cos\lambda_s x, \\ W_{2n}(x) &= a_n \sin\lambda_s x, \\ \Psi_{2n}(x) &= b_n \cos\lambda_s x. \end{aligned} \tag{28}$$

Solution of the Initial-Value Problem. The complex equation of motion

$$T = \Phi\exp(i\nu t), \tag{29}$$

in the case $\nu = \nu_n$ can be rewritten in the form

$$T_n = \Phi_n \exp(i\nu_n t), \tag{30}$$

where Φ_n is the Fourier coefficient.

Free vibration of beams is represented in the form of a Fourier series based on the complex eigenfunctions [17], i.e.,

$$\begin{bmatrix} w_{1n} \\ w_{2n} \\ \psi_{2n} \end{bmatrix} = \begin{bmatrix} \sum\limits_{n=1}^{\infty} W_{1n}(x) \\ \sum\limits_{n=1}^{\infty} W_{2n}(x) \\ \sum\limits_{n=1}^{\infty} \Psi_{2n}(x) \end{bmatrix} \Phi_n \exp(i\nu_n t). \tag{31}$$

From the system of equations (4), after performing algebraic transformations, adding the equations together, and then integrating them on the sides from 0 to l, we establish the property of orthogonality of the eigenfunctions for two beams coupled together by a two-directional viscoelastic interlayer:

$$\int_0^l [i(\nu_n + \nu_m)(\mu_1 W_{1n} W_{1m} + \mu_2 W_{2n} W_{2m} + \Xi_2 \Psi_{2n}\Psi_{2m}) +$$

$$+ c(W_{1n} - W_{2n})(W_{1m} - W_{2m})]dx = N_n \delta_{mn}, \tag{32}$$

where δ_{mn} is the Kronecker delta and

$$N_n = \int_0^l [2iv_n(\mu_1 W_{1n}^2 + \mu_2 W_{2n}^2 + \Xi_2 \Psi_{2n}^2 + c(W_{1n} - W_{2n})^2] dx. \qquad (33)$$

The following initial conditions form the basis for the solution of the problem of free vibrations:

$$\begin{aligned} &w_1(x,0) = w_{01}, \quad w_2(x,0) = w_{02}, \quad \psi_2(x,0) = \psi_{02}, \\ &w_1^0(x,0) = w_{01}^0, \quad w_2^0(x,0) = w_{02}^0, \quad \psi_2^0(x,0) = \psi_{02}^0. \end{aligned} \qquad (34)$$

Applying conditions (34) to series (31) and taking into account the property of orthogonality (32), we obtain the following formula for the Fourier coefficients:

$$\Phi_n = \frac{U_n}{N_n}, \qquad (35)$$

where

$$U_n = \int_0^l \{\mu_1(iv_n W_{1n} w_{01} + W_{1n} w_{01}^0) + \mu_2(iv_n W_{2n} w_{02} + W_{2n} w_{02}^0) +$$

$$+ \Xi_2(iv_n \Psi_{2n} \psi_{02} + \Psi_{2n} \psi_{02}^0) + c[(W_{1n} - W_{2n})(w_{01} - w_{02})]\} dx. \qquad (36)$$

Substituting (28), (30), and (35) in (31) and performing trigonometric and algebraic transformations, we determine the final free vibration of beams [2–4]:

$$\begin{aligned} w_1 &= \sum_{n=1}^{\infty} e^{-\eta_n t} |W_{1n}| |\Phi_n| [\cos(\omega_n t + \chi_{1n} + \varphi_n), \\ w_2 &= \sum_{n=1}^{\infty} e^{-\eta_n t} |W_{2n}| |\Phi_n| [\cos(\omega_n t + \chi_{2n} + \varphi_n), \\ \psi_2 &= \sum_{n=1}^{\infty} e^{-\eta_n t} |\Psi_{2n}| |\Phi_n| [\cos(\omega_n t + \theta_{2n} + \varphi_n), \end{aligned} \qquad (37)$$

where

$$\begin{aligned} &|W_{1n}| = \sqrt{X_{1n}^2 + Y_{1n}^2}, \ |W_{2n}| = \sqrt{X_{2n}^2 + Y_{2n}^2}, \ |\Psi_{2n}| = \sqrt{\Lambda_{2n}^2 + \Omega_{2n}^2}, \\ &\chi_{1n} = \arg W_{1n}, \ \chi_{2n} = \arg W_{2n}, \ \theta_{2n} = \arg \Psi_{2n}, \\ &|\Phi_n| = \sqrt{C_n^2 + D_n^2}, \ \varphi_n = \arg \Phi_n, \end{aligned} \qquad (38)$$

and

$$\begin{aligned} &X_{1n} = \mathrm{Re} W_{1n}, \quad Y_{1n} = \mathrm{Im} W_{2n}, \quad X_{1n} = \mathrm{Re} W_{2n}, \quad Y_{2n} = \mathrm{Im} W_{2m}, \\ &\Lambda_{2n} = \mathrm{Re}\, \Psi_{2n}, \quad \Omega_{2n} = \mathrm{Im}\, \Psi_{2n}, \quad C_n = \mathrm{Re} \Phi_n, \quad D_n = \mathrm{Im} \Phi_n. \end{aligned} \qquad (39)$$

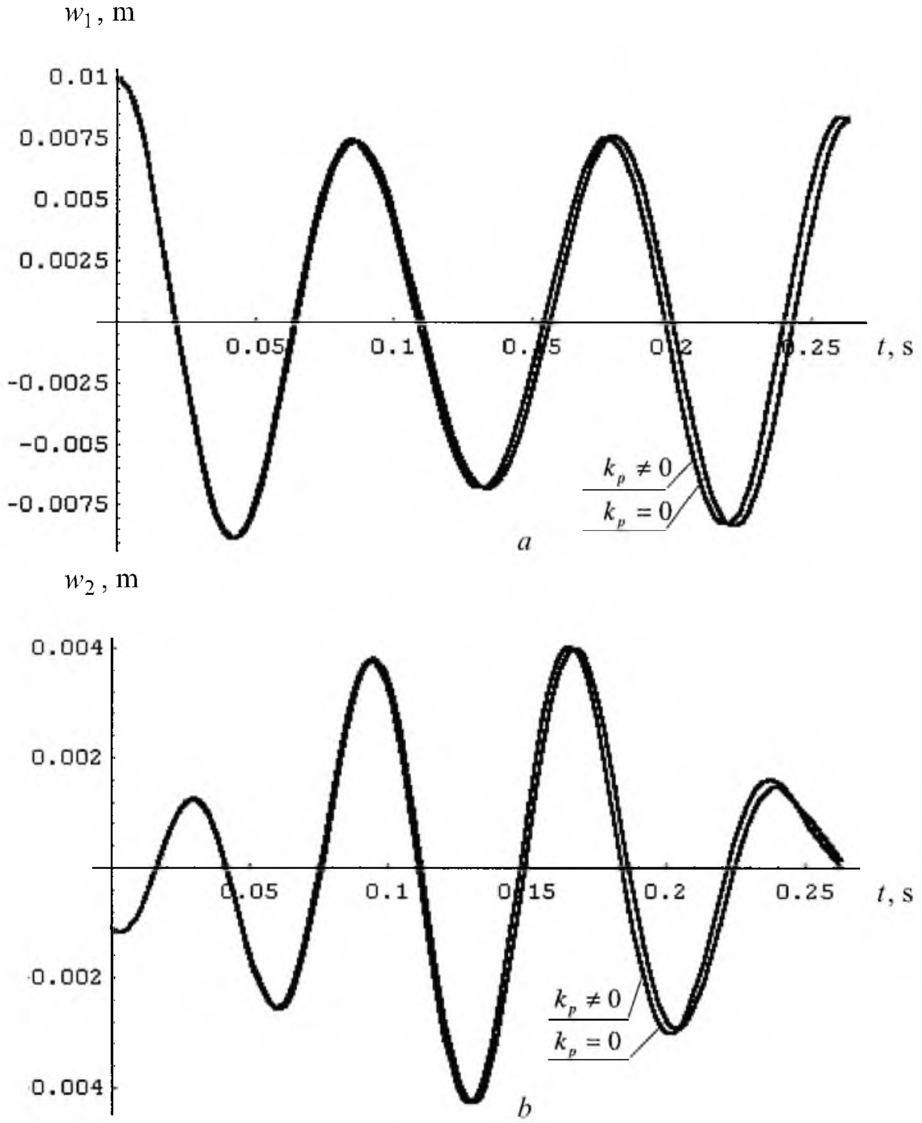

Fig. 2. Distribution of the dynamic displacements $w_1(x,t)$ (a) and $w_2(x,t)$ (b) of beams I and II for the axial force $P = -4 \cdot 10^5$ N and $x = 0.5l$.

Numerical Results. On the basis of the method developed, the investigation of sandwich system was carried out. Calculations were performed for the following data: $E_1 = E_2 = E = 2.1 \cdot 10^{11}$ N $\cdot$ m^{-2}, $E_0 = 10^8$ N $\cdot$ m^{-2}, $k = (E_0 b_0)/h_0$, $k_p = k/[2(1+\nu_0)]$, $l = 6$ m, $\rho_1 = \rho_2 = 7.8 \cdot 10^3$ N $\cdot$ s^2 / m^4, $k' = 0.84$, $P = \{-4 \cdot 10^5, 4 \cdot 10^5\}$ N, $G_2 = E_2/[2(1+\nu_0)]$, $c = 2 \cdot 10^2$ N $\cdot$ s $\cdot$ m^{-2}, $b_1 = b_2 = b_0 = 0.07$ m, $h_1 = 0.1$ m, $h_2 = 0.2$ m, $h_0 = 0.2$ m, $F_1 = b_1 h_1$, $F_2 = b_2 h_2$, $I_1 = (b_1 h_1^3)/12$, $I_2 = (b_2 h_2^3)/12$, $\nu_0 = 0.2$.

In order to find the Fourier coefficient Φ_n (35), the following initial conditions were assumed:

$$w_{01} = 0.01\sin\left(\frac{\pi x}{l}\right),\ w_{01}^{0} = 0,$$
$$w_{02}^{0} = -0.001\sin\left(\frac{\pi x}{l}\right),\ w_{02}^{0} = 0, \qquad (40)$$
$$\psi_{02} = -0.001\cos\left(\frac{\pi x}{l}\right),\ \psi_{02}^{0} = 0.$$

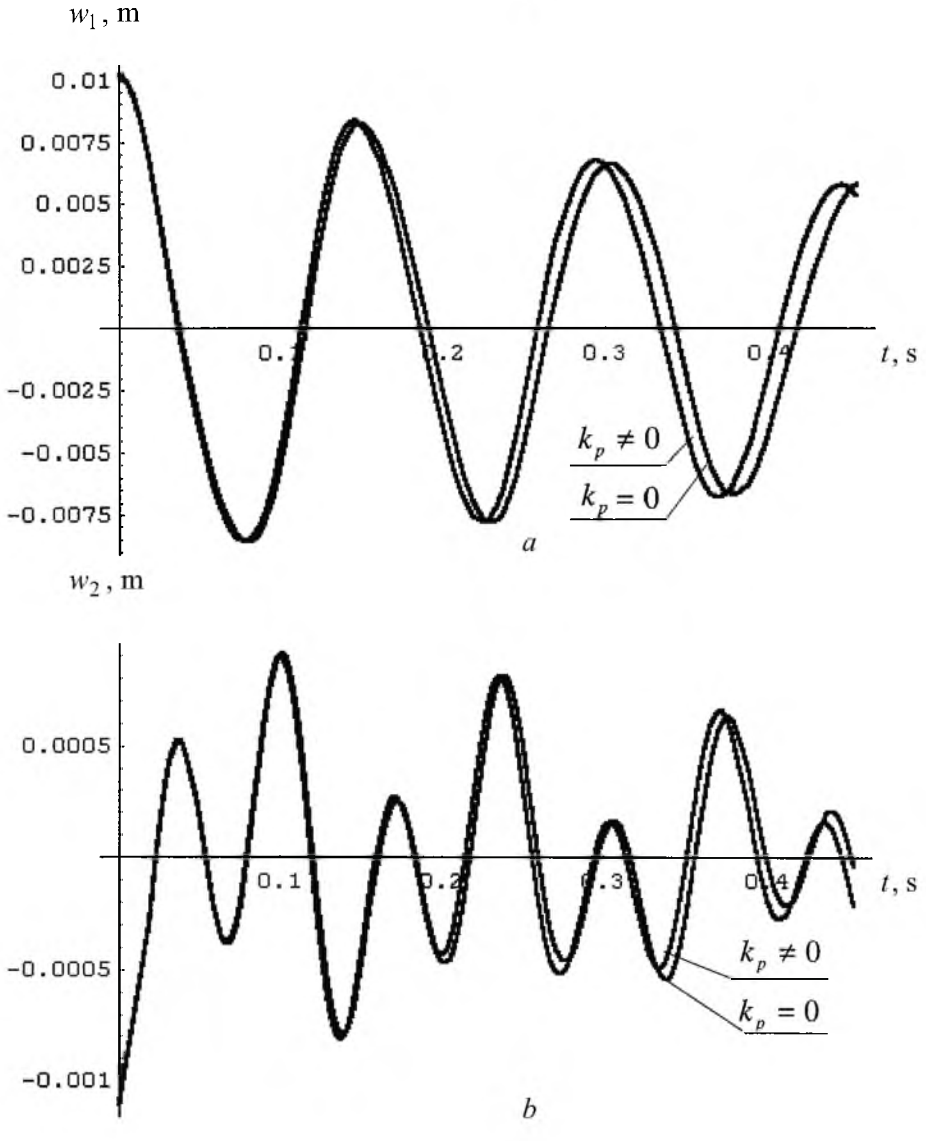

Fig. 3. Distribution of the dynamic displacements $w_1(x,t)$ (a) and $w_2(x,t)$ (b) of beams I and II for the axial force $P = 4 \cdot 10^5$ N and $x = 0.5l$.

The effect of various axial forces in the sandwich beams with damping is shown in Figs. 2–3. The distributions of displacements for external layers described according to the Bernoulli–Euler model are displayed in Figs. 2a and

3a. The distributions of displacements for lower external layers represented using the Timoshenko model are displayed in Figs. 2b and 3b. Figures 2–3 show the motion of the sandwich beams with time t for $x = 0.5l$. Calculations of the dynamic displacements for a two-directional interlayer $k \neq 0$ and $k_p \neq 0$ are compared with a one-directional interlayer $k \neq 0$ and $k_p = 0$. It follows from the comparison of the results for one- and two-directional interlayers that, as time t increases, the dynamic displacements rapidly decay in the sandwich beams for which the interlayer corresponds to a two-directional Winkler base $k_p \neq 0$. This difference becomes more substantial in the case of loading of sandwich constructions by compressing forces.

The dynamic displacements $w_1(x,t)$ and $w_2(x,t)$ of beams I and II for various loadings by axial forces rapidly decay with time t in the sandwich beams for which the interlayer corresponds to a two-directional Winkler base $k_p \neq 0$.

Р е з ю м е

Запропоновано аналітичний метод розв'язку задач про вільні коливання зі згасанням шаруватих балок, що складаються з двох зовнішніх шарів, з'єднаних внутрішнім в'язкопружним шаром. Останній розглядається в якості двонапрямленої вінклерівської основи. Верхній зовнішній шар, що навантажується осьовою постійною силою, описується на основі моделі Бернуллі–Ейлера. Нижній зовнішній шар моделюється за моделлю Тимошенка. Вільні коливання описуються однорідною системою зв'язаних диференціальних рівнянь в частинних похідних. Після розділення змінних у вихідній системі диференціальних рівнянь розв'язується крайова задача. У результаті отримано три комплексних рівняння для визначення частот і мод вільних коливань. Задача про вільні коливання розглянута для довільних початкових умов і різних осьових сил.

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