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ON THE RATE OF CONVERGENCE OF BARRIER OPTION PRICES IN BINOMIAL MARKET TO THOSE IN CONTINUOUS TIME MARKET

We estimate the rate of convergence of barrier option price in a discrete time binomial market to such in a continuous time market.

1. Introduction

A barrier option is a derivative with a payoff that depends on the fact whether asset price crosses certain level during certain time interval. Thus, payment for barrier option depends on the behavior of the price asset during all the time interval, i.e. barrier option is a particular case of exotic option.

The simplest barrier options are calls and puts that are knocked out or knocked in by the underlying asset itself. The payoff of a knock-out option is made if underlying asset price does not cross the barrier, such options are of two types: if asset price does not cross the barrier below, then such an option is called “up-and-out”, if from above – “down-and-out”. Payoff of a knock-in option is made if underlying asset price crosses the barrier, they also are of two types accordingly: “up-and-in” and “down-and-in”. Altogether there are eight types of barrier options.

For example, the payoff function of up-and-in option is given by

$$C = \begin{cases} (S_T - K)^+, & \text{if } \max_{0\le t\le T} S_t \ge H, \\ 0 & \text{else,} \end{cases}$$

where H is a barrier level ($H > S_0$ and $H > K$), K is a strike price. Payoffs for the rest options are determined in the same way. Barrier options are among the most popular path-dependent option traded in exchanges worldwide and also over-the-counter markets.

Invited lecture.

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The problem of pricing and hedging barrier option in the models with continuous time is rather complete, and analytical formulae for the prices of such options are known only in the most easy cases. Therefore the problem of asymptotic estimation of the prices of such options arises. The simplest asymptotic methods is the method of time discretization, which can be described in the following way. Time interval is divided into m equal parts and now the asset price model with discrete time is considered. In such a formulation we can approximately calculate option price using Monte Carlo simulations, modelling the path of the underlying asset price. From the other side, the opposite problem could arise: let we have analytical formula for option price in continuous time model. Then the demand may come to estimate the price of the option with payoff realized when the asset price crosses the barrier level, and this price is observed only in certain time moments (for example, daily when stock exchange is closing).

From the practical point of view, when we approximately estimate the price of the option it is important to know the quality of such an estimation, i.e. the order of the error.

In [1] authors introduce a simple continuity correction for approximate pricing of discrete barrier option. Their method uses formulae for the prices of continuous barrier options but shifts the barrier to correct for discrete monitoring. Compared with using the unadjusted continuous price, their formula reduces the error from $O(\frac{1}{\sqrt{m}})$ to $o(\frac{1}{\sqrt{m}})$, as the number of monitoring points m increases. The correction is justified both theoretically and experimentally.

Theorem 1. [1] *Let $V(H)$ be the price of a continuously monitored knock-in or knock-out down call or put with barrier H, and let $V_m(H)$ be the price of the corresponding discrete monitored barrier option. Then*

$$V_m(H) = V(He^{\pm\beta\sigma\sqrt{T/m}}) + o(\frac{1}{\sqrt{m}}),$$

where $+$ applies if $H > S_0$, and $-$ applies if $H < S_0$, $\beta = -\zeta(1/2)/\sqrt{2\pi} \approx 0.5826$, with ζ the Riemann zeta function.

The paper [6] extends an approximation by Broadie et al. in [1] for discretely monitored barrier options by covering more cases and giving a simpler proof. The paper [4] also continues the work of Broadie and determine formulae to estimate the price of discrete up-and-out/in calls, down-and-out/in puts and double barrier option. The methods used here lead to slightly different barrier correction formulae. In [2] the rate of convergence for lookback options and other exotic options is obtained.

The model considered in [8] investigates the rate of convergence of option price in discrete market, but this price is not fair in the sense that it might be not unique. Discrete market, generated by the increments of geometric

Brownian motion, is not complete, so there are many "fair prices". Thus it would be better to have result for convergence of the unique price in complete market. That because in our work we consider discrete binomial market and investigate the rate of convergence of fair price of barrier option in such market to correspondent price on continuous market. We have proved that the rate of convergence is $\ln n/\sqrt{n}$, where n is the number of periods in the binomial market.

2. Main result

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with filtration $\{\mathcal{F}_t, t \geq 0\}$, $\{W_t, t \geq 0\}$ is standard $\mathcal{F}_t$-Brownian motion on it. Consider Black–Scholes financial market model, where we have two assets: riskless (bond), whose price at the moment t equals

$$B_t = B_0 \exp\Big\{ \int_0^t r_s ds \Big\},$$

and a risky asset (stock), whose price is

$$S_t = S_0 \exp\Big\{ \int_0^t \mu_s ds + \sigma W_t \Big\},$$

where W_t is standard Brownian motion defined before. Volatility $\sigma > 0$ is assumed to be constant. For simplicity, we assume that P itself is a martingale measure for discounting process of risky asset price, i.e. $\mu_t = r_t - \sigma^2/2$. Besides, we demand the interest rate r_t to be Lipschitz continuous, i.e. for every $t, s \in [0, T]$

$$|r_t - r_s| \leq C|t - s|, \tag{1}$$

where C is a constant.

In the market with continuous time the fair option price is defined as the expectation of discounting payoff for the option given martingale measure. Let I_A denote the indicator of an event A, $M_T = \max\{S_t, t \in [0, T]\}$, $m_t = \min\{S_t, t \in [0, T]\}$. Then, for instance, European up-and-out call option price is given by

$$V(H) = E\bigg(\exp\Big\{ - \int_0^T r_t dt \Big\} (S_T - K)^+ I_{\{M_T < H\}} \bigg),$$

where $K > 0$ is a strike price, $H > S_0$ is a barrier, and European down-and-in put option price is given by

$$V(H) = E\bigg(\exp\Big\{ - \int_0^T r_t dt \Big\} (K - S_T)^+ I_{\{m_T \leq H\}} \bigg),$$

where $H < S_0$ is a barrier. In Merton's paper [7] an explicit form for the price of knock-out call option is established, when the risk-neutral interest rate r is constant.

Now consider a binomial market model with discrete time, which is constructed as follows. Divide time interval $[0, T]$ into $n \geq 1$ parts, define $\Delta = \frac{T}{n}$, $t_i = i\Delta$, $i = 0, \dots, n$. Let ξ_i, $i = 0, \dots, n-1$ be independent identically distributed random variables, such that $P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}$. The risky asset price in the binomial market model is defined as

$$S^b_{t_i} = S_0 \exp\Big\{\sum_{j=0}^{i-1}(\mu_j\Delta + \sigma\xi_j\sqrt{\Delta})\Big\}, i = 1, \dots, n;$$

on $[t_i, t_{i+1})$ we put $S^b_t = S^b_{t_i}$, and set the interest rate to be equal to r_{t_i}. Instead of Brownian motion, the role of "random driver" of financial market in the binomial model is played by a random walk $\{\Xi_i\}$, defined as

$$\Xi_i := \sum_{j=0}^{i-1} \xi_j.$$

An analogue of European up-and-out call option in the binomial model has the payoff function $(S^b_T - K)^+ I_{\{M^b_T < H\}}$, consequently, the price is

$$V^b_n(H) = E\bigg(\exp\Big\{-\sum_{i=0}^{n-1} r_{t_i}\Delta\Big\}(S^b_T - K)^+ I_{\{M^b_T < H\}}\bigg),$$

where $M^b_T = \max_{0 \leq i \leq n} S^b_{t_i} = \max_{t\in[0,T]} S^b_t$.

The following is the main result about convergence of price in binomial model to the one in continuous model. We need the following result.

Theorem 2. *The difference of European up-and-out call options fair prices in discrete binomial and continuous models under the assumption* (1) *satisfies*

$$V(H) - V^b_n(H) = O\Big(\frac{\ln n}{\sqrt{n}}\Big),\ n \to \infty.$$

Proof. In the following C will denote a generic positive constant, which may depend only on σ, the Lipschitz continuity parameter of r_t, H, K, S_0, i.e. the inputs of our problem.

In order to prove our result, we will use an approximation result from [8]. In the discrete time market define a discretized version of S:

$$S^d_{t_i} = S_0 \exp\Big\{\sum_{j=0}^{i-1}(\mu_j\Delta + \sigma Z_j\sqrt{\Delta})\Big\}, i = 1, \dots, n,$$

where $Z_j = (W_{t_{j+1}} - W_{t_j})/\sqrt{\Delta}$, and consider European up-and-out call option with a payoff $(S_T^d - K)_+ I_{\{M_T^d < H\}}$, where $M_T^d = \max_{0\le i\le n} S_{t_i}$. Its fair price is

$$V_n^d(H) = E\biggl(\exp\Bigl\{-\sum_{i=0}^{n-1} r_{t_i}\Delta\Bigr\}(S_T^d - K)^+ I_{\{M_T^d<H\}}\biggr).$$

It is proved in [8] that $V(H) - V_n^d(H) = O(1/\sqrt{n})$, $n\to\infty$. Thus, it is enough to prove that $V_n^d(H) - V_n^b(H) = O(\ln n/\sqrt{n})$, $n\to\infty$.

It is clear that

$$|V_n^d(H) - V_n^b(H)| \le C\left|E\bigl((S_T^d - K)^+ I_{\{M_T^d<H\}}\bigr) - E\bigl((S_T^b - K)^+ I_{\{M_T^b<H\}}\bigr)\right|. \tag{2}$$

Now we apply the result of Komlós, Major and Tusnády [5]. It says that for any given $\lambda > 0$ it is possible to construct independent random variables $\eta_i \overset{d}{=} \xi_i$ and independent standard random variables ζ_i, $0\le i\le n-1$, such that for some positive constants K

$$P\Bigl(\max_{0\le i\le n-1}|S_i - T_i| > K\ln n + x\Bigr) \le Ke^{-\lambda x}, \tag{3}$$

where

$$S_i = \sum_{j=0}^{i}\eta_j, \quad T_i = \sum_{j=0}^{i}\zeta_j.$$

Note that (3) implies $E(\max_{0\le i\le n-1}|S_i - T_i|^2) \le C\ln^2 n$. Indeed, denoting $R = \max_{0\le i\le n-1}|S_i - T_i|$, we have

$$\begin{aligned} E(R^2) &\le (2K+2)^2\ln^2 n + E(R^2 I_{\{R>(2K+2)\ln n\}}) \\ &\le C\ln^2 n + \int_0^\infty P(R^2 > (2K+2)^2\ln^2 n + x)dx \\ &\le C\ln^2 n + \int_0^\infty P(R > (K+1)\ln n + x/2)dx \\ &\le C\ln^2 n + Kn^{-\lambda}\int_0^\infty e^{-\lambda\sqrt{x/2}}dx \le C\ln^2 n. \end{aligned}$$

In the following we will assume without loss of generality $K\lambda > 1/2$.

As long as $\{\xi_i,\ i = 0,\dots,n-1\} \overset{d}{=} \{\eta_i,\ i=0,\dots,n-1\}$ and $\{Z_i,\ i = 0,\dots,n-1\} \overset{d}{=} \{\zeta_i,\ i=0,\dots,n-1\}$, in order to estimate the difference $|V_n^d(H) - V_n^b(H)|$ we can assume that $\xi_i = \eta_i$ and $Z_i = \zeta_i$, because this will not change the expectations in (2). Now write

$$|V_n^d(H) - V_n^b(H)| \le C(I_1 + I_2),$$

where

$$\begin{aligned}I_1 &= \big|E([(S_T^d-K)^+-(S_T^b-K)^+]I_{\{M_T^b<H\}})\big|\\&\le E(|(S_T^d-K)^+-(S_T^b-K)^+|I_{\{M_T^b<H\}})\le E(|S_T^d-S_T^b|I_{\{M_T^b<H\}}),\\I_2 &= \big|E((S_T^d-K)^+[I_{\{M_T^d<H\}}-I_{\{M_T^b<H\}}])\big|\\&\le CE\big(|I_{\{M_T^d<H\}}-I_{\{M_T^b<H\}}|\big)\\&\le C\big(P(M_T^d<H,M_T^b\ge H)+P(M_T^d\ge H,M_T^b<H)\big).\end{aligned}$$

Processes S^d and S^b are of the form S_0e^x, hence from inequality $|e^x-e^y|\le(e^x+e^y)|x-y|$ we obtain

$$I_1\le CE\left(|S_T^b+S_T^d|\sigma\sqrt{\Delta}\left|\sum_{j=0}^{n-1}(Z_j-\xi_j)\right|I_{\{M_T^b<H\}}\right)$$

Using the Cauchy–Bunyakovsky inequality, we get:

$$I_1\le C\sigma\sqrt{\Delta}\big(E(|S_T^d+S_T^b|^2I_{\{M_T^b<H\}})\big)^{1/2}\times\left(E\Big[\sum_{j=0}^{n-1}\big(Z_j-\xi_{j+l}\big)\Big]^2\right)^{1/2}.$$

Now

$$\begin{aligned}E(|S_T^d+S_T^b|^2I_{\{M_T^b<H\}})&\le 2E\big[\big((S_T^d)^2+(S_T^b)^2\big)I_{\{M_T^b<H\}}\big]\\&\le C\big(E(S_0\exp\{2CT+2\sigma TW_T\})+H^2\big)\le C,\end{aligned}$$

as $\exp\{\sigma TW_T\}$ is integrable, and $|\mu_t|$ is bounded. On the other hand, as it was pointed above,

$$E\Big[\sum_{j=0}^{n-1}\big(Z_j-\xi_{j+l}\big)\Big]^2\le C\ln^2 n,$$

thus we have

$$I_1\le C\sqrt{\Delta}\ln n\le C\frac{\ln n}{\sqrt{n}}.$$

Now turn to I_2. Both probabilities are estimated in a similar manner, so we will estimate only the first one. Write

$$\begin{aligned}&P(M_T^d<H,M_m^b\ge H)\\&\le P(H-\delta\le M_T^d<H,M_T^b\ge H)+P(M_T^d<H-\delta,M_T^b\ge H)\\&\le P(H-\delta\le M_T^d<H)+P(M_T^d<H-\delta,M_T^b\ge H)=:P_1+P_2.\end{aligned}$$

It is easy to see that M_T^d has a bounded density, so $P_1\le C\delta$.

Now we observe that $P(H-\delta \le M_T^d < H, M_T^b \ge H)$ implies that for some i $S_T^d < H-\delta < H \le S_T^b$, so, by taking logarithms, we have

$$\sqrt{\Delta}\sum_{j=0}^{i-1}(\xi_j - Z_j) > C\delta,$$

which implies

$$\sum_{j=0}^{i-1}(\xi_j - Z_j) > C\delta\sqrt{n}.$$

Now take $\delta = 2K\ln n/\sqrt{n}$. With this choice we have from (3) $P_2 \le Cn^{-\lambda K} \le C\ln n/\sqrt{n}$. Summing up, we have $I_2 \le C\ln n/\sqrt{n}$, and the assertion of the theorem follows. □

3. Modelling

As in [8], we give an example showing how fast the price in discrete binomial model converges to correspondent price in continuous model.

Consider the drift function of the form:

$$\mu_t = \begin{cases} \mu_1, & 0 \le t < T/2, \\ \mu_2, & T/2 \le t \le T. \end{cases}$$

This function (and corresponding interest rate r_t) does not satisfy the condition of continuity (1), which we have impose on it. But, if we look into the proof of Theorem 2, it is not difficult to see that it is enough to have the condition (1) fulfilled only for $t = t_i, s \in [t_i, t_{i+1})$, which is true for such a function.

According to [3] we have that for Brownian motion X_t with initial value x and constant drift coefficient μ simultaneous density of the distribution of maximum M_t on interval $[0,t]$, of the points T_t of the maximum and of the values X_t is given by

$$P(X_t \in dz, M_t \in dy, T_t \in ds) = \frac{(y-x)(y-z)}{\pi\sqrt{s^3(t-s)^3}}\times$$
$$\exp\Big(-\frac{(y-x)^2}{2s} - \frac{(y-z)^2}{2(t-s)} - \mu(x-z) - \frac{\mu^2 t}{2}\Big)dz\,dy\,ds$$
$$=: f_{t,x,\mu}(z,y,s)dz\,dy\,ds$$

when $x \le y$ and $z \le y$; when $x > y$ or $z > y$ it equals to zero. Noting $\nu(T) = \exp\big\{-\int_0^T r_t dt\big\} = \exp\Big\{-\frac{T}{2}(\mu_1+\mu_2+\sigma^2)\Big\}$ and using the fact, that $Z_t = \frac{1}{\sigma}\ln S_t$ is a Brownian motion with the drift $\nu_1 = \frac{\mu_1}{\sigma}$ on $[0, \frac{T}{2}]$ and

$\nu_2 = \frac{\mu_2}{\sigma}$ on $[\frac{T}{2}, T]$, we can get the European up-and-out call option fair price as

$$V(H) = E\Big(\nu(T)(S_T - K)^+ I_{\{\tau(H,S)>T\}}\Big) = \nu(T)E\Big((S_T - K)^+ I_{\{\sup\limits_{[0,T]} S_t<H\}}\Big) =$$

$$\nu(T)E\Big(E\big((S_T - K)^+ I_{\{\sup_{[\frac{T}{2},T]} S_t<H\}} \,\big|\, \mathcal{F}_{\frac{T}{2}}\big) I_{\{\sup_{[0,\frac{T}{2}]} S_t<H\}}\Big) =$$

$$\nu(T)E\Big(\int_0^{\frac{T}{2}} \int_{Z_{\frac{T}{2}}}^{\frac{1}{\sigma}\ln H} \int_{-\infty}^{y} (e^{\sigma z} - K)^+ f_{\frac{T}{2}, Z_{\frac{T}{2}}, \nu_2}(z, y, s) dz\, dy\, ds\; I_{\{\sup_{[0,\frac{T}{2}]} S_t<H\}}\Big) =$$

$$\nu(T) \int_0^{\frac{T}{2}} \int_{Z_0}^{\frac{1}{\sigma}\ln H} \int_{-\infty}^{v} f_{\frac{T}{2}, Z_0, \nu_1}(x, v, u) \times$$

$$\int_0^{\frac{T}{2}} \int_x^{\frac{1}{\sigma}\ln H} \int_{-\infty}^{y} (e^{\sigma z} - K)^+ f_{\frac{T}{2}, x, \nu_2}(z, y, s) dz\, dy\, ds\, dx\, dv\, du.$$

The last integral is rather difficult to calculate because of its high dimension. Nevertheless, integrals in y and v can be evaluated in closed form, with the use of the standard normal distribution function; we do not give the result of this integrating — formulae are very intricate — and give only the final estimation for the integral.

n	$V_n^b(H)$	$V_n^b(H) - V(H)$	$(V_n^b(H) - V(H))n^{1/2}/\ln n$
10	0,5675	0,0932	0,1279
20	0,5001	0,0257	0,0383
50	0,5772	0,1028	0,1858
100	0,5438	0,0694	0,1507
200	0,4991	0,0237	0,0658
500	0,4921	0,0177	0,0638
1000	0,4906	0,0162	0,0742
2000	0,4785	0,0041	0,0241

Table 1: Price $V(H)$ of the European up-and-out call option in continuous model and the price $V_m^b(H)$ of the same option in discrete binomial model.

Let take the following meanings of parameters: $S_0 = 100$, $\sigma = 0.1$, $K = 100$, $H = 105$, $T = 0{,}2$, $\mu_1 = 0{,}1$, $\mu_2 = 0{,}2$. Then with accuracy 10^{-4}

$$V(H) = 0{,}4744.$$

To estimate the order of the rate of convergence for the option prices with discrete time, using Monte Carlo simulations for the estimation of mathematical expectation, we will generate 100000 trajectories of asset price (50000 trajectories for $m = 1000, 2000$). The results we have got are noted in the table 1. We should note that the option prices with discrete time are bigger and decreasing when size of partition increasing. This property is natural, because in the case when the quantity of the points in our division increases, the moment set in which we examine does asset price cross given level or not also increases. There is no clear evidence however from this data whether the estimate for the rate of convergence is sharp.

Conclusions

We have proved that barrier option fair prices in discrete binomial Black–Scholes model with non-constant drift coefficient converges to corresponding price in continuous model, and the rate of convergence could be estimated as $O(\frac{\ln n}{\sqrt{n}})$, where n is the number of operational moments in the discrete binomial model. Thus is a result for convergence of the unique price in a complete market. Numerical example is presented.

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