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EXACT NON-RUIN PROBABILITIES IN ARITHMETIC CASE

Using the Wiener-Hopf method, for the model with arithmetic distributions of waiting times T_i and claims Z_i in ordinary renewal process, an exact non-ruin probabilities for an insurance company in terms of the factorization of the symbol of the discrete Feller-Lundberg equation, are obtained. The delayed stationary process is introduced and generating function for delay is given. It is proved that the stationary renewal process in arithmetic case is ordinary if and only if, when the inter-arrival times have the shifted geometrical distribution. A formula for exact non-ruin probabilities in delayed stationary process is obtained. Illustrative examples when the distributions of T_i and Z_i are shifted geometrical or negative binomial with positive integer power are considered. In these cases the symbol of the equation is rational functions what allows us to obtain the factorization in explicit form.

1. Introduction

Let $F(u)$ be the distribution function of claims $Z_j(=Z)$, a sequence of independent identically distributed (i.i.d.) random variables with expectation $EZ_j=\mu$, $K(u)$ be the distribution of waiting time $T_j(=T)$, a sequence of i.i.d. random variables, $ET_j=1/\alpha$, and $c>\alpha\mu$ be the gross premium rate, $j\in\mathbf{N}$. Random variables Z_j and T_j are supposed to be mutually independent.

Regardless of the continuous or discrete distributions, probability of solvency of an insurance company, $\varphi(u)$, with initial capital u, in *ordinary renewal process* satisfies the Feller-Lundberg integral equation, [1,4]:

$$(L\varphi)(u)\equiv\varphi(u)-\int_0^\infty dK(v)\int_0^{u+cv}\varphi(u+cv-z)\,dF(z)=0. \qquad (1)$$

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Note, that the initial capital u of the company can accept both the non-negative or negative integer values. The non-ruin probability $\varphi(u)$ in the latter case means the probability that the company begins the activity having a debt $(-u)$, however to the moment of the first claim the capital of the company becomes positive and is hereinafter not below of zero.

Therefore we are interested by the solution $\varphi(u)$ which is a monotone nondecreasing function of u, satisfying the conditions

$$\varphi(u) \nearrow 1 \quad \text{when} \quad u \to +\infty, \quad \text{and} \quad \varphi(u) \searrow 0 \quad \text{when} \quad u \to -\infty. \tag{2}$$

In this work we investigate the case when $K(x)$ and $F(x)$ are arithmetic distributions with step $d = 1$, and $c \in \mathbf{N}$.

Let $\mathbf{m}$ be the space of all real bounded number sequences $\varphi = \{\varphi_n\}_{-\infty}^{\infty}$ which after introduction of the norm

$$\|\varphi\| = \sup_n |\varphi_n| \tag{3}$$

turns into the Banach space. Let $\mathbf{c}$ be the subspace of all convergent sequences $\varphi \in \mathbf{m}$, and $\mathbf{c}^0$ be the subspace of all sequences convergent to zero. Introduce the norm in $\mathbf{c}$ and $\mathbf{c}^0$ by the formula (3) as for $\mathbf{m}$. Then $\mathbf{c}$ and $\mathbf{c}^0$ become the Banach spaces.

Let $g_T(z) = \sum_{n=1}^{\infty} p_n z^n$ and $g_Z(z) = \sum_{n=1}^{\infty} q_n z^n$ be the generating functions of the random variables Z and T, respectively. In the arithmetic case, we can rewrite (1)-(2) in the discrete form

$$A\varphi \equiv \varphi_u - \sum_{v=1}^{\infty} q_v \sum_{k=1}^{u+cv} \varphi_{u+cv-k} p_k = 0, \quad u \in \mathbf{Z}, \quad c \in \mathbf{N}, \tag{4}$$

$$\varphi_u \nearrow 1 \quad \text{when} \quad u \to +\infty, \quad \text{and} \quad \varphi_u \searrow 0 \quad \text{when} \quad u \to -\infty. \tag{5}$$

Similar so called compound binomial model was earlier considered by A. Melnikov in the monograph [5], which can be interpreted as model with random variable T having the shifted geometrical distribution with generating function

$$g_T(z) = \frac{pz}{1 - qz}, \quad p + q = 1, \tag{6}$$

$c = 1$, Z has an arbitrary arithmetic distribution with $\mathbf{P}\{Z = 0\} = 0$ and $\alpha = q$. A.Melnikov reduces the solution of the problem to the solution of a system of infinite number of linear algebraic equations with the Toeplitz matrix of coefficients using some recurrence relations. The solution of the problem was received in the terms of a generating function.

In this paper we use the Wiener-Hopf method for the solution of the problem (4)-(5).

2. The Wiener-Hopf method in arithmetic case.

We shall seek the solution of the problem (4)-(5) in the form of the infinite-dimensional vector $\varphi = \{\varphi_n\}_{-\infty}^{\infty}$.

Let $\mathbf{S}$ be the unit circle in the complex plane, $\mathbf{S} = \{t : |t| = 1\}$, $\mathbf{B}^+ = \{z : |z| < 1\}$, $\mathbf{B}^- = \{z : |z| > 1\}$. Associate with the vector φ the formal Laurent trasform,

$$\varphi(t) \sim \sum_{k=-\infty}^{+\infty} \varphi_k t^k, \quad t \in \mathbf{S},$$

where the series

$$\varphi^+(z) = \sum_{k=0}^{+\infty} \varphi_k z^k, \quad -\varphi^-(z) = \sum_{k=-1}^{-\infty} \varphi_k z^k, \tag{7}$$

converge at $z \in \mathbf{B}^+$ and $z \in \mathbf{B}^-$, respectively, and there exist everywhere on $\mathbf{S}$, except $z = 1$, the boundary values $\varphi^+(t)$ unbounded at $t = 1$, and $\varphi^-(t)$ bounded on $\mathbf{S}$ (without singularities). It is obvious that

$$\varphi(t) = \varphi^+(t) - \varphi^-(t), \quad t \in \mathbf{S}. \tag{8}$$

Theorem 1. *The symbol $a(t)$ of the operator A is equal*

$$a(t) = 1 - g_T(t^{-c})\, g_Z(t), \quad t \in \mathbf{S}. \tag{9}$$

Proof. Using the formulas (8) and going over to the generating functions in (4), we obtain the homogeneous Riemann boundary-value problem on $\mathbf{S}$

$$\varphi^+(t) - \varphi^-(t) - g_T(t^{-c})\, g_Z(t)\, \varphi^+(t) = 0, \quad t \in \mathbf{S}, \tag{10}$$

or

$$\varphi^+(t) = \frac{1}{a(t)} \varphi^-(t), \quad t \in \mathbf{S}, \tag{11}$$

where $a(t)$ is given by the formula (9).

Remark the following properties of the symbol.

Property 1. *The symbol $a(t)$ is differentiable function on* $\mathbf{S}$.

This follows from the existence of the expectations of the random variables T and Z.

Property 2. *The point $t = 1$ is the unique root of the symbol $a(t)$ on* $\mathbf{S}$, *and this root is of the first order.*

The notation '$-\varphi^-(t)$' is the tribute to the tradition of the theory of boundary problem for analytical function to present a function on a contour in the form (8) (the Sokhotsky-Plemelj formula).

Uniqueness follows from the fact that the step of the random variables T and Z is assumed to be equal 1, [2].

The first order of the root $t = 1$ follows from the L'Hospital rule and the condition $c > \alpha\mu$,

$$\lim_{t\to 1} \frac{a(t)}{t-1} = \lim_{t\to 1} \Big(g'_T(t^{-c}) c t^{-c-1}\, g_Z(t) - g_T(t^{-c})\, g'_Z(t) \Big) = \frac{c}{\alpha} - \mu > 0.$$

So, the Riemann problem (11) is singular (non-Noether).

Property 3.

$$|1 - a(t)| \le 1 \quad \text{for} \quad t \in \mathbf{S}. \tag{12}$$

The inequality (12) follows from the property of the generating functions. Geometrical sense of this inequality is that the plot of the symbol $a(t)$, $t \in \mathbf{S}$, is situated in the circle of the radius 1 with the center in the point $z = 1$ and is tangent to the axis of ordinates at the point $z = 0$.

Let $[\arg a(t)]_S$ be the increment of the argument of $a(t)]$ when t passes $\mathbf{S}$ in positive direction (counter-clockwise).

Property 4. *The function $b(t) = a(t)/(1-t)$, $t \in \mathbf{S}$, has not the roots on $\mathbf{S}$ and*

$$\operatorname{ind}_S b(t) = \frac{1}{2\pi}[\arg b(t)]_S = -1, \tag{13}$$

Proof. On the one hand we have

$$\operatorname{ind}_S a(t) = -\frac{1}{2}. \tag{14}$$

Really, let Γ be the plot of the function $a(t)$. Then

$$\Big(a(e^{i\tau})\Big)'\big|_{\tau=0} = i(c/\alpha - \mu),$$

what means that the point $a(t) = a(e^{i\tau})$ on the plot Γ bypasses through the origin of coordinates in negative direction when $\tau \in [0, 2\pi]$ bypasses from 0 to 2π. From this follows (14), since the $a(t)$ at $t = 1$ has the root of the first order and the plot of function $a(t)$ is smooth.

On the other hand we have

$$\operatorname{ind}_S a(t) = \operatorname{ind}_S (1-t) + \operatorname{ind}_S b(t) = \frac{1}{2} + \operatorname{ind}_S b(t),$$

whence, in view of (14), we have (13).

From the above Properties and results of [3,6] follows

Property 5. *For the symbol $a(t)$ the factorization exists*

$$a(t) = (1-t)\, a^+(t)\, t^{-1}\, a^-(t), \tag{15}$$

where

$$a^+(z) \neq 0, \quad |z| \leq 1, \quad a^+(1) = 1,$$
$$a^-(z) \neq 0, \quad |z| \geq 1, \quad a^-(1) = \frac{c}{\alpha} - \mu.$$

The formulas for $a^{\pm}(t)$ are given in [3] in terms of the Cauchy type integrals. The value $a^-(1) = \frac{c}{\alpha} - \mu$ is obtained from the condition $a^+(1) = 1$ by the L'Hospital rule applied to the symbol $a(t)$ at $t = 1$.

Theorem 2. *The equation* (4) *has two linear independent solution generated by the factorization* (12), *one in the space* $\mathbf{c}_+$ *and another (irrelevant) in* $\mathbf{c}_+^0$.

Proof. Proof follows from the solvability of discrete Wiener-Hopf equation with the symbol of entire order, [6], Theorem 1.5.

The desired solution of the problem (4)-(5) is generated by the functions

$$\varphi^+(z) = \frac{1}{(1-z)\, a^+(z)}, \quad z \in \mathbf{B}^+, \quad -\varphi^-(z) = \frac{a^-(z)}{z}, \quad z \in \mathbf{B}^-,$$

from which the probabilities φ_u are obtained by the expansions of the functions $\varphi^+(z)$ and $-\varphi^-(z)$ into a Taylor series about z and $1/z$, respectively.

Note that $-\varphi^-(1) = \frac{c}{\alpha} - \mu$.

The second irrelevant solution of the equation (4) is generated by the functions

$$\phi^+(z) = \frac{1}{a^+(z)}, \quad |z| \leq 1, \quad -\phi^-(z) = \frac{(1-z)a^-(z)}{z}, \quad |z| \geq 1.$$

Note that we can attach to the solution $\phi^+(z) - \phi^-(z)$ certain probabilistic sense. Really, the following obvious statement takes place.

Let $g(z)$ be the generating function of a random variable X. Then the generating function of the distribution function of X is given by the formula

$$G(z) = \frac{g(z)}{1-z}.$$

Comparing two solutions of the Riemann boundary problem we see that they are connected by the relations

$$\varphi^{\pm}(z) = \frac{\phi^{\pm}(z)}{1-z}, \quad z \in \mathbf{B}^{\pm}.$$

The solution $\varphi(t) = \varphi^+(t) - \varphi^-(t)$, $t \in \mathbf{S}$, can be considered as a generating function for the distribution function of some imaginary random variable X. The component $\varphi^+(t)$ can be interpreted as a generating function for the distribution function of the random variable X^+ with

$\mathbf{P}\{X^+ = n\} = \varphi_n$, $n = 0, 1, \ldots$. Then $\phi^+(t)$ is a generating function of the random variable X^+ with

$$\mathbf{P}\{X^+ = n\} = \phi_n^+ = \varphi_n - \varphi_{n-1}, \quad n \in \mathbf{N}, \quad \mathbf{P}\{X^+ = 0\} = \phi_0^+ = \varphi_0 - 0 = \varphi_0.$$

Note that $\sum_{n=0}^{\infty} \phi_n^+ = a^+(1) = 1$. Namely like this we selected the factor $a^+(t)$ in (16). To the component $-\varphi^-(t)$ we can not attach any probabilistic sense. The coefficient of the decomposition of this function into power series about $1/z$ are

$$\phi_{-n}^- = \varphi_{-n} - \varphi_{-n-1}, \quad n \in \mathbf{N}, \quad \phi_0^- = -\varphi_{-1} (< 0!).$$

But if we consider the function $\phi(t) = \phi^+(t) - \phi^-(t)$ having the formal decomposition into the Laurant series

$$\phi(t) = \sum_{n=-\infty}^{+\infty} \phi_n t^n = \sum_{n=-\infty}^{+\infty} (\varphi_n - \varphi_{n-1}) t^n, \quad t \in \mathbf{S},$$

we obtain the generating function of the random variable X. Here

$$\phi_{\pm n} = \phi_n^{\pm}, \quad n \in \mathbf{N}, \quad \phi_0 = \phi_0^+ + \phi_0^- = \varphi_0 - \varphi_{-1}.$$

Note that analogous fact takes place also in the non-arithmetic case. In that case the equation (1) also has two linear independent solutions connected by the relation

$$\phi(u) = \phi^+(u) - \phi^-(u) = \varphi'(u) = \varphi^{+'}(u) - \varphi^{-'}(u), \quad u \in \mathbf{R},$$

[6], Theorem 2.3. So $\varphi(u)$ and $\phi(u)$ can be interpreted as the distribution function and density of some imaginary random variable X, respectively.

3. Delayed renewal processes and stationarity in arithmetic case.

In addition to the sequence T_n there is defined a non-negative lattice variable S_0 with a generating function $g_0(z) = \sum_{n=1}^{\infty} r_n z^n$. The variables $S_n = S_0 + T_1 + \ldots + T_n$ are called *renewal epochs*, [2]. The renewal process $\{S_n\}$ is called delayed if $S_0 \neq 0$. The expected number $V(m) = \sum_{k=0}^{\infty} \mathbf{P}\{S_k \leq m\}$ of renewal epochs in $[0, m]$, $m \in \mathbf{N}$, has the generating function

$$g_V(z) = \frac{g_0(z)}{(1-z)(1-g_T(z))}. \tag{16}$$

$a^+(t)$ in (16) is selected up to constant factor $C \neq 0$. If $a^+(t)$ is multiplied by C, then $a^-(t)$ is divided into C.

The derivatives are understood in the generalized sense.

The expected number of renewal epochs within $[m, m+1]$, $m \in \mathbf{N}$, tends to α,

$$V(m+1) - V(m) \to \alpha.$$

It follows that $V(m) \sim \alpha m$ as $m \to \infty$. It is natural to ask whether $g_0(z)$ can be chosen as to get the identity $V(m) = \alpha m$, $m \in \mathbf{N}$, meaning a constant renewal rate. Noticing that the generating function for the sequence $\{\alpha m\}$ equals

$$g_V(z) = \frac{\alpha z}{(1-z)^2},$$

from (16) we obtain that $g_0(z)$ satisfies now the equation

$$g_V(z) = \frac{1}{1-z}\, g_0(z) + g_T(z)\, g_V(z)$$

and thus equals

$$g_0(z) = (1 - g_T(z))\, g_V(z)\, (1-z) = \frac{\alpha z(1 - g_T(z))}{1-z}. \tag{17}$$

This $g_0(z)$ is a generating function of a probability distribution and so the answer is affirmative:

With the initial random variable S_0 having the generating function (17) *the renewal rate is constant, $V(m) = \alpha m, \quad m \in \mathbf{N}$.*

The following statement takes place:

The stationary renewal process in arithmetic case is ordinary if and only if, when the inter-arrival times T_n have the shifted geometrical distribution (6).

This shifted geometrical distributions is analog of exponential distribution for T_n in non-arithmetic case.

For the problem (4)-(5) we introduce the *accompanying delayed stationary renewal process* $\{S_n\}$ with generating function for S_0, given by (17). Then the generating function $\varphi_s(z)$ for non-ruin probabilities of the company in such process is built as follows

$$\varphi_s(t) = g_0(t^{-c})\, g_Z(t)\, \varphi^+(t) = \frac{\alpha t^{-c}(1 - g_T(t^{-c}))\, g_Z(t)\, \varphi^+(t)}{1 - t^{-c}} =$$

$$= \frac{\alpha(1 - g_T(t^{-c}))\, g_Z(t)\, \varphi^+(t)}{t^c - 1}, \quad t \in \mathbf{S}, \tag{18}$$

similarly as it is done in the non-arithmetic case.

Solve the jump problem for $\varphi_s(t)$, [3],

$$\varphi_s(t) = \varphi_s^+(t) - \varphi_s^-(t),$$

where $\varphi_s^+(t)$ is the boundary value on $\mathbf{S}$ of an analytic function in $\mathbf{B}^+$, having the pole of the first order at $t = 1$, and $\varphi_s^-(t)$ is the boundary value on $\mathbf{S}$ of an analytic function in $\mathbf{B}^-$, lacking the singularities on $\mathbf{S}$ and $\varphi_s^-(\infty) = 0$. Whence the probabilities φ_n^s, $n \in \mathbf{Z}$, are obtained by the decompositions of the functions $\varphi_s^+(z)$ and $-\varphi_s^-(z)$ into power series about z and $1/z$, respectively.

Using (10) we can simplify the formula (18), excluding $g_T(t^{-c})$,

$$\varphi_s(t) = \frac{\alpha(1 - g_Z(t))\varphi^+(t) - \alpha\varphi^-(t)}{1 - t^c}, \quad t \in \mathbf{S}. \tag{19}$$

The case $\mathbf{c = 1}$. Multiplying both sides of the equation (19) on $(1 - t)$,

$$(1 - t)\varphi_s(t) = \alpha(1 - g_Z(t))\varphi^+(t) - \alpha\varphi^-(t), \quad t \in \mathbf{S}, \tag{20}$$

and project both parts of this equation on the space of the functions $\Phi^-(z)$ holomorphic in $\mathbf{B}^-$, $\Phi^-(\infty) = 0$, having a boundary values on $\mathbf{S}$ without singularities. Equating the coefficients in the decompositions of these projections into the power series about $1/z$, we obtain the system of equations for φ_{-n}^s, $n \in \mathbf{N}$,

$$\begin{cases} \varphi_{-1}^s - \varphi_{-2}^s &= \alpha\varphi_{-1} \\ \varphi_{-2}^s - \varphi_{-3}^s &= \alpha\varphi_{-2} \\ \dots\dots\dots\dots\dots\dots\dots & \\ \varphi_{-n}^s - \varphi_{-(n+1)}^s &= \alpha\varphi_{-n} \\ \dots\dots\dots\dots\dots\dots\dots & \end{cases}$$

Summing these equations, we obtain

$$\varphi_{-1}^s = \alpha \sum_{n=-1}^{-\infty} \varphi_n = -\alpha\varphi^-(1) = \alpha\left(\frac{c}{\alpha} - \mu\right) = 1 - \alpha\mu, \quad (c = 1),$$

and the values φ_{-n}^s form the geometric progression with the denominator $1 - \alpha$, $\varphi_{-(n+1)}^s = (1 - \alpha)\varphi_{-n}^s$, $n \in \mathbf{N}$.

Note that in non-arithmetic case $1 - \alpha\mu$ is the value of $\varphi_s(0)$.

Using the value φ_{-1}^s, we can construct the truncated form $\varphi_s^+(z)$ of $\varphi_s(z)$. Observing that the right part of (20) can be considered as the generating function of some imaginary random variable X with $\mathbf{P}\{X = n\} = \varphi_n^s - \varphi_{n-1}^s$, $n \in \mathbf{Z}$, $\varphi_s^+(z)$ as the truncation of $\varphi_s(z)$ that is the generating function of distribution function of X, and projecting both parts of the equation (20) on the space of the functions $\Phi_0^+(z)$ holomorphic in $\mathbf{B}^+$, we obtain

$$\varphi_s^+(z) = \frac{\varphi_{-1}^s + \alpha(1 - g_Z(z))\,\varphi^+(z)}{1 - z}, \quad z \in \mathbf{B}^+. \tag{21}$$

Here the summand φ_{-1}^s appears to annihilate the corresponding term in the expression $\varphi_0^s - \varphi_{-1}^s = \mathbf{P}\{X = 0\}$, if we wish to remain in the space of analytical functions in $\mathbf{B}^+$.

Assume now that the inter-arrival times T_n have the shifted geometrical distribution (6), i.e., *the initial renewal process is stationary* and $\varphi_s(t) = \varphi(t)$. Using (21) and $\varphi^+(z) = \varphi_s^+(s)$ we obtain the equation for $\varphi_s^+(z)$

$$\varphi_s^+(z) = \frac{\varphi_{-1}^s + \alpha(1 - g_Z(z))\varphi_s^+(z)}{1 - z}, \quad z \in \mathbf{B}^+.$$

whence

$$\varphi_s^+(z) = \frac{\varphi_{-1}^s}{1 - z - \alpha(1 - g_Z(z))}, \quad z \in \mathbf{B}^+. \tag{22}$$

In [5] the formula for $\varphi_s^+(z)$ is written in some cumbersome form.

Project both parts of the equation (20) on the space of the functions $\Phi^-(z)$ holomorphic outside of $\mathbf{S}$, $\Phi^-(\infty) \neq 0$, having a boundary values on $\mathbf{S}$ without singularities. Equating the coefficients in the decompositions of these projections into the power series about $1/z$, we obtain the series of equations for φ_{-n}^s, $n \in \mathbf{N} \cup \{0\}$,

$$\begin{cases} \varphi_0^s - \varphi_{-1}^s &= \alpha\varphi_0^s \\ \varphi_{-1}^s - \varphi_{-2}^s &= \alpha\varphi_{-1}^s \\ \dots\dots\dots\dots & \dots\dots \\ \varphi_{-n}^s - \varphi_{-(n+1)}^s &= \alpha\varphi_{-n}^s \\ \dots\dots\dots\dots & \dots\dots \end{cases}$$

Summing these equations, we obtain

$$\varphi_0^s = \alpha\varphi_0^s - \alpha\varphi^-(1),$$

whence we receive the formula derived by A.Melnikov [5]

$$\varphi_0^s = \frac{1 - \alpha\mu}{1 - \alpha} = \frac{1 - q\mu}{1 - q}.$$

The case $\mathbf{c} > \mathbf{1}$. Multiplying both sides of the equation (19) on $(1 - t^c)$,

$$(1 - t^c)\varphi_s(t) = \alpha(1 - g_Z(t))\varphi^+(t) - \alpha\varphi^-(t), \quad t \in \mathbf{S}. \tag{23}$$

Observing that the right part of (23) can be considered as the distinctive generating function of some imaginary random variable X with $\mathbf{P}\{X = n\} = \varphi_n^s - \varphi_{n-c}^s$, $n \in \mathbf{Z}$, and $\varphi_s^+(z)$ as the truncation of $\varphi_s(z)$ that is the generating function of distribution function of X. Projecting both parts of the equation (20) on the space of the functions $\Phi_0^+(z)$ holomorphic in $\mathbf{B}^+$, we obtain

$$\varphi_s^+(z) = \frac{\varphi_{-1}^s z^{c-1} + \varphi_{-2}^s z^{c-2} + \dots + \varphi_{-c}^s + \alpha(1 - g_Z(z))\,\varphi^+(z)}{1 - z^c}, \quad z \in \mathbf{B}^+. \tag{24}$$

This formula is the generalization of (22). Here the summand $\varphi^s_{-1}z^{c-1} + \varphi^s_{-2}z^{c-2} + \cdots + \varphi^s_{-c}$ appears to annihilate corresponding subtrahends in the generating function for the distribution function of the imaginary random variable X, if we wish to remain in the space of analytical functions in $\mathbf{B}^+$.

Unfortunately we have not succeeded to obtain the explicit formulas for $\varphi^s_{-1}, \varphi^s_{-2}, \cdots, \varphi^s_{-c}$. Their numerical values it is possible obtain from the decomposition of $\varphi^-_s(z)$ into power series about $1/z$. Note only that using the reasoning as at the derivation of the formula for φ^s_{-1} in the case of $c = 1$, we can receive the expression for the sum

$$\varphi^s_{-1} + \varphi^s_{-2} + \cdots + \varphi^s_{-c} = c - \alpha\mu.$$

Then the sums of the form

$$\varphi^s_{nc-1} + \varphi^s_{nc-2} + \cdots + \varphi^s_{nc-c}, \quad n \in \mathbf{N},$$

makes up the geometrical progression with the denominator $1 - \alpha$.

Assume now that the inter-arrival times T_n have the shifted geometrical distribution (6). $\varphi_s(t) = \varphi(t)$.

Using (24) and $\varphi^+(z) = \varphi^+_s(s)$ we obtain the equation for $\varphi^+_s(z)$

$$\varphi^+_s(z) = \frac{\varphi^s_{-1}z^{c-1} + \varphi^s_{-2}z^{c-2} + \cdots + \varphi^s_{-c} + \alpha(1 - g_Z(z))\varphi^+_s(z)}{1 - z^c}, \quad z \in \mathbf{B}^+.$$

whence

$$\varphi^+_s(z) = \frac{\varphi^s_{-1}z^{c-1} + \varphi^s_{-2}z^{c-2} + \cdots + \varphi^s_{-c}}{1 - z^c - \alpha(1 - g_Z(z))}, \quad z \in \mathbf{B}^+. \tag{25}$$

This formula is the generalization of (22).

4. Examples.

Especially simply the Wiener-Hopf method works when the random variables T_k and Z_k are shifted uniform discrete, binomial, geometrical, negative binomial (with entire exponent). In these cases the symbol $a(t)$ is a rational function that allows us obtain its factorization in explicit form. For example, in the case of the shifted Poisson distribution, the problem of factorization is reduced to solution of some transcendental equation.

Example 1. Consider from the point of view of the Wiener-Hopf method the example examined by A. Melnikov [5].

Let T be shifted geometric random variable (6), Z constant variable with the generating functions

$$g_T = \frac{z}{5 - 4z}, \quad g_Z = z^2,$$

respectively, and $c = 1$. Then

$$\alpha = 0.2, \quad \mu = 2, \quad \varphi_{-1} = 0.6, \quad \varphi_0 = 0.75,$$

$$a(t) = \frac{(1-t)(4-t)}{5t-4} = (1-t)\,\frac{4-t}{3}\,t^{-1}\,\frac{3}{5-\frac{4}{t}}, \quad t \in \mathbf{S}.$$

Here

$$a^+(t) = \frac{4-t}{3}, \quad a^-(t) = \frac{3}{5-\frac{4}{t}}, \quad t \in \mathbf{S}.$$

$$\varphi^+(z) = \varphi_s^+(z) = \frac{3}{(1-z)(4-z)}, \quad z \in \mathbf{B}^+,$$

$$-\varphi^-(z) = -\varphi_s^-(z) = \frac{3}{5z-4}, \quad z \in \mathbf{B}^-.$$

The formula (22) for $\varphi_s^+(z)$ gives the same result.

Expanding the functions $\varphi^+(z)$ and $-\varphi^-(z)$ in the series about z and $1/z$, respectively, we obtain the solution of the problem (4)-(5).

The following table gives the probabilities of solvency and the results for irrelevant solution with accuracy 0.000001.

u	φ_u	ϕ_u	u	φ_u	ϕ_u
-9	.100663	.201324e-1	1	.937500	.187500
-8	.125829	.251658e-1	2	.984375	.468750e-1
-7	.157286	.314574e-1	3	.996094	.117188e-1
-6	.196608	.393216e-1	4	.999023	.292969e-2
-5	.245760	.491520e-1	5	.999756	.732422e-3
-4	.307200	.614400e-1	6	.999939	.183105e-3
-3	.384000	.768000e-1	7	.999985	.457764e-4
-2	.480000	.960000e-1	8	.999996	.114441e-4
-1	.600000	.120000	9	.999999	.286102e-5
0	.750000	.150000	10	.999999	.715256e-6

Here $\phi_0 = \phi_0^+ + \phi_0^- = \varphi_0 - \varphi_{-1} = .750000 - .600000 = .150000$.

Our results coincides with the ones of A. Melnikov if take into account that in [4] the probabilities $1 - \varphi_u$ are computed only for $u = 0, 1, \ldots, 10$.

Example 2. Let T be shifted negative binomial variable, Z be shifted geometric random variable with the generating functions

$$g_T = \frac{16z}{(5-z)^2}, \quad g_Z = \frac{9z}{10-z},$$

respectively, and $c = 1$. Then

$$\alpha = \frac{2}{3}, \quad \mu = \frac{10}{9}, \quad a(t) = \frac{(25t^2 - 91t + 10)(t-1)}{(5t-1)^2(t-10)},$$

$$a^+(t) = \frac{.280730(t-10)}{(t-3.52658)}, \quad a^-(t) = \frac{7.01825t(t-.11342)}{(5t-1)^2}, \quad t \in \mathbf{S}.$$

$$\varphi^+(z) = \frac{(z-3.52657)}{(1-z).280730(z-10)}, \quad z \in \mathbf{B}^+,$$

$$-\varphi^-(z) = \frac{7.01825(z-.11342)}{(5z-1)^2}, \quad z \in \mathbf{B}^-.$$

The generating function $\varphi_s(z)$ for non-ruin probabilities in the delayed stationary renewal process is built as follows

$$\varphi_s(t) = \frac{1.68438(1-25t)t}{(5t-1)^2(1-t)(t-3.52658)}, \quad t \in \mathbf{S}.$$

Expanding this function in the sum of partial fractions and using the Sokhotski-Plemelj decomposition we obtain

$$\varphi_s^+(z) = \frac{1}{1-z} + \frac{.740738}{z-3.52658}, \quad z \in \mathbf{B}^+,$$

$$-\varphi_s^-(z) = \frac{.4(.648149z-.0789956)}{z^2-.4z+.04}, \quad z \in \mathbf{B}^-.$$

Expanding the functions $\pm\varphi^\pm(z)$ and $\pm\varphi_s^\pm(z)$ in the series about z and $1/z$, respectively, we obtain the solution of the problem (4)-(5). The following table gives the probabilities of solvency in ordinary and stationary processes with accuracy 0.000001.

u	φ_u	φ_u^s	u	φ_u	φ_u^s
-9	.320745e-5	.27377e-5	1	.942163	.940440
-8	.144817e-4	.1239e-4	2	.983598	.983111
-7	.646313e-4	.5548e-4	3	.995348	.995211
-6	.284269e-3	.24499e-3	4	.998680	.998642
-5	.122691e-2	.10629e-2	5	.999624	.999615
-4	.516238e-2	.4504e-2	6	.999892	.999891
-3	.209510e-1	.1847e-1	7	.999968	.999969
-2	.804505e-1	.7210e-1	8	.999989	.999991
-1	.280730	.259260	9	.999995	.999997
0	.796041	.789956	10	.999997	.999999

Example 3. Let T be shifted negative binomial variable, Z be shifted geometric random variable with the generating functions

$$g_T = \frac{81z}{(10-z)^2}, \quad g_Z = \frac{z}{2-z},$$

respectively, and $c = 2$. Then $\alpha = \frac{9}{11}$, $\mu = 2$,

$$a(t) = \frac{(20t^3-16t^2-11t-2)(5t-1)(t-1)}{(10t^2-1)^2(-2+t)}, \quad t \in \mathbf{S},$$

$$a^+(t) = \frac{.280730(t-10)}{(t-3.52658)}, \quad a^-(t) = \frac{7.01825t(t-.11342)}{(5t-1)^2}, \quad t \in \mathbf{S}.$$

$$\varphi^+(z) = \frac{.2875080(z-2)}{(1-z)(z-1.287508)},$$

$$-\varphi^-(z) =$$

$$\frac{.2875080(5z-1)(z+.2437537-.1351057I)(z+.2437537+.1351057I)}{(10z^2-1)^2}.$$

The generating function $\varphi_s(t)$ for non-ruin probabilities in the delayed stationary renewal process is built as follows

$$\varphi_s(t) = \frac{.2352338(1-100t^2)t}{(10t^2-1)^2(1-t)(t-1.287508)}, \quad t \in \mathbf{S}.$$

Expanding this function in the sum of partial fractions and using the Sokhotski-Plemelj decomposition we obtain

$$\varphi_s^+(z) = \frac{1}{1-z} + \frac{.7153470}{z-1.287508},$$

$$-\varphi_s^-(z) = \frac{62.01608}{z+.3162917} - \frac{61.97134}{z+.3161638} - \frac{94.09140}{z-.3160941} + \frac{94.33134}{z-.3163616}.$$

Expanding the functions $\pm\varphi^\pm(z)$ and $\pm\varphi_s^\pm(z)$ in the series about z and $1/z$, respectively, we obtain the solution of the problem (4)-(5). The following table gives the probabilities of solvency in ordinary and stationary processes with accuracy 0.0000001.

u	φ_u	φ_u^s	u	φ_u	φ_u^s
-2	.4133037e-2	.7898426e-1	10	.9557890	.9556118
-1	.1437540e-1	.2846521	11	.9656615	.9655239
0	.4466115	.4443941	12	.9733296	.9732226
1	.5701864	.5684641	13	.9792853	.9792022
2	.6661664	.6648286	14	.9839110	.9838465
3	.7407135	.7396743	15	.9875037	.9874536
4	.7986136	.7978065	16	.9902943	.9902553
5	.8435844	.8429575	17	.9924615	.9924314
6	.8785129	.8780260	18	.9941449	.9941215
7	.9056416	.9052635	19	.9954525	.9954342
8	.9267125	.9264187	20	.9964679	.9964538
9	.9430780	.9428498	21	.9972566	.9972457

Note that

$$\varphi_{-1}^s + \varphi_{-1}^s = c - \alpha\mu = 2 - \frac{9}{11} \cdot 2 \approx .3636364.$$

The computation carried out in *Maple-V*.

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