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OKSANA BANNA AND YULIYA MISHURA

APPROXIMATION OF FRACTIONAL BROWNIAN MOTION WITH ASSOCIATED HURST INDEX SEPARATED FROM 1 BY STOCHASTIC INTEGRALS OF LINEAR POWER FUNCTIONS

In this article we present the best uniform approximation of the fractional Brownian motion in space $L_\infty([0,T];L_2(\Omega))$ by martingales of the following type $\int\limits_0^t a(s)dW_s$, where W is a Wiener process, a is a function defined by $a(s) = k_1 + k_2 s^\alpha$, $k_1, k_2 \in \mathbb{R}$, $s \in [0,T]$, $\alpha = H - 1/2$, H is the Hurst index, separated from 1, associated with the fractional Brownian motion.

1. Introduction

A fractional Brownian motion (fBm) with associated Hurst index $H \in (0,1)$ is a Gaussian process $\{B_t^H,\ t \geq 0\}$ with mean $\mathrm{E}B_t^H = 0$ and covariance $\mathrm{E}B_t^H B_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$, such that $B_0^H = 0$. We consider only the case when the Hurst index $H \in (\frac{1}{2},\ 1)$. From [6] we know that fBm $\{B_t^H,\ \ t \in [0,\ T]\}$ admits the following representation $B_t^H = \int\limits_0^t z(t,s)dW_s$, where $\{W_t,\ \ t \in [0,\ T]\}$ is a Wiener process, $z(t,s) = \left(H - \frac{1}{2}\right)\ c_H\, s^{1/2-H} \int\limits_s^t u^{H-1/2}(u-s)^{H-3/2}du$, $c_H = \left(\frac{2H\cdot\Gamma\left(\frac{3}{2} - H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\right)^{1/2}$, $\Gamma(x)$, $x > 0$ is the Gamma function. We will use the following notation $\alpha = H - \frac{1}{2}$.

It is known that the fBm with index $H \in (\frac{1}{2},\ 1)$ is not a semimartingale, actually it is neither a martingale nor a process of bounded variation. However in [1], [5], and [7] it is shown that in some metric space fBm can be approximated by semimartingales and processes of bounded variation. Hence in this article we present the approximation of the fractional Brownian motion in the space $L_\infty([0,T];L_2(\Omega))$ by martingales, namely, by the

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stochastic integrals of the particular Wiener process $\int_0^t a(s)dW_s$, where W is a standard Wiener process, a is a non-random function $a \in L_2[0,T]$. In fact, we consider only a particular case when $a(s)$ is a linear power function, and under this restriction we derive some interesting and even surprising results. In the previous articles we presented the approximation of the fBm with constant functions, power functions of the type $a(s) = k \cdot s^{\alpha}$, $k \in \mathbb{R}$ and functions $a(s) = k \cdot s^{\gamma}$, $k \in \mathbb{R}$, $\gamma > 0$ [2], and power functions with a negative power $a(s) = k \cdot s^{-\alpha}$, $k > 0$ [4]. Here we present the approximation of fBm with functions $a(s) = k_1 + k_2 \cdot s^{\alpha}, k_1,\ k_2 \in \mathbb{R}$, for $H < H_0$, where $H_0 \approx 0,94005165$. In [3] this approximation is derived in empirical way (below it is shown how to define H_0), as well as the minimum and the function where this minimum is reached are presented. Note that this minimum is not equal to zero. The best approximation of the fractional Brownian motion for $H \geq H_0$ is presented in [3]. Namely, it is proved that the minimum of the function $\max\limits_{t\in[0,T]} \mathrm{E}(B_t^H - M_t)^2$, where $M_t = \int\limits_0^t a(s)dW_s$, $t \in [0,\ T]$ is a stochastic integral of Wiener process $\{W_t,\quad t \in [0,\ T]\}$ (Wiener integral), is reached at certain k_1 and k_2. We briefly outline an algorithm of how to find this minimum. For a function $a(s) = k_1 + k_2 \cdot s^{\alpha}, k_1,\ k_2 \in \mathbb{R}$

$$\begin{aligned}\mathrm{E}(B_t^H - M_t)^2 &= t^{2H}\left(1 - k_2 \cdot \tfrac{c_H}{H} + \tfrac{k_2^2}{2H}\right) - 2k_1 \cdot \tfrac{t^{\alpha+1}}{\alpha+1}\left(c_3(H) - k_2\right) + k_1^2 t \\ &=: f(t, k_1, k_2),\end{aligned}$$

where $c_3(H) = c_1(H)(\alpha+1)$, $c_1(H) = \alpha\, c_H \cdot \frac{1}{\alpha+1} \cdot B(1-\alpha,\, \alpha)$.

First consider the derivative over t of $f(t, k_1, k_2)$:

$$\tfrac{\partial f}{\partial t} = t^{2\alpha}\left(2H - 2k_2 \cdot c_H + k_2^2\right) + 2t^{\alpha}k_1\left(k_2 - c_3(H)\right) + k_1^2. \tag{1}$$

This quadratic polynomial of t has its discriminant

$$\frac{D}{4} = k_1^2\left(c_3^2(H) - 2H - 2k_2\left(c_3(H) - c_H\right)\right). \tag{2}$$

We equate this discriminant with zero and find k_2:

$$k_2 = \tfrac{2H - c_3^2(H)}{2(c_H - c_3(H))} =: k_2^0.$$

Discriminant D in (2) is linearly dependent on k_2, in fact, it is decreasing with respect to k_2, hence:

a) if $k_2 > k_2^0$ then $D < 0$;

b) if $k_2 = k_2^0$ then $D = 0$;

c) if $k_2 < k_2^0$ then $D > 0$, and, in this case, if $k_1 \geq 0$ then $\frac{\partial f}{\partial t} = 0$ at points $t_1 = t_1(k_1, k_2)$ and $t_2 = t_2(k_1, k_2)$, where

$$t_1(k_1, k_2) = \left(\tfrac{|k_1|\left(c_3(H) - k_2 - \sqrt{c_3^2(H) - 2H - 2k_2(c_3(H) - c_H)}\right)}{2H - 2k_2 c_H + k_2^2}\right)^{\frac{1}{\alpha}} =: |k_1|^{\frac{1}{\alpha}} \cdot c_1(k_2),$$

$$t_2(k_1,k_2)=\left(\frac{|k_1|\left(c_3(H)-k_2+\sqrt{c_3^2(H)-2H-2k_2(c_3(H)-c_H)}\right)}{2H-2k_2c_H+k_2^2}\right)^{\frac{1}{\alpha}}.$$

These roots exist and are positive if and only if $k_1(c_3(H)-k_2)>0$. However we consider only the case when $k_2<k_2^0$, and $k_2^0<0$, so the roots are positive if and only if $k_1>0$. Further, we investigate the behavior of $f(t,k_1,k_2)$ with respect to $k_1,\ k_2$. If $k_1\le 0$ then $\frac{\partial f}{\partial t}>0$ for all t.

If $k_1>0$ then $\frac{\partial f}{\partial t}>0$ for $0<t<x_1^{\frac{1}{\alpha}}$, and the function $f(t,k_1,k_2)$ is increasing; $\frac{\partial f}{\partial t}<0$ for $x_1^{\frac{1}{\alpha}}<t<x_2^{\frac{1}{\alpha}}$, $f(t,k_1,k_2)$ is decreasing; $\frac{\partial f}{\partial t}>0$ for $t>x_2^{\frac{1}{\alpha}}$, $f(t,k_1,k_2)$ is increasing.

Consider the following cases:

I. For $k_1\le 0,\ k_2\in\mathbb{R}$

$$\min_{k_1\le 0,k_2\in\mathbb{R}}\max_{t\in[0,T]}f(t,k_1,k_2)=\min_{k_1\le 0,k_2\in\mathbb{R}}f(T,k_1,k_2)=T^{2H}\left(1-\frac{c_H^2}{2H}\right).$$

It is proved that the minimum is reached at $k_1=0,\ k_2=c_H$.

Use the proposition in [3], showing that function $f(T,k_1,k_2)$ has its minimum at (k_1^*,k_2^*), where

$$\begin{aligned}k_1^*&=\frac{(\alpha+1)(c_3(H)-c_H)T^\alpha}{\alpha^2},\\ k_2^*&=\frac{(\alpha+1)(c_H(\alpha+1)-c_1(H)(2\alpha+1))}{\alpha^2}.\end{aligned}\tag{3}$$

Moreover in [3] there are graphs displaying the dependence of k_2^0 and k_2^* on H, and it is easy to see that if $H<H_0$ then we have $k_2^*<k_2^0$, and for $H>H_0$ $k_2^*>k_2^0$. Hence H_0 is a unique point of intersection of these graphs.

II. For $k_1>0,\ k_2\ge k_2^0$ we have

a) if $H>H_0$ then $k_2^*>k_2^0$, hence

$$\begin{aligned}\min_{k_1>0,k_2>k_2^0}f(T,k_1,k_2)&=\min_{k_1\in\mathbf{R},k_2\in\mathbf{R}}f(T,k_1,k_2)=f(T,k_1^*,k_2^*)=\\ &=T^{2\alpha+1}-T^{2\alpha+1}c_H^2\left(B^2(1-\alpha,\alpha)-\frac{2B(1-\alpha,\alpha)}{\alpha}+\frac{(\alpha+1)^2}{\alpha^2(2\alpha+1)}\right).\end{aligned}$$

b) if $H<H_0$ then $k_2^*<k_2^0$, hence for the minimum we have:

$$\begin{aligned}\min_{k_1\ge 0,k_2\ge k_2^0}\max_{t\in[0,T]}f(t,k_1,k_2)&=\min_{k_1\ge 0,k_2\ge k_2^0}f(T,k_1,k_2)\\ &=f(T)\Big|_{k_1=\left(c_1(H)-\frac{k_2^0}{\alpha+1}\right)T^\alpha,\,k_2=k_2^0}\\ =-\left(c_1(H)-\frac{k_2^0}{\alpha+1}\right)^2\cdot T^{2\alpha+1}&+T^{2\alpha+1}-2k_2^0\frac{c_H}{2\alpha+1}T^{2\alpha+1}+(k_2^0)^2\cdot\frac{T^{2\alpha+1}}{2\alpha+1}.\end{aligned}$$

III. Let $k_1 > 0$, $k_2 < k_2^0$. Since the function $f(t, k_1, k_2)$ is increasing on $[0, t_1]$, $[t_2, +\infty)$ and decreasing on $[t_1, t_2]$, we have:

a) if $t_1 > T$ then $f(t, k_1, k_2)$ is increasing on $[0, T]$, and $\max\limits_{t\in[0,T]} f(t, k_1, k_2) = f(T, k_1, k_2)$;

b) if $t_1 < T$ then $\max\limits_{t\in[0,T]} f(t, k_1, k_2) = \max\{f(T, k_1, k_2),\ f(t_1, k_1, k_2)\}$;

c) if $t_1 = T$ then $\max\limits_{t\in[0,T]} f(t, k_1, k_2) = f(T, k_1, k_2) = f(t_1, k_1, k_2)$.

Hence,

$$\min_{k_1\geq 0, k_2<k_2^0} \max_{t\in[0,T]} f(t, k_1, k_2) = \min_{k_1\geq 0, k_2<k_2^0} \begin{cases} \max\{f(T, k_1, k_2), f(t_1, k_1, k_2)\}, & t_1 < T \\ f(T, k_1, k_2), & t_1 \geq T \end{cases} .$$

However, as mentioned above, in [3] it is proved that for $H > H_0$ and $k_2^* \geq k_2^0$ function $\max\limits_{t\in[0,T]} f(t, k_1, k_2)$ has its minimum at $k_1 = k_1^*, k_2 = k_2^*$, which is equal to

$$T^{2\alpha+1} - T^{2\alpha+1} c_H^2 \left(B^2(1-\alpha, \alpha) - \tfrac{2B(1-\alpha,\alpha)}{\alpha} + \tfrac{(\alpha+1)^2}{\alpha^2(2\alpha+1)} \right).$$

Hence for $H > H_0$ one may ignore this case.

The remaining case, when $H < H_0$ and $k_2^* < k_2^0$, is considered in this work. As $H_0 < 1$, we call these values of Hurst index *separated from* 1. We will call the point (k_1^*, k_2^*) minimal one. The paper is organized as follows: in Section 2 we consider some auxiliary results demonstrating that it is impossible to achieve the required minimum for $H < H_0$ in the minimal point or in some connected point. In Section 3 we demonstrate analytically and with the help of the graph how to achieve minimum in our case.

2. Some approximation results for Hurst index, separated from 1, for minimal point and for connected points

Consider the following expression:

$$\begin{aligned} &f(t_1(k_1,\ k_2),\ k_1,\ k_2) \\ &\quad = k_1^{\frac{2H}{\alpha}} \left(c_1(k_2)\right)^{2H} a(k_2) - 2k_1 \frac{k_1^{\frac{\alpha+1}{\alpha}} \left(c_1(k_2)\right)^{\alpha+1}}{\alpha+1} b(k_2) + k_1^2 k_1^{\frac{1}{\alpha}} c_1(k_2) \\ &\quad =: k_1^{\frac{2H}{\alpha}} \varphi(k_2). \end{aligned}$$

Here $a(k_2) = 1 - k_2 \cdot \frac{c_H}{H} + \frac{k_2^2}{2H} > 0$, $b(k_2) = c_3(H) - k_2$, $\varphi(k_2) > 0$, $k_2 < k_2^0 < 0$.

Now we present a number of "negative" propositions concerned minimal point and connected points.

Lemma 2.1. *If $k_2^* < k_2^0$, then the minimum reached in case III is less than the minimum in cases II b) and I.*

Proof. First we show that the minimum reached in case II b) is less than the one reached in case I for $k_2^* < k_2^0$.

Since $c_H > 0 > k_2^0$ then in the neighborhood of $(k_1, k_2) = (0, c_H)$ function $f(t, k_1, k_2)$ is increasing with respect to t. Hence in this neighborhood $\max_{[0,T]} f(t, k_1, k_2) = f(T, k_1, k_2)$. In [3] it is shown that the point $(0, c_H)$ is not a point of local minimum of $f(T, k_1, k_2)$ (although it is a point of minimum of this function on the half-plane $\{k_1 \le 0\}$). Therefore this point is not the point of local minimum for $\max_{[0,T]} f(t, k_1, k_2)$ and thus there exists a point (k_1, k_2), $k_1 > 0$, $k_2 > c_H$, such that $\max_{[0,T]} f(t, k_1, k_2) < \max_{[0,T]} f(t, 0, c_H)$.

Now we show that the minimum reached in case III, is less than the minimum in case II b) (for $k_2^* < k_2^0$).

Since in case II b) the minimum is reached at $k_1 = \left(c_1(H) - \frac{k_2^0}{\alpha+1}\right) T^\alpha > 0$, $k_2 = k_2^0$, and for $k_2 = k_2^0$ we have $t_1 = t_2$, we denote $t_0 = t_1\left(\left(c_1(H) - \frac{k_2^0}{\alpha+1}\right) T^\alpha,\, k_2^0\right) = t_2\left(\left(c_1(H) - \frac{k_2^0}{\alpha+1}\right) T^\alpha,\, k_2^0\right)$.

We show that $t_0 < T$. Indeed,

$$t_1(k_1, k_2) = \left(\frac{|k_1|\left(c_3(H) - k_2 - \sqrt{c_3^2(H) - 2H - 2k_2(c_3(H) - c_H)}\right)}{2H - 2k_2 c_H + k_2^2}\right)^{\frac{1}{\alpha}}.$$

As $k_2 = k_2^0$, the discriminant (2) is equal to zero, hence $c_3^2(H) - 2H - 2k_2(c_3(H) - c_H) = 0$. To simplify the denominator of $t_1(k_1, k_2)$ we add the discriminant (2) to the denominator. Note that for $k_2 = k_2^0$ this discriminant is equal to zero. We have

$$\begin{aligned} &2H - 2k_2 c_H + k_2^2 + c_3^2(H) - 2H - 2k_2(c_3(H) - c_H) \\ &\qquad = c_3^2(H) - 2k_2 c_3(H) + k_2^2 = (c_3(H) - k_2)^2. \end{aligned}$$

The expression for $t_1(k_1, k_2^0)$ is reduced to :

$$t_1(k_1, k_2^0) = \left(\frac{k_1(c_3(H) - k_2^0)}{(c_3(H) - k_2^0)^2}\right)^{\frac{1}{\alpha}} = \left(\frac{k_1}{c_3(H) - k_2^0}\right)^{\frac{1}{\alpha}}.$$

Then

$$t_0 = \left(\frac{c_1(H) - \frac{k_2^0}{\alpha+1}}{c_3(H) - k_2^0}\right)^{\frac{1}{\alpha}} \cdot T = \left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha}} \cdot T < T.$$

As $t_0 < T$, $f\left(t_0, \left(c_1(H) - \frac{k_2^0}{\alpha+1}\right) T^\alpha,\, k_2^0\right) < f\left(T, \left(c_1(H) - \frac{k_2^0}{\alpha+1}\right) T^\alpha,\, k_2^0\right)$, hence

$$f(t_1, k_1, k_2) < f(T, k_1, k_2) \tag{4}$$

in some "one-side" neighborhood:

$$\{(k_1, k_2) | \left| k_1 - \left(c_1(H) - \frac{k_2^0}{\alpha+1}\right) T^\alpha \right| < r, k_2^0 - r < k_2 \le k_2^0\}$$

of the point $\left(\left(c_1(H) - \frac{k_2^0}{\alpha+1}\right) T^\alpha,\, k_2^0\right)$. Since t_1 is continuously dependent on k_1 and k_2, $f(t_1, k_1, k_2)$ is also continuously dependent on k_1 and k_2, and there exists $r > 0$ such that in the defined above "one side" neighborhood inequality (4) holds. In this neighborhood $\max_{[0,T]} f(t, k_1, k_2) = f(T, k_1, k_2)$. Taking into account that for $k_2 \ge k_2^0$ function $f(t, k_1, k_2)$ is increasing with respect to t, we get $\max_{[0,T]} f(t, k_1, k_2) = f(T, k_1, k_2)$ in a "normal" neighborhood $\left| k_1 - \left(c_1(H) - \frac{k_2^0}{\alpha+1}\right) T^\alpha \right| < r, |k_2 - k_2^0| < r$. However point $\left(\left(c_1(H) - \frac{k_2^0}{\alpha+1}\right) T^\alpha,\, k_2^0\right)$ is not a point of local minimum of $f(T, k_1, k_2)$. Hence it is not a point of local minimum of $\max_{[0,T]} f(t, k_1, k_2)$. □

Lemma 2.2. *If $k_2^* < k_2^0, k_1 > 0$ then the following inequality holds*

$$t_1(k_1^*,\ k_2^*) < T.$$

Proof. Let $k_2^* < k_2^0$. Since $t_1(k_1,\ k_2) = \left(\frac{|k_1|\left(c_3(H) - k_2 - \sqrt{c_3^2(H) - 2H - 2k_2(c_3(H) - c_H)}\right)}{2H - 2k_2 c_H + k_2^2}\right)^{\frac{1}{\alpha}}$, we can multiply and divide this fraction by the conjugate expression to enumerator $c_3(H) - k_2 + \sqrt{c_3^2(H) - 2H - 2k_2\left(c_3(H) - c_H\right)}$, and get $t_1^{-\alpha}(k_1,\ k_2) = \frac{c_3(H) - k_2 + \sqrt{c_3^2(H) - 2H - 2k_2(c_3(H) - c_H)}}{k_1}$. In the same vein we get $t_2^{-\alpha}(k_1,\ k_2) = \frac{c_3(H) - k_2 - \sqrt{c_3^2(H) - 2H - 2k_2(c_3(H) - c_H)}}{k_1}$, moreover

$$t_1^{-\alpha}(k_1,\ k_2) > t_2^{-\alpha}(k_1,\ k_2). \tag{5}$$

Add $t_1^{-\alpha}(k_1,\ k_2)$ to both left and right hand sides of (5), to get $\frac{2}{t_1^\alpha(k_1,\, k_2)} > \frac{1}{t_1^\alpha(k_1,\, k_2)} + \frac{1}{t_2^\alpha(k_1,\, k_2)} = \frac{2(c_3(H) - k_2)}{k_1}$.

Substitute $k_1 = k_1^*$, $k_2 = k_2^*$, and derive

$$\frac{2}{t_1^\alpha(k_1^*,\, k_2^*)} > \frac{2(c_3(H) - k_2^*)}{k_1^*} = \frac{2\left(c_3(H) - \frac{(\alpha+1)^2 c_H - (2\alpha+1) c_3(H)}{\alpha^2}\right)}{T^\alpha \left(\frac{(\alpha+1)(c_3(H) - c_H)}{\alpha^2}\right)} =$$

$$= \frac{2}{T^\alpha} \frac{(\alpha+1)^2 c_3(H) - (\alpha+1)^2 c_H}{(\alpha+1)(c_3(H) - c_H)} = \frac{2}{T^\alpha}(\alpha+1) > \frac{2}{T^\alpha}.$$

From here $t_1^\alpha(k_1^*,\ k_2^*) < T^\alpha$, that is, $t_1(k_1^*,\ k_2^*) < T$ for $k_2^* < k_2^0$. $\square$

So we are looking for $\min\limits_{k_1\geq 0, k_2<k_2^0} \max\{f(T,k_1,k_2), f(t_1,k_1,k_2)\}$. We now strengthen this result.

Lemma 2.3. *If $k_2^* < k_2^0, k_1 > 0$ then the following inequality holds $t_2(k_1^*,\ k_2^*) < T$.*

Proof. Since $f(T,k_1,k_2) = T^{2\alpha+1}a(k_2) - 2k_1\frac{T^{\alpha+1}}{\alpha+1}b(k_2) + Tk_1^2$, and $(k_1^*,\ k_2^*)$ is the point of minimum of $f(T,k_1,k_2)$, we have $k_1^* = \frac{T^\alpha}{\alpha+1}b(k_2^*)$.

Note that $f(T,k_1,k_2) > 0$ for all k_1, k_2 therefore the discriminant of the quadratic function $f(T,k_1,k_2)$ with the positive highest coefficient is negative:

$$\frac{D}{4} = T^{2(\alpha+1)}\left(\frac{b^2(k_2)}{(\alpha+1)^2} - a(k_2)\right) < 0. \tag{6}$$

Find the derivative of $f(t,k_1,k_2)$ with respect to t:

$$\tfrac{\partial f(t,k_1,k_2)}{\partial t} = (2\alpha+1)t^{2\alpha}a(k_2) - 2k_1t^\alpha b(k_2) + k_1^2.$$

Since $t_1(k_1,\ k_2)$ and $t_2(k_1,\ k_2)$ are solutions of equation $\frac{\partial f(t,k_1,k_2)}{\partial t} = 0$, we have $t_{1,2}^\alpha(k_1^*,\ k_2^*) =: x_{1,2}$ are roots of quadratic polynomial

$$x^2(2\alpha+1)a(k_2^*) - 2k_1^*xb(k_2^*) + (k_1^*)^2. \tag{7}$$

We substitute T^α instead of x and show that the quadratic polynomial is positive at T^α. Indeed,

$$\begin{aligned}
\frac{\partial f(T,k_1^*,k_2^*)}{\partial T} &= T^{2\alpha}(2\alpha+1)a(k_2^*) - 2k_1^*T^\alpha b(k_2^*) + (k_1^*)^2 \\
&= T^{2\alpha}(2\alpha+1)a(k_2^*) - 2\frac{T^\alpha b(k_2^*)}{\alpha+1}T^\alpha b(k_2^*) + \left(\frac{T^\alpha b(k_2^*)}{\alpha+1}\right)^2 \\
&= T^{2\alpha}\left((2\alpha+1)a(k_2^*) + \left(\frac{-(2\alpha+1)}{(\alpha+1)^2}\right)b^2(k_2^*)\right) \\
&= T^{2\alpha}(2\alpha+1)\left(a(k_2^*) - \frac{b^2(k_2^*)}{(\alpha+1)^2}\right) > 0,
\end{aligned}$$

as according to (6) the expression in the brackets is positive.

Since the quadratic polynomial with positive highest coefficient (7) has a positive value at T^α, point T^α is not between its roots, that is $T^\alpha \notin [t_1^\alpha(k_1^*,k_2^*),\ \ t_2^\alpha(k_1^*,k_2^*)]$. In Particular by lemma 2.1 since $T > t_1(k_1^*,\ k_2^*)$, we get $T > t_2(k_1^*,\ k_2^*)$. $\square$

Lemmas 2.2 and 2.3 demonstrate that it is impossible to compare in a simple way the values at the points $t_1(k_1, k_2)$ and T because point of minimum $t_2(k_1, k_2)$ is situated between these two points. We have just shown that one should look for $\min\limits_{k_1 \geq 0, k_2 < k_2^0} \max\{f(T, k_1, k_2), f(t_1, k_1, k_2)\}$. The case would be simplified if at least one of the inequalities $f(T, k_1, k_2) \leq f(t_1, k_1, k_2)$ or $f(T, k_1, k_2) \geq f(t_1, k_1, k_2)$ hold for all k_1, k_2. However we are dealing with a more complicated case as shows the following

Theorem 2.1. *For any $k_2 < k_2^0$ the equation $f(t_1(k_1,\, k_2),\ \ k_1,\, k_2) = f(T,\, k_1,\, k_2)$ (with unknown k_1) has two solutions $\widetilde{k_1}$ and $\overline{k_1}$, $0 < \widetilde{k_1} < \overline{k_1}$, moreover*

$$\frac{\partial}{\partial k_1} \left(f(t_1(k_1,\, k_2),\ k_1,\, k_2) - f(T,\, k_1,\, k_2) \right) |_{k_1 = \widetilde{k_1}} > 0,$$

$$\frac{\partial}{\partial k_1} \left(f(t_1(k_1,\, k_2),\ k_1,\, k_2) - f(T,\, k_1,\, k_2) \right) |_{k_1 = \overline{k_1}} = 0.$$

If $k_1 < k_1^{(1)}$ then $f(t_1(k_1,\, k_2),\ k_1,\, k_2) < f(T,\, k_1,\, k_2)$ and vice versa.

Proof. Equate $f(t_1(k_1,\, k_2),\ k_1,\, k_2)$ with $f(T,\, k_1,\, k_2)$ to get

$$k_1^{\frac{2H}{\alpha}} \varphi(k_2) = T^{2H} a(k_2) - 2k_1 \frac{T^{\alpha+1}}{\alpha+1} b(k_2) + k_1^2 T. \tag{8}$$

Multiplying the left and right hand sides of (8) by $\frac{2H}{\alpha\, k_1}$ we get:

$$\frac{2H}{\alpha}\, k_1^{\frac{2H}{\alpha} - 1} \varphi(k_2) = \frac{2H}{\alpha} \cdot \frac{T^{2H} a(k_2)}{k_1} - 4\frac{H \cdot T^{\alpha+1}}{\alpha(\alpha+1)} b(k_2) + \frac{2H}{\alpha} k_1 T. \tag{9}$$

Now differentiate (8) with respect to k_1:

$$\frac{2H}{\alpha}\, k_1^{\frac{2H}{\alpha} - 1} \varphi(k_2) = -\frac{2T^{\alpha+1}}{\alpha+1} b(k_2) + 2k_1 T. \tag{10}$$

Since in (9) and (10) left hand sides are equal, we may equate their right hand sides to get:

$$\frac{2H}{\alpha} \cdot \frac{T^{2H} a(k_2)}{k_1} - 4\frac{H \cdot T^{\alpha+1}}{\alpha(\alpha+1)} b(k_2) + \frac{2H}{\alpha} k_1 T + \frac{2T^{\alpha+1}}{\alpha+1} b(k_2) - 2k_1 T = 0.$$

This implies

$$\frac{2H}{\alpha} \cdot \frac{T^{2H} a(k_2)}{k_1} + \frac{T^{\alpha+1}}{\alpha+1} b(k_2) \left(2 - 4\frac{H}{\alpha} \right) + \left(\frac{2H}{\alpha} - 2 \right) k_1 T = 0. \tag{11}$$

Multiply (11) by $\alpha\, k_1$:

$$2H \cdot T^{2H} a(k_2) - 2k_1 T^{\alpha+1} b(k_2) + k_1^2 T = 0. \tag{12}$$

Find discriminant D_1 of the quadratic in k_1 polynomial $f(T,\, k_1,\, k_2)$:

$$\frac{D_1}{4} = \frac{T^{2\alpha+2}}{(\alpha+1)^2} b^2(k_2) - T^{2H+1} a(k_2) < 0.$$

Find discriminant D_2 of the quadratic in k_1 polynomial (12), that is, of the derivative of the difference between $f(t_1(k_1,\, k_2),\ k_1,\, k_2)$ and $f(T,\, k_1,\, k_2)$ in the points of intersection:

$$\frac{D_2}{4} = T^{2\alpha+2} b^2(k_2) - 2HT^{2\alpha+2} a(k_2).$$

By identical transformation we get:

$$\begin{aligned}\frac{D_2}{4} &= T^{2\alpha+2}\left((c_3(H) - k_2)^2 - (2H - 2k_2 \cdot c_H + k_2^2)\right) \\ &= T^{2\alpha+2}\left(c_3^2(H) - 2k_2 c_3(H) - 2H + 2k_2 c_H\right) \\ &= T^{2\alpha+2}\left(c_3^2(H) - 2H + 2k_2(c_H - c_3(H))\right).\end{aligned}$$

Since

$$k_2^0 = \tfrac{c_3^2(H)-2H}{2(c_3(H)-c_H)},$$

inequality $k_2 < k_2^0$ holds if and only if $D_2 > 0$.

In this case (12) has roots

$$\overline{k_0} = T^{\alpha}\left(b(k_2) - \sqrt{b^2(k_2) - 2Ha(k_2)}\right),$$

and

$$\overline{k_1} = T^{\alpha}\left(b(k_2) + \sqrt{b^2(k_2) - 2Ha(k_2)}\right).$$

Now, $t_1(k_1,\, k_2) = k_1^{\frac{1}{\alpha}} \cdot c_1(k_2)$, where $c_1(k_2) = \left(\frac{b(k_2)-\sqrt{b^2(k_2)-2Ha(k_2)}}{2Ha(k_2)}\right)^{\frac{1}{\alpha}}$. We want to find when $t_1(k_1,\, k_2) = T$. We have

$$\begin{aligned}k_1 = \left(\frac{T}{c_1(k_2)}\right)^{\alpha} &= \frac{T^{\alpha} \cdot 2Ha(k_2)}{b(k_2) - \sqrt{b^2(k_2) - 2Ha(k_2)}} \\ &= T^{\alpha}\left(b(k_2) + \sqrt{b^2(k_2) - 2Ha(k_2)}\right) = \overline{k_1},\end{aligned}$$

and therefore we checked that if $k_1 = \overline{k_1}$ then the following equality $f(t_1(k_1,\, k_2),\ k_1,\, k_2) = f(T,\, k_1,\, k_2)$ holds. Hence point $\overline{k_1}$ is the intersection point of $f(t_1(k_1,\ k_2),\ k_1,\ k_2)$ and $f(T,\ k_1,\ k_2)$, moreover the derivative at this point $\frac{\partial(f(t_1)-f(T))}{\partial k_1}=0$. There is another point of intersection less than $\overline{k_2}$ and such that the derivative $\frac{\partial(f(t_1)-f(T))}{\partial k_1}$ at this point is positive. To prove this fact, consider at first the value $k_2=k_2^0$. Denote $g(k_1,k_2)=k_1^{\frac{2H}{\alpha}}\varphi(k_2)-T^{2H}a(k_2)+2k_1\frac{T^{\alpha+1}}{\alpha+1}b(k_2)-k_1^2T$. It is very easy to see that the second derivative of this function equals $g''_{(k_1,k_2)}(k_1,k_2)=\frac{2H}{\alpha}\left(\frac{2H}{\alpha}-1\right)k_1^{\frac{1}{\alpha}}\varphi(k_2)-2T$, that is equivalent to the equality $k_1\varphi^\alpha(k_2)=\frac{T\alpha^{2\alpha}}{H^\alpha(H+1)^\alpha}$. So, this derivative has the unique zero point of the form $\widehat{k}_1=\left(\frac{T\alpha^2}{H(H+1)\varphi(k_2)}\right)^\alpha$. It means that the first derivative decreases for $k_1<\widehat{k}_1$ and it increases for $k_1>\widehat{k}_1$. At the same time for $k_2=k_2^0$ it is very easy to see that $g(\widehat{k}_1,k_2^0)=0$, $g'_{k_1}(\widehat{k}_1,k_2^0)=0$. It means that $\widehat{k}_1$ is the unique zero point of both the functions $g(k_1,k_2^0)$ and g'_{k_1}. If we calculate the value of the first derivative $g'_{k_1}(\widehat{k}_1,k_2)$ it will be equal, up to constant multiplier, to $\frac{b\varphi^\alpha H^\alpha(\alpha+1)^\alpha-\alpha^{2\alpha}}{\varphi^\alpha H^\alpha(\alpha+1)^\alpha}$. The first derivative $\varphi'(k_2)=c_1(k_2)^{2H}\frac{k_-c_H}{H}+2\frac{c_1^{\alpha+1}}{\alpha+1}$, therefore we can calculate that $(\varphi^\alpha b)'(k_2)>0$ for $\frac{c_3(H)c_H-2H}{c_3(H)-c_H}<k_2<k_2^0$, $(\varphi^\alpha b)'(k_2)<0$ for $k_2<\frac{c_3(H)c_H-2H}{c_3(H)-c_H}$ and $(\varphi^\alpha b)'(k_2)\to 0$ as $k_2\to-\infty$. It means that the first derivative $g'_{k_1}(\widehat{k}_1,k_2)$ is negative for all $k_2<k_2^0$. In tu

rn, it means that the function $g(\widehat{k}_1,k_2)$ is positive, at the point $\overline{k_1}>\widehat{k}_1$ it equals zero and at zero point it is negative. It means that there exists one more point of intersection of the functions $f(T,\ k_1,\ k_2)$ and $f(t_1(k_1,\ k_2),\ k_1,\ k_2)$. Denote this point $\widetilde{k}_1$ and we complete the proof. □

Now we pass to the investigation of the minimization problem concerned with minimal point. At first, state an auxiliary result.

Lemma 2.4. *Function*

$$\zeta(t):=\frac{(T^{2H}-t^{2H})(T-t)}{(T^{\alpha+1}-t^{\alpha+1})^2} \tag{13}$$

is strictly monotonically decreasing on $(0,T)$ *with respect to* t.

Proof. It is sufficient to check that the derivative of the right hand side of (13) with respect to t is negative. We prove this using the following transformations:

$$
\begin{aligned}
\frac{\partial\zeta(t)}{\partial t} &= -(2\alpha+1)t^{2\alpha}T^{\alpha+2} - T^{3\alpha+2} + (2\alpha+2)t^{2\alpha+1}T^{\alpha+1} \\
&\quad + (2\alpha+1)t^{3\alpha+1}T + T^{2\alpha+1}t^{\alpha+1} - (2\alpha+2)t^{3\alpha+2} + 2(\alpha+1)t^{\alpha}T^{2\alpha+2} \\
&\quad - 2(\alpha+1)t^{3\alpha+1}T - 2(\alpha+1)t^{\alpha+1}T^{2\alpha+1} + 2(\alpha+1)t^{3\alpha+2} \\
&= T(T^{\alpha}-t^{\alpha})\Big((2\alpha+1)t^{\alpha}T^{\alpha}(T-t) - T^{2\alpha+1} + t^{2\alpha+1}\Big).
\end{aligned}
$$

Since $T(T^{\alpha}-t^{\alpha})>0$, it is sufficient to prove that

$$
(2\alpha+1)t^{\alpha}T^{\alpha}(T-t) - T^{2\alpha+1} + t^{2\alpha+1} < 0. \tag{14}
$$

Divide (14) by $T^{2\alpha+1}$. By substitution $\frac{t}{T} =: x$ we transform this inequality to:

$$
x^{2\alpha+1} - (2\alpha+1)x^{\alpha+1} + (2\alpha+1)x^{\alpha} - 1 < 0, \quad x \in (0,1). \tag{15}
$$

Since at point $x=0$ expression (15) is equal to -1, and for $x=1$ it is equal to zero, it is sufficient to check that the function is increasing on $[0,1]$, that is, that its derivative is positive. Indeed, take the derivative of (15) with respect to x, and divide it by $2\alpha+1$ to get

$$
x^{2\alpha} - (\alpha+1)x^{\alpha} + \alpha x^{\alpha-1}, \quad x \in (0,1). \tag{16}
$$

Divide (16) by $x^{\alpha-1} \in (0,1)$:

$$
x^{\alpha+1} - (\alpha+1)x + \alpha, \quad x \in (0,1). \tag{17}
$$

Since at $x=0$ the value of (17) is equal to α, and for $x=1$ it is equal to zero it is sufficient to prove that the function is decreasing on $(0,1)$, namely, that its derivative is negative. Differentiate (17) with respect to x: $(\alpha+1)x^{\alpha} - (\alpha+1) < 0$ for $x^{\alpha} < 1$, and the lemma is complete. $\square$

Now we prove that it is impossible to achieve the required minimum at the point connected with T and minimal point (k_1^*, k_2^*).

Lemma 2.5. *Let $a(s)$ be a function of type $a(s) = k_1 + k_2 \cdot s^{\alpha}, \quad k_1, k_2 \in \mathbb{R}, \quad s \in [0,T]$. If $H < H_0$ so that $k_2^* < k_2^0$ then the inequality holds: $f(T, k_1^*, k_2^*) < f(t_1(k_1^*, k_2^*), k_1^*, k_2^*)$.*

Proof. Consider quadratic polynomials on $k_1 > 0$ for any fixed $0 < t < T$:

$$
k_1^2 T - \tfrac{2k_1}{\alpha+1} b(k_2^*) T^{\alpha+1} + T^{2H} a(k_2^*) = f(T, k_1, k_2^*),
$$

$$
k_1^2 t - \tfrac{2k_1}{\alpha+1} b(k_2^*) t^{\alpha+1} + t^{2H} a(k_2^*) = f(t, k_1, k_2^*).
$$

Here $a(k_2) = 1 - k_2 \cdot \frac{c_H}{H} + \frac{k_2^2}{2H} > 0$, $b(k_2) = c_3(H) - k_2$, $k_2 < k_2^0 < 0$.

Consider the difference

$$\begin{aligned}\Delta = \Delta f(T, t, k_1, k_2^*) &:= f(T, k_1, k_2^*) - f(t, k_1, k_2^*)\\ &= k_1^2(T-t) - \frac{2k_1}{\alpha+1} b(k_2^*)(T^{\alpha+1} - t^{\alpha+1}) + a(k_2^*)(T^{2H} - t^{2H}).\end{aligned}$$

Divide Δ by $(T-t)$, and get a quadratic polynomial

$$\Delta_1 = k_1^2 - \frac{2k_1}{\alpha+1} b(k_2^*) \frac{T^{\alpha+1} - t^{\alpha+1}}{T-t} + a(k_2^*) \frac{T^{2H} - t^{2H}}{T-t}.$$

Its discriminant is equal to

$$D = \frac{b^2(k_2^*)}{(\alpha+1)^2} \left(\frac{T^{\alpha+1} - t^{\alpha+1}}{T-t} \right)^2 - a(k_2^*) \frac{T^{2H} - t^{2H}}{T-t}.$$

Consider possible cases:

1) $D < 0$. Then $f(T, k_1, k_2^*) > f(t, k_1, k_2^*)$ for all $k_1 > 0$, in particular $f(T, k_1^*, k_2^*) > f(t, k_1^*, k_2^*)$.

2) $D > 0$. Then there exist two roots $k_1^{1,2} = \frac{b(k_2^*)}{\alpha+1} \frac{T^{\alpha+1} - t^{\alpha+1}}{T-t} \pm \sqrt{D}$,

moreover $f(T, k_1, k_2^*) < f(t, k_1, k_2^*)$ for $k_1^1 < k_1 < k_1^2$ and an opposite inequality holds for $k_1 < k_1^1$, $k_1 > k_1^2$.

3) $D = 0$. Then $f(T, k_1, k_2^*) > f(t, k_1, k_2^*)$ for all k_1 but $k_1 = \frac{b(k_2^*)}{\alpha+1} \frac{T^{\alpha+1} - t^{\alpha+1}}{T-t}$. For this value of k_1 we have an equality.

Transform D so that we could easier investigate its sign. Instead of D, first consider

$D_3 = \frac{b^2(k_2^*)}{a(k_2^*)(\alpha+1)^2} - \frac{\varphi(T,t)}{\psi(T,t)}$, where $\varphi(T, t) = \frac{T^{2H} - t^{2H}}{T-t}$, $\psi(T, t) = \left(\frac{T^{\alpha+1} - t^{\alpha+1}}{T-t} \right)^2$.

It is easy to see $\frac{\varphi(T,t)}{\psi(T,t)} = \zeta(t)$ (see Lemma 2.4).

If $t = 0$ then $\zeta(t) = 1$, and since $a(k_2^*) > \frac{b^2(k_2^*)}{(\alpha+1)^2}$, we have $D_3 < 0$.

If $t \to T$ then

$$\begin{aligned}\lim_{t \uparrow T} \zeta(t) &= \lim_{t \uparrow T} \frac{(T^{2H} - t^{2H})(T-t)}{(T^{\alpha+1} - t^{\alpha+1})^2}\\ &= \lim_{t \uparrow T} \frac{-2Ht^{2H-1}T - T^{2H} + (2H+1)t^{2H}}{-2(\alpha+1)t^\alpha T^{\alpha+1} + 2(\alpha+1)t^{2\alpha+1}}\\ &= \lim_{t \uparrow T} \frac{-2H(2H-1)t^{2H-2}T + (2H+1)2Ht^{2H-1}}{-2(\alpha+1)\alpha\, t^{\alpha-1}T^{\alpha+1} + 2(\alpha+1)(2\alpha+1)t^{2\alpha}}\\ &= \frac{-4H^2 + 2H + 4H^2 + 2H}{-2(H+\frac{1}{2})(H-\frac{1}{2}) + 2(H+\frac{1}{2})2H}\\ &= \frac{4H}{(H+\frac{1}{2})(4H - 2H + 1)} = \frac{2H}{(H+\frac{1}{2})^2},\end{aligned}$$

and $b^2(k_2^*) - 2Ha(k_2^*) > 0$ hence, $D_3 > 0$.

Generally D_3 is strictly monotonically increasing with respect to t, as by Lemma 2.4 $\zeta(t)$ is strictly monotonically decreasing with respect to t. $f(t_1(k_1,\, k_2),\ k_1,\, k_2) = f(T,\, k_1,\, k_2)$

Hence there exists a unique t_0 such that for $0 < t < t_0$, $D_3 < 0$, and for $t_0 < t \le T$, $D_3 > 0$. Moreover, if $0 < t < t_0$, then at k_1^*

$$\begin{aligned}\Delta = \Delta f(T, t, k_1^*, k_2^*) &:= f(T, k_1^*, k_2^*) - f(t, k_1^*, k_2^*)\\ &= (k_1^*)^2\, T - \frac{2k_1^*}{\alpha+1} b(k_2^*) T^{\alpha+1} + T^{2\alpha+1} a(k_2^*)\\ &> (k_1^*)^2\, t - \frac{2k_1^*}{\alpha+1} b(k_2^*) t^{\alpha+1} + t^{2\alpha+1} a(k_2^*).\end{aligned}$$

We do not know whether $D_3 > 0$ or $D_3 < 0$ for $t = t_1(k_1^*, k_2^*)$. Assume that we verified one of the two inequalities: either $0 < t_1(k_1^*, k_2^*) < t_0$ or $t_1(k_1^*, k_2^*) > t_0$ but $k_1^* < k_1^1$. The latter inequality is equivalent to:

$$\frac{b(k_2^*)}{\alpha+1} T^\alpha < \frac{b(k_2^*)(T^{\alpha+1} - (t_1(k_1^*, k_2^*))^{\alpha+1})}{(\alpha+1)(T - t_1(k_1^*, k_2^*))} - \sqrt{D},$$

or

$$D < \frac{b^2(k_2^*)}{(\alpha+1)^2} (t_1(k_1^*, k_2^*))^2 \frac{(T^\alpha - (t_1(k_1^*, k_2^*))^\alpha)^2}{(T - t_1(k_1^*, k_2^*))^2}.$$

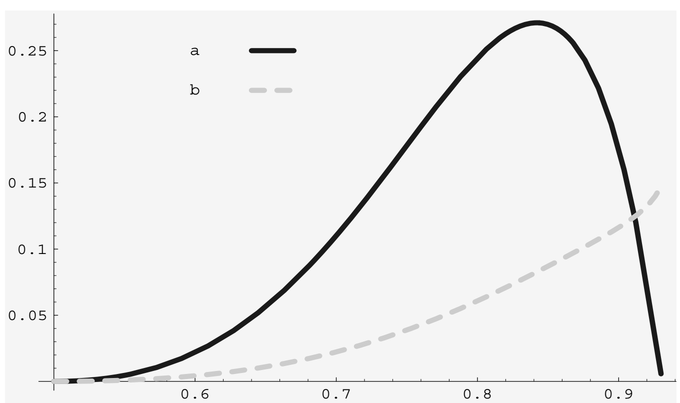

Fig.1

Then in both cases we would have inequality $f(T, k_1^*, k_2^*) >$ $f(t_1(k_1^*, k_2^*), k_1^*, k_2^*)$. Since both of these statements are very complicated to check them analytically we suggest to verify them graphically with graphs (a) and (b) presented in Figure 1 (we used Mathematica tool to plot the graphs). Graph (a) (firm line) shows D at point $t = t_1(k_1^*, k_2^*)$ against H, whereas graph (b) (dotted line) plots $\frac{b^2(k_2^*)}{(\alpha+1)^2}(t_1(k_1^*, k_2^*))^2 \frac{(T^\alpha - (t_1(k_1^*, k_2^*))^\alpha)^2}{(T - t_1(k_1^*, k_2^*))^2}$

against H. Graph (a) shows that $D > 0$ at point $t = t_1(k_1^*, k_2^*)$ consequently $D_3 > 0$ at the same point since they are connected via positive multiplier. Therefore $t_1(k_1^*, k_2^*) > t_0$ and we cannot give positive conclusion by this way. Also, from graph (b) it is clear that

$$D > \frac{b^2(k_2^*)}{(\alpha+1)^2}(t_1(k_1^*, k_2^*))^2 \frac{(T^\alpha - (t_1(k_1^*, k_2^*))^\alpha)^2}{(T - t_1(k_1^*, k_2^*))^2}$$

for any $H < H_0$ that is $k_1^* < k_1^1$ so the second possible positive conclusion does not hold.

The lemma is complete. □

Corollary 2.1. For the solution of minimization problem we must calculate the value

$$\min_{k_1 \geq 0, k_2 < k_2^0} \max\{f(T, k_1, k_2), f(t_1, k_1, k_2)\}.$$

3. Approximation results for fBm with Hurst index, separated from 1, and the integrals with integrands of the form $a(s) = k_1 + k_2 \cdot s^\alpha, k_1,\ k_2 \in \mathbb{R}$

At last we state the "positive" result concerning the minimal point and the minimal value

$$\min_{k_1 \geq 0, k_2 < k_2^0} \max\{f(T, k_1, k_2), f(t_1, k_1, k_2)\}.$$

Theorem 3.1. *The minimal value* $\min_{k_1 \geq 0, k_2 < k_2^0} \max\{f(T, k_1, k_2), f(t_1, k_1, k_2)\}$ *is achieved at the point* $k_1^0 = \frac{b(\widehat{k}_2)T^\alpha}{\alpha+1}$, $\widehat{k}_2 = \frac{c_H(H+1/2)^2 - 2Hc_3(H)}{\alpha^2}$. *This value equals* $T^{2H}\Big(a(\widehat{k}_2) - \frac{b^2(\widehat{k}_2)}{(\alpha+1)^2}\Big)$.

Proof. Consider the function $f(T, k_1, k_2) = T^{2H}a(k_2) - 2k_1\frac{T^{\alpha+1}}{\alpha+1}b(k_2) + k_1^2 T$. Evidently, the minimal value of this function for fixed k_2 is achieved at the point $k_1 = \frac{b(k_2)T^\alpha}{\alpha+1}$ and equals $T^{2H}\Big(a(k_2) - \frac{b^2(k_2)}{(\alpha+1)^2}\Big)$. In turn, this formula gives a quadratic polynomial function of k_2 and achieves its minimum at the point $\widehat{k}_2$. It is evident from Figure 2 that for $H < H_0$ we have the inequality $\widehat{k}_2 < k_2^0$. Moreover, we can see from Figure 3 that $f(T, k_1^0, \widehat{k}_2) = f(t_1(k_1^0, \widehat{k}_2), k_1^0, \widehat{k}_2) = (k_1^0)^{2H/\alpha}\varphi(\widehat{k}_2)$. Without any doubt, the value $f(T, k_1^0, \widehat{k}_2)$ is a required minimal value. It completes the proof. □

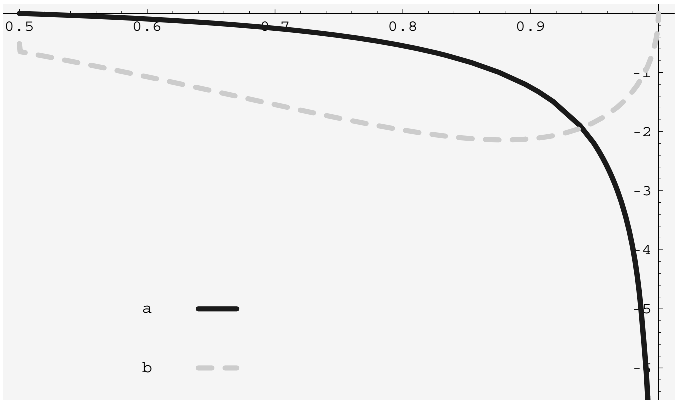

Fig.2

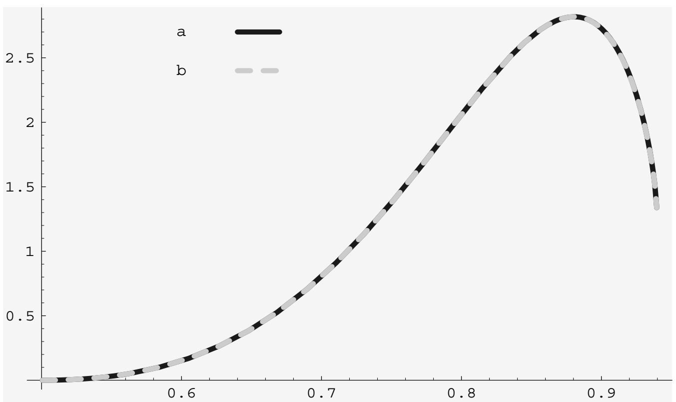

Fig.3

Here are values of the functions shown on the Figure 3.

H	a	b
0.51	0.00106525	0.00106525
0.55	0.0313579	0.0313579
0.59	0.117903	0.117903
0.63	0.28104	0.28104
0.67	0.539574	0.539574
0.71	0.905731	0.905731
0.75	1.37669	1.37669
0.79	1.9206	1.9206
0.83	2.45322	2.45322
0.87	2.797	2.797
0.91	2.59085	2.59085
0.95	0.981965	0.981965

4. Conclusion

It was shown in [3] that in the case when $a(s)$ is a function of type $a(s) = k_1 + k_2 \cdot s^\alpha, k_1,\ k_2 \in \mathbb{R}, s \in [0,T]$, and $H > H_0$ is such that $k_2^* > k_2^0$, then the minimum of the function $\max\limits_{t\in[0,T]} f(t,k_1,k_2)$ is reached at $k_1 = k_1^*, k_2 = k_2^*$ and is equal to $T^{2\alpha+1} - T^{2\alpha+1}c_H^2 \left(B^2(1-\alpha,\alpha) - \frac{2B(1-\alpha,\alpha)}{\alpha} + \frac{(\alpha+1)^2}{\alpha^2(2\alpha+1)}\right)$. In the present paper we obtain the approximation results for the same type of function, for $H < H_0$ and $k_2^* < k_2^0$. We demonstrate that in this case, in contrary to the case $H > H_0$, it is impossible to achieve minimum at the minimal point $k_1 = k_1^*, k_2 = k_2^*$ or in some connected point, and demonstrate analytically and with the help of the graph where the minimum "is situated" and how to calculate it. So, this paper completes the problem of minimization of the distance between fBm and Wiener integrals with the integrands of the form $a(s) = k_1 + k_2 \cdot s^\alpha, k_1,\ k_2 \in \mathbb{R}$ in the space $L_\infty([0,T]; L_2(\Omega))$.

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Department of Probability Theory and Mathematical Statistics, Kyiv National Taras Shevchenko University, Kyiv, Ukraine
E-mail address: bannaya@mail.univ.kiev.ua, myus@univ.kiev.ua