Theory of Stochastic Processes
Vol. 14 (30), no. 2, 2008, pp. 145–154

UDC 519.21

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NONHOMOGENEOUS DIFFUSION PROCESSES IN A HALFSPACE WHOSE BEHAVIOUR ON THE BOUNDARY IS DESCRIBED BY GENERAL WENTZEL BOUNDARY CONDITION

Using analytical methods, we consider the problem of constructing a nonhomogeneous multidimensional diffusion process in a halfspace with given diffusion characteristics at the inner points and general Wentzel boundary conditions.

1. Introduction. In this paper, we consider the problem of constructing a multiplicative operator family that describes a multidimensional nonhomogeneous diffusion process in a domain with general Wentzel boundary conditions [1]. Analytical methods are used to solve this problem. Within these methods, the desired operator family can be determined using the solution of the corresponding boundary-value problem, where the boundary conditions as well as the equation in the domain is described by a second-order parabolic linear partial differential equation with a term that contains the directional derivative on the normal to the boundary. In turn, we have obtained the classical solvability of the Wentzel parabolic problem by using a simple-layer potential [2]. Here, we will confine ourselves to a model problem, where the diffusion process is given in a domain that is the upper half-space in an Euclidian space. Note that the stated problem with its special cases previously was studied using different approaches mostly for homogeneous processes [3–7]. As for the Wentzel parabolic boundary problem, it was studied, besides [2], also in [8–10], by using different methods.

2. Basic notations and definitions. Let $\mathbb{R}^n$, $n \geq 2$, be the n-dimensional Euclidian space; $\mathbb{R}_t^{n+1} = [0,t) \times \mathbb{R}^n$, $0 < t \leq T$, $T > 0$ is fixed; $\mathbb{R}_t^n = [0,t) \times \mathbb{R}^{n-1}$; $x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$ a point in $\mathbb{R}^n$; $x' = (x_1, \dots, x_{n-1})$ a point in $\mathbb{R}^{n-1}$; (s,x) a point in $\mathbb{R}_t^{n+1}$; (s,x') a point in $\mathbb{R}_t^n$; $|x|^2 = \sum\limits_{i=1}^{n} x_i^2$; $|x'|^2 = \sum\limits_{i=1}^{n-1} x_i^2$; $(x,y) = \sum\limits_{i=1}^{n} x_i y_i$; $(x', y') = \sum\limits_{i=1}^{n-1} x_i y_i$.

We will use the following notations for the differential operators: D_s^r and D_x^p are the partial derivatives with respect to s of order r and any partial derivative with respect to x of order p, respectively, where r and p are integer nonnegative numbers; $D_t^1 = D_t$, $D_i \equiv \frac{\partial}{\partial x_i}$, $D_{ij} = D_{ji} \equiv \frac{\partial^2}{\partial x_i \partial x_j}$, $i,j = 1, \dots, n$; $\nabla' = (D_1, \dots, D_{n-1})$.

Similarly as in [11, p. 16], $H_{s}^{\frac{k+\lambda}{2}}{}_{,x}^{k+\lambda}(\overline{B}) \equiv H^{\frac{k+\lambda}{2}, k+\lambda}(\overline{B})$ ($k = 0,1,2$, $\lambda \in (0,1)$, B is a domain in the space $\mathbb{R}_t^{n+1}$ or $\mathbb{R}_t^n$, $\overline{B}$ is the closure of B) mean the respective Hölder spaces; $\underset{0}{H}{}^{\frac{k+\lambda}{2}, k+\lambda}(\overline{B})$ is a set of functions from $H^{\frac{k+\lambda}{2}, k+\lambda}(\overline{B})$ that (in case $k = 2$ also with the derivative with respect to s) vanishes when $s = t$.

2000 *AMS Mathematics Subject Classification.* Primary 60J60.

Key words and phrases. Nonhomogeneous diffusion process, general Wentzel boundary condition, analytical method, parabolic potentials.

By $||w||_{H^{\frac{k+\lambda}{2},\,k+\lambda}(\overline{B})}$, we denote the norm of a function w in $H^{\frac{k+\lambda}{2},\,k+\lambda}(\overline{B})$. Also we use the Hölder spaces $H^{k+\lambda}(\mathbb{R}^n)$, $H^{k+\lambda}(\mathbb{R}^{n-1})$, aggregates of continuous functions $C^k(B)$, $C^{1,\,2}(B)$, and the Banach space of bounded measurable functions $\mathcal{B}(\mathbb{R}^n)$ with the norm $||\varphi|| = \sup\limits_{x\in\mathbb{R}^n} |\varphi(x)|$.

In $\mathbb{R}^n$, we consider the domain $D = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$ with the boundary $S = \mathbb{R}^{n-1}$, and, in $\overline{\mathbb{R}}^{n+1}_t$, we consider $\Omega_t = \{(t,x) \in \mathbb{R}^{n+1}_t \,|\, x_n > 0\}$ with the side boundary $\Sigma_t = \overline{\mathbb{R}}^n_t$. By $\nu(x') = (0,\dots,0,1) \in \mathbb{R}^n$, we denote the inner unit normal vector to S at $x' \in \mathbb{R}^{n-1}$. Everywhere below, C and c are positive constants that do not depend on (s,x), and their specific values are not interesting for us.

3. Parabolic potentials. Regularizer. In a layer $\mathbb{R}^{n+1}_T$, let us consider a second-order uniformly parabolic operator with real coefficients,

(1)
$$Lu \equiv \frac{1}{2}\sum_{i,j=1}^{n} b_{ij}(s,x)D_{ij}u(s,x) + \sum_{i=1}^{n} a_i(s,x)D_iu(s,x) + D_su(s,x), \qquad (s,x) \in \mathbb{R}^{n+1}_t.$$

Assume that the coefficients of the operator L are defined in $\overline{\mathbb{R}}^{n+1}_T$, and the following assumptions are true:

A1) $\sum\limits_{i,j=1}^{n} b_{ij}(s,x)\xi_i\xi_j \geq \delta_0|\xi|^2$, $b_{ij} = b_{ji}$, $\delta_0 > 0$, $\forall (s,x) \in \overline{\mathbb{R}}^{n+1}_T$, $\forall \xi \in \mathbb{R}^n$;

A2) $b_{ij},\, a_i \in H^{\frac{\lambda}{2},\,\lambda}(\overline{\mathbb{R}}^{n+1}_t)$, $i,j = 1,\dots,n$.

Assumptions A1), A2) guarantee the existence of a fundamental solution (f.s.)

$$g(s,x,t,y), \qquad 0 \leq s < t \leq T,\ x,y \in \mathbb{R}^n,$$

for the operator L ([4, 11]):

$$g(s,x,t,y) = g_0(s,x,t,y) + g_1(s,x,t,y), \tag{2}$$

where

$$\begin{aligned} g_0(s,x,t,y) &= g_0^{(t,y)}(t-s, x-y) \\ &= (2\pi(t-s))^{-n/2}(\det b(t,y))^{-1/2} \exp\left\{-\frac{(b^{-1}(t,y)(y-x), y-x)}{2(t-s)}\right\}, \end{aligned}$$

$b(t,y) = (b_{ij}(t,y))^n_{i,j=1}$, $b^{-1}(t,y) = (b^{ij}(t,y))^n_{i,j=1}$ is the inverse matrix to $b(t,y)$, g_1 is the integral term with a "weaker" singularity than that of g_0 at $s \to t$, and $g \equiv 0$ if $s \geq t$. A function $g(s,x,t,y)$ is nonnegative continuous with respect to the aggregate of the variables, with fixed $t \in (0,T]$, $y \in \mathbb{R}^n$, as a function of the arguments $(s,x) \in [0,t) \times \mathbb{R}^n$, it satisfies the equation $Lu = 0$ and the condition

$$\lim_{s\uparrow t} \int\limits_{\mathbb{R}^n} g(s,x,t,y)\varphi(y)dy = \varphi(x), \tag{3}$$

for every $t \in (0,T]$, $x \in \mathbb{R}^n$, and a bounded continuous function $\varphi(x)$ on $\mathbb{R}^n$.

Among other properties of f.s. g, we also note the following ones:

1) $\int\limits_{\mathbb{R}^n} g(s,x,t,y)\,dy = 1$ for all $0 \leq s < t \leq T$, $x \in \mathbb{R}^n$;

2) $\int\limits_{\mathbb{R}^n} g(s,x,t,y)g(t,y,u,z)\,dy = g(s,x,u,z)$ for all $0 \leq s < t \leq T$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$;

3) for $0 \leq s < t \leq T$, $x \in \mathbb{R}^n$, $\Theta \in \mathbb{R}^n$, the next assumptions hold:

$\int\limits_{\mathbb{R}^n} (y-x,\Theta)g(s,x,t,y)dy = \int\limits_s^t d\tau \int\limits_{\mathbb{R}^n} (a(\tau,y),\Theta)g(s,x,t,y)dy,$

$$\int_{\mathbb{R}^n}(y-x,\Theta)^2 g(s,x,t,y)dy=\int_s^t d\tau\int_{\mathbb{R}^n} g(s,x,\tau,z)(b(\tau,z)\Theta,\Theta)dz+$$
$$+2\int_s^t d\tau\int_{\mathbb{R}^n} g(s,x,\tau,z)(a(\tau,z),\Theta)(z-x,\Theta)dz;$$

4) there exist the positive constants C and c such that, for the functions g and g_1 with $0\le s<t\le T$, x, $y\in\mathbb{R}^n$, the estimations

$$|D_s^r D_x^p g(s,x,t,y)|\le C(t-s)^{-\frac{n+2r+p}{2}}\exp\left\{-c\,\frac{|x-y|^2}{t-s}\right\},\quad 2r+s\le 2, \tag{4}$$

$$|D_s^r D_x^p g_1(s,x,t,y)|\le C(t-s)^{-\frac{n+2r+p-\lambda}{2}}\exp\left\{-c\,\frac{|x-y|^2}{t-s}\right\},\quad 2r+s\le 2, \tag{5}$$

hold.

Let us consider the integral representing a parabolic simple-layer potential:

$$u_1(s,x,t)=\int_s^t d\tau\int_{\mathbb{R}^{n-1}} g(s,x,\tau,z')V(\tau,z')dz', \tag{6}$$

where $0\le s<t\le T$, $x\in\mathbb{R}^n$, $V(t,z')$ is a bounded measurable function given on $\Sigma=\overline{\mathbb{R}}^n_T$. From properties of f.s. g and (4), it follows that $u_1(s,x,t)$, as a function of the arguments $(s,x)\in[0,t)\times\mathbb{R}^n$, is continuous for $0\le s<t\le T$, $x\in\mathbb{R}^n$ and satisfies the equation $Lu_1=0$ in the domain $(s,x)\in[0,t)\times(\mathbb{R}^n\backslash S)$ and the zero initial condition $\lim\limits_{s\uparrow t}u(s,x,t)=0$.

Let, for $(s,x')\in\overline{\mathbb{R}}^n_T$, the conormal vector $N(s,x')=(N_1(s,x'),\dots,N_n(s,x'))$,

$$N_i(s,x')=\sum_{j=1}^n b_{ij}(s,x')\nu_j(x'),\qquad i=1,\dots,n,$$

be defined. Then if $V\in \underset{0}{H}^{\frac{\lambda}{2},\lambda}(\overline{\mathbb{R}}^n_t)$, then $u_1\in \underset{0}{H}^{\frac{1+\lambda}{2},1+\lambda}(\overline{\mathbb{R}}^n_t)$ [12]; and, for the conormal derivative of the function u_1, a jump formula holds ([4, p. 59], [11, p. 459]):

$$\frac{\partial u_1(s,t,x')}{\partial N(s,x')}=\lim_{\substack{x\to x'\\ x\in\mathbb{R}^n_+}}\frac{\partial u_1(s,t,x)}{\partial N(s,x')}=-V(s,x')+\int_s^t d\tau\int_{\mathbb{R}^{n-1}}\frac{\partial g(s,x',\tau,z')}{\partial N(s,x')}V(\tau,z')dz',\tag{7}$$
$$(s,x')\in\mathbb{R}^n_t.$$

The integral on the right-hand side of (7) is called the direct value of the conormal derivative of a simple-layer potential. Its existence follows from the inequality ($0\le s<t\le T$, x', $z'\in\mathbb{R}^{n-1}$)

$$\left|\frac{\partial g(s,x',\tau,z')}{\partial N(s,x')}\right|\le C(\tau-s)^{-\frac{n+1-\lambda}{2}}\exp\left\{-c\,\frac{|x'-z'|^2}{\tau-s}\right\}. \tag{8}$$

Now we will define a boundary operator $\mathcal{E}$ that further will be used as the regularizer of a first-kind Volterra integral equation equivalent in some sense to the boundary-value problem formulated below in Section 4. To this end, in $\mathbb{R}^n_T$, we consider the parabolic operator

$$L'\equiv\frac{1}{2}\sum_{i,j=1}^{n-1}h_{ij}(s,x')D_{ij}-D_t,\qquad (s,x')\in\mathbb{R}^n_T,$$

whose coefficients are defined by the relation

$$h_{ij} = b_{ij} - \frac{b_{in} b_{jn}}{b_{nn}}, \qquad i, j = 1, \ldots, n-1.$$

From assumptions A1), A2), it follows that the operator L' is uniformly parabolic, the functions $h_{ij}(s, x')$ are Hölder with respect to two variables. Moreover, for the operator L', there exists f.s.

$$h(s, x', t, y') = h_0(s, x', t, y') + h_1(s, x', t, y'), \qquad 0 \le s < t \le T, \quad x',\, y' \in \mathbb{R}^{n-1}, \tag{9}$$

where h_0 and h_1, as in (2), mean, respectively, the main and additional (integral) terms of f.s. h.

For f.s. h, properties 1)–4) of f.s. g from (2) hold with obvious changes.

Note also a connection between the functions $g(s, x', t, y')$ and $h(s, x', t, y')$. Using formula (2) and the definition of f.s. h, we easily obtain the equality

$$g(s, x', t, y') = (2\pi b_{nn}(t, y')(t-s))^{-1/2} \left[h(s, x', t, y') - h_1(s, x', t, y') \right] + g_1(s, x', t, y'), \qquad 0 \le s < t \le T, \quad x',\, y' \in \mathbb{R}^{n-1}. \tag{10}$$

Consider an integro-differential operator $\mathcal{E} : \Psi_0 \to \mathcal{E}\Psi_0$ that acts on functions Ψ_0 from $\mathbb{R}^n_T$ in the following way:

$$\mathcal{E}(s, x', t)\Psi_0 = \sqrt{\frac{2}{\pi}} \left\{ \frac{\partial}{\partial s} \int_s^t (\tau - s)^{-1/2} d\tau \int_{\mathbb{R}^{n-1}} h(\hat{s}, x', \tau, y') \Psi_0(\tau, y') dy' \right\} \Big|_{\hat{s}=s}, \qquad (s, x') \in [0, t) \times \mathbb{R}^{n-1}. \tag{11}$$

From results obtained in [12], it follows that $\mathcal{E}$ is a linear bounded operator that acts from $\underset{0}{H}^{\frac{1+\lambda}{2}, 1+\lambda}(\overline{\mathbb{R}}^n_t)$ into $\underset{0}{H}^{\frac{\lambda}{2}, \lambda}(\overline{\mathbb{R}}^n_t)$, for which there exists an inverse operator $\mathcal{E}^{-1}$. Additionally, we determine that the operator $\mathcal{E}$ converts the functions from $\underset{0}{H}^{\frac{2+\lambda}{2}, 2+\lambda}(\overline{\mathbb{R}}^n_t)$ into $\underset{0}{H}^{\frac{1+\lambda}{2}, 1+\lambda}(\overline{\mathbb{R}}^n_t)$. In addition, $\mathcal{E}$ is a regularizer in the case of the first boundary-value problem [12]. Namely, by using relations (10), (11), and (3), properties 1)–3), and inequalities (4) and (5), we obtain

$$\mathcal{E}(s, x', t)u_1 = -\tilde{V}(s, x') + \int\limits_s^t d\tau \int\limits_{\mathbb{R}^{n-1}} R(s, x', \tau, y') V(\tau, y') dy', \qquad \forall\, V \in \underset{0}{H}^{\frac{\lambda}{2}, \lambda}(\overline{\mathbb{R}}^n_T), \tag{12}$$

where $\tilde{V}(s, x') = (b_{nn}(s, x'))^{-1/2} V(s, x')$, and estimation (8) holds for the kernel R.

Using f.s. g from (2), we can determine two more potentials that are used to solve the Cauchy problem for a generalized second-order parabolic equation. There are the Poisson potential

$$u_2(s, x, t) = \int\limits_{\mathbb{R}^n} g(s, x, t, y) \varphi(y) dy, \qquad (s, x) \in \mathbb{R}^n_t \tag{13}$$

, and the volume potential

$$u_3(s, x, t) = \int\limits_s^t d\tau \int\limits_{\mathbb{R}^n} g(s, x, \tau, z) f(\tau, z) dz, \qquad (s, x) \in \mathbb{R}^n_t,$$

where $\varphi(y)$, $y \in \mathbb{R}^n$, and $f(\tau, z)$, $(\tau, z) \in \overline{\mathbb{R}}^{n+1}_T$, are the given functions.

Assume that φ is bounded and continuous in $\mathbb{R}^n$, and $f \in H^{\frac{\lambda}{2}, \lambda}(\overline{\mathbb{R}}^{n+1}_t)$. Then one can affirm (see [11, Ch. IV, §14]) that the functions u_i, $i = 2, 3$, are continuous in $\overline{\mathbb{R}}^{n+1}_t$ and satisfy the equation $Lu_2 = 0$, $Lu_3 = -f$ in the domain $(s, x) \in [0, t) \times \mathbb{R}^n$

also as the initial conditions $\lim\limits_{s\uparrow t} u_2(s,x,t) = \varphi(x)$, $\lim\limits_{s\uparrow t} u_3(s,x,t) = 0$, $x \in \mathbb{R}^n$. Also $u_3 \in H^{\frac{2+\lambda}{2},2+\lambda}(\overline{\mathbb{R}}_t^{n+1})$ and, in the case where $\varphi \in H^{2+\lambda}(\mathbb{R}^n)$, $u_2 \in H^{\frac{2+\lambda}{2},2+\lambda}(\overline{\mathbb{R}}_t^{n+1})$.

4. Problem statement and its solution. Assume that, in $\overline{\mathcal{D}} = \overline{\mathbb{R}}_+^n$, a generating second-order differential operator of some nonhomogeneous diffusion process that acts on the set of all twice continuously differentiable functions with compact carriers $C_k^2(\overline{\mathcal{D}})$ is given as

$$\mathcal{L}\varphi(x) = \frac{1}{2}\sum_{i,j=1}^{n} b_{ij}(s,x) D_{ij}\varphi(x) + \sum_{i=1}^{n} a_i(s,x) D_i\varphi(x), \tag{14}$$

where $b_{ij}(s,x)$, $a_i(s,x)$ are bounded continuous functions on $\overline{\Omega}_T$, $b(s,x) = (b_{ij}(s,x))_{i,j=1}^n$ is a symmetric nonnegative definite matrix. Assume also that the Wentzel boundary operator [1] is given, i.e., a mapping from $C_k^2(\overline{\mathcal{D}})$ into the space of all continuous functions on $\Sigma_T = \overline{\mathbb{R}}_T^n$ has the form

$$\mathcal{L}_0\varphi(x) = \frac{1}{2}\sum_{k,l=1}^{n-1} \beta_{kl}(s,x) D_{kl}\varphi(x) + \sum_{k=1}^{n-1} \alpha_k(s,x) D_k\varphi(x) + q(s,x)\frac{\partial\varphi(x)}{\partial x_n} - \rho(s,x)\mathcal{L}\varphi(x), \quad (s,x) \in \Sigma_T, \tag{15}$$

where $\beta_{kl}(s,x)$, $\alpha_k(s,x)$, $q(s,x)$, $\rho(s,x)$ are bounded continuous functions on Σ_T such that $\beta(s,x) = (\beta_{kl}(s,x))_{k,l=1}^{n-1}$ is a symmetric nonnegative definite matrix, $q(s,x) \geq 0$, $\rho(s,x) \geq 0$.

We will set up the problem to construct a multiplicative operator family T_{st}, $0 \leq s < t \leq T$, that describes a continuous nonbreaking Feller process at $\overline{\mathcal{D}}$ such that it coincides with a diffusion process controlled by the operator $\mathcal{L}$ at the inner points of $\mathcal{D}$, and its behaviour on the boundary S is determined by the boundary condition

$$\mathcal{L}_0\varphi(x) = 0, \qquad x \in S. \tag{16}$$

We will use analytical methods to solve the problem. It means (see [3, 4]) that the required operator family will be determined using a solution of the following parabolic boundary-value problem with respect to $T_{st}\varphi(x) = u(s,x,t)$:

$$Lu = \mathcal{L}u + D_s u = 0, \qquad (s,x) \in \Omega_t, \tag{17}$$

$$\lim_{s\uparrow t} u(s,x,t) = \varphi(x), \qquad x \in \overline{\mathbb{R}}_+^n, \tag{18}$$

$$L_0 u = \frac{1}{2}\sum_{k,l=1}^{n-1} \beta_{kl}(s,x) D_{kl}u + \sum_{k=1}^{n-1} \alpha_k(s,x) D_k u + q(s,x) D_n u + \rho(s,x) D_s u = 0, \quad (s,x) \in \mathbb{R}_t^n. \tag{19}$$

Solving the Wentzel parabolic boundary-value problem using the potential method. We will study the classical solvability of problem (17)–(19) assuming that, for the coefficients of the operator L from (17), assumptions A1), A2) hold. Moreover, for the coefficients of the operator L_0 from (19), excepting the mentioned assumptions, the following assumptions hold:

B1) $\sum\limits_{k,l=1}^{n-1} \beta_{kl}(s,x')\xi_k\xi_l \geq \mu_0|\xi'|^2$, $\mu_0 > 0$, $\forall\,(s,x') \in \overline{\mathbb{R}}_T^n$, $\forall\,\xi' \in \mathbb{R}^{n-1}$;

B2) β_{kl}, $\alpha_k, q \in H^{\frac{\lambda}{2},\lambda}(\overline{\mathbb{R}}_T^n)$, $\rho \equiv 1$, $\inf\limits_{s,x'} q(s,x') > 0$.

Also we assume that the function φ from (18) is smooth enough and satisfies the fitting condition

$$\frac{1}{2}\sum_{k,l=1}^{n-1}\beta_{kl}(s,x')D_{kl}\varphi(x')+\sum_{k=1}^{n-1}\alpha_k(s,x')D_k\varphi(x')+q(s,x')D_n\varphi(x')-\mathcal{L}\varphi(x')=0,$$

$$s=t, \qquad x'\in\mathbb{R}^{n-1}. \tag{20}$$

Theorem 1. *Let, for the coefficients of the operators L and L_0 from (1) and (19), conditions A1), A2) and B1), B2) hold, respectively. Then, for every function $\varphi \in H^{2+\lambda}(\mathbb{R}^n)$ from (18) that satisfies the fitting condition (20), problem (17)–(19) has the unique solution*

$$u \in H^{\frac{2+\lambda}{2},\,2+\lambda}(\overline{\Omega}_t), \tag{21}$$

and the estimation

$$||u||_{H^{\frac{2+\lambda}{2},\,2+\lambda}(\overline{\Omega}_t)} \le C||\varphi||_{H^{2+\lambda}(\mathbb{R}^n)} \tag{22}$$

is true.

Proof. We will look for a solution of problem (17)–(19) of the form

$$u(s,x,t)=u_1(s,x,t)+u_2(s,x,t), \tag{23}$$

where the functions u_1 and u_2 are defined by formulas (6) and (13), respectively, and the density V from the simple-layer potential (6) is unknown. In order to find it, we will use the boundary condition (19). Consider V to be a function of the arguments s, x, t and also consider *a priori* that, as a function dependent on (s,x), it belongs to the Hölder class $\underset{0}{H}^{\frac{\lambda}{2},\,\lambda}(\overline{\mathbb{R}}^n_t)$. Separating the conormal derivative in the formula for L_0u in (19) and executing simple transformations, we obtain the equality

$$L_0'u=\sum_{k,l=1}^{n-1}\beta_{kl}(s,x')D_{kl}u+\sum_{k=1}^{n-1}\alpha_k^{(0)}(s,x')D_ku+D_su=-\Theta^{(0)}(s,x',t), \quad (s,x')\in\mathbb{R}^n_t, \tag{24}$$

where

$$\alpha_k^{(0)}(s,x')=\alpha_k(s,x')-q(s,x')\,\frac{b_{kn}(s,x')}{b_{nn}(s,x')},\; k=1,\dots,n-1,$$

$$\Theta^{(0)}(s,x',t)=\frac{q(s,x')}{b_{nn}(s,x')}\frac{\partial u(s,x',t)}{\partial N(s,x')}=$$
$$=\frac{q(s,x')}{b_{nn}(s,x')}\left[\frac{\partial u_2(s,x',t)}{\partial N(s,x')}-V(s,x',t)+\int_s^t d\tau\int_{\mathbb{R}^{n-1}}\frac{\partial g(s,x',\tau,z')}{\partial N(s,x')}V(\tau,z',t)dz'\right].$$

Further we will consider equality (24) to be an independent parabolic equation in $\mathbb{R}^n_t$ for the function $u(s,x',t)$. As follows from the conditions of Theorem 1, additional assumption about V, and mentioned properties of the potentials (see Section 3) in this equation, its coefficients and the right-hand side $\Theta^{(0)}$ belong to the Hölder class $H^{\frac{\lambda}{2},\,\lambda}(\overline{\mathbb{R}}^n_t)$. The conditions of the theorem guarantee the existence of f.s. for the operator L_0'. We will denote it by $\Gamma(s,x',t,y')$ ($0\le s<t\le T$, $x',y'\in\mathbb{R}^{n-1}$). Hence, we may conclude that there exists a unique classical solution of Eq. (24) that satisfies the boundary-initial condition $\lim\limits_{s\uparrow t}u(s,x',t)=\varphi(x')$, $x'\in\mathbb{R}^{n-1}$. In addition,

$$u\in H^{\frac{2+\lambda}{2},\,2+\lambda}(\overline{\mathbb{R}}^n_t), \tag{25}$$

and this solution can be written in the form

$$u(s,x',t)=\int\limits_{\mathbb{R}^{n-1}}\Gamma(s,x',t,y')\,\varphi(y')dy'+\int\limits_{s}^{t}d\tau\int\limits_{\mathbb{R}^{n-1}}\Gamma(s,x',\tau,z')\,\Theta^{(0)}(\tau,z',t)dz',\tag{26}$$
$$(s,x')\in\mathbb{R}^n_t.$$

Thus, we have two different expressions for the function $u(s,x',t)$: relation (23), where one must substitute $(s,x)=(s,x')\in\mathbb{R}^n_t$, and relation (26). Equating the right-hand sides of these equations, we obtain the integral equation for V,

$$\int\limits_{s}^{t}d\tau\int\limits_{\mathbb{R}^{n-1}}[g(s,x',\tau,z')+K_0(s,x',\tau,z')]\;V(\tau,z',t)dz'=\Psi_0(s,x',t),\quad (s,x')\in\mathbb{R}^n_t,\tag{27}$$

where

$$\Psi_0(s,x',t)=\int\limits_{\mathbb{R}^{n-1}}\Gamma(s,x',t,y')\,\varphi(y')dy'-\int\limits_{\mathbb{R}^{n}}g(s,x',t,y)\,\varphi(y)dy+$$
$$+\int\limits_{s}^{t}d\tau\int\limits_{\mathbb{R}^{n-1}}\Gamma(s,x',\tau,z')\,\frac{q(\tau,z')}{b_{nn}(\tau,z')}\frac{\partial u_2(\tau,z',t)}{\partial N(\tau,z')}dz'.$$

For the kernel K_0, whose explicit form can be easily calculated when $0\le s<t\le T$, $x',z'\in\mathbb{R}^{n-1}$, the estimation

$$|K_0(s,x',\tau,z')|\le C(\tau-s)^{-\frac{n-1}{2}}\exp\left\{-c\,\frac{|x'-z'|^2}{\tau-s}\right\}.\tag{28}$$

holds.

Considering the expression for the function Ψ_0 from (27), relation (20), and properties of parabolic potentials (see Section 3), we establish that $\Psi_0\in \underset{0}{H}^{\frac{2+\lambda}{2},\,2+\lambda}(\overline{\mathbb{R}}^n_t)$.

Equation (27) is a Volterra equation of the first kind. Applying the operator $\mathcal{E}$ from (11) to both sides of it and taking (12) and (28) into account, one can easily ascertain that this equation will transform into an equivalent second-type integral Volterra equation of the form

$$V(s,x',t)+\int\limits_{s}^{t}d\tau\int\limits_{\mathbb{R}^{n-1}}K(s,x',\tau,z')V(\tau,z',t)dz'=\Psi(s,x',t),\quad (s,x')\in\mathbb{R}^n_t,\tag{29}$$

where $\Psi(s,x',t)=(b_{nn}(s,x'))^{1/2}\mathcal{E}(s,x',t)\Psi_0$, therewith $\mathcal{E}\Psi_0\in \underset{0}{H}^{\frac{1+\lambda}{2},\,1+\lambda}(\overline{\mathbb{R}}^n_t)$, $\Psi\in \underset{0}{H}^{\frac{\lambda}{2},\,\lambda}(\overline{\mathbb{R}}^n_t)$, and the kernel K satisfies inequality (8), when $0\le s<t\le T,\quad x',z'\in\mathbb{R}^{n-1}$.

Solving Eq. (29) by the convergence method, we find V. At this, we check that $V\in \underset{0}{H}^{\frac{\lambda}{2},\,\lambda}(\overline{\mathbb{R}}^n_t)$, and estimation (22) holds for the norm $||V||_{\underset{0}{H}^{\frac{\lambda}{2},\,\lambda}(\overline{\mathbb{R}}^n_t)}$.

It remains only to validate condition (21) for the constructed solution and the statement of Theorem 1 considering the uniqueness. To prove this, it is enough to notice that the solution of problem (17)–(19) constructed with the use of formulas (23) and (29) can be interpreted as a solution of the first boundary-value parabolic problem

$$Lu(s,x,t)=0,\qquad (s,x)\in\Omega_t,$$
$$\lim_{s\uparrow t}u(s,x,t)=\varphi(x),\qquad x\in\overline{\mathbb{R}}^n_+,$$
$$u(s,x,t)=v(s,x,t),\qquad (s,x)\in\Sigma_t,$$

under the fitting condition

$$v(s,x,t)|_{s=t} = \varphi(x), \qquad D_s u(s,x,t)|_{s=t} = D_s v(s,x,t)|_{s=t}, \quad x \in \mathbb{R}^{n-1},$$

where the function $v(s,x,t)$, $(s,x) \in \Sigma_t$ is defined using relation (26). Then (see, e.g., [11, Ch. IV, §5]) the conditions of Theorem 1 together with conditions (20) and (25) guarantee the existence of the unique solution of the problem that belongs to class (21) and satisfies estimation (22). Theorem 1 is proved.

Remark.

If it is unnecessary to satisfy the fitting condition (20) for initial function φ from (18) in the statement of Theorem 1, then the solution of problem (17)–(19) determined by formulas (23) and (29) satisfies the condition

$$u \in C^{1,2}(\Omega_t) \cap C(\overline{\Omega}_t),$$

therewith the function u with its possible derivatives are bounded with respect to the variable x.

Construction of the process. From Theorem 1, it follows that, over functions φ from the class $H^{2+\lambda}(\mathbb{R}^n)$, a multiplicative operator family T_{st}, $0 \le s < t \le T$ is defined, whose analytical representation is given by the formula

$$T_{st}\varphi(x) = T_{st}^{(1)}\varphi(x) + T_{st}^{(2)}\varphi(x), \qquad (s,x) \in \overline{\mathbb{R}}_t^{n+1}, \tag{30}$$

where $T_{st}^{(1)}\varphi(x) = u_1(s,x,t)$, $T_{st}^{(2)}\varphi(x) = u_2(s,x,t)$, u_1 and u_2 are defined by formulas (6) and (13), respectively, the function $V(s,x,t)$ is from the simple-layer potential; and $T_{st}^{(1)}\varphi(x)$ is a solution of the integral equation (29). Notice that the fulfilment of the property

$$T_{st} = T_{su}T_{ut} \qquad (s < u < t) \tag{31}$$

for the operators T_{st} is a simple corollary from the statements of Theorem 1 considering the uniqueness of a solution of problem (17)–(19).

Now we prove that the operators T_{st} can be applied to functions φ from the space $\mathcal{B}(\mathbb{R}^n)$. Let $\varphi \in \mathcal{B}(\mathbb{R}^n)$. Then the existence of the second addend in (30) follows from the obvious inequality

$$|T_{st}^{(2)}\varphi(x)| \le C\,||\varphi|| \tag{32}$$

that holds in every domain of the form $0 \le s < t \le T$, $x \in \mathbb{R}^n$ with some constant C. To prove the existence of the function $T_{st}^{(1)}\varphi(x)$ from (30), we will study, at first, the integral equation (29). In this equation, its right-hand side $\Psi(s,x',t)$ up to the factor $(b_{nn}(s,x'))^{1/2}$ is a result of the action of the operator $\mathcal{E}$ from (11) on the function Ψ_0 from (27). Combining the first two terms in the expression for Ψ_0 into a single one, we denote it by Ψ_{01}. Then, using properties 1)–3) applied to f.s. h from (9), we can easily obtain the formula

(33)

$$\begin{aligned}
&\left(\frac{\pi}{2b_{nn}(s,x')}\right)^{1/2}\Psi(s,x',t)\\
&\quad= \frac{1}{2}\int_s^t (\tau-s)^{-3/2}d\tau \int_{\mathbb{R}^{n-1}} h(s,x',\tau,y')[\Psi_{01}(\tau,y',t) - \Psi_{01}(\tau,x',t)\\
&\qquad\qquad - (y'-x', \nabla'_{x'}\Psi_{01}(\tau,x',t))]dy'\\
&\qquad + \frac{1}{2}\int_s^t (\tau-s)^{-3/2}[\Psi_{01}(\tau,x',t) - \Psi_{01}(s,x',t)]d\tau + (t-s)^{-1/2}\Psi_{01}(s,x',t)
\end{aligned}$$

$$+\int_s^t d\alpha \int_{\mathbb{R}^{n-1}} \frac{q(\alpha,z')}{b_{nn}(\alpha,z')}\,\frac{\partial u_2(\alpha,z',t)}{\partial N(\alpha,z')}\,dz' \times \left\{ \frac{\partial}{\partial s} \int_s^{\alpha} (\tau-s)^{-1/2} d\tau \int_{\mathbb{R}^{n-1}} h(\hat{s},x',\tau,y')\Gamma(\tau,y',\alpha,z')\,dy' \right\}\Bigg|_{\hat{s}=s}.$$

In turn, the expression in the brackets in the last term can be written in the form similar to the representation of the function $\mathcal{E}(s,x',t)\Psi_{01}$. Consequently, using the basic inequalities for f.s. g, Γ, and h, we obtain the estimate

$$|\Psi(s,x,t)| \le C\,||\varphi||(t-s)^{-1/2}, \qquad 0 \le s < t \le T, \quad x' \in \mathbb{R}^{n-1}. \tag{34}$$

It is obvious that the same estimate also holds for the solution of Eq. (29) found by the convergence method. Applying inequalities (4) and (34) to f.s. g and the density V, respectively, we prove the existence of the function $T_{st}^{(1)}\varphi(x)$ and, therefore, the existence of the function $T_{st}\varphi(x)$, as well as the fulfilment of estimation (32) for them.

Further, with regard for relation (33), it is easy to verify that if $\lim_{n\to\infty}\varphi_n(x) = \varphi(x)$ for all $x \in \mathbb{R}^n$ and $\sup_{n,x}|\varphi_n(x)| < \infty$), then $\lim_{n\to\infty}\Psi(s,x',t,\varphi_n) = \Psi(s,x',t,\varphi)$, consequently $\lim_{n\to\infty} V(s,x',t,\varphi_n) = V(s,x',t,\varphi)$ for all $0 \le s < t \le T$, $x' \in \mathbb{R}^{n-1}$. Together with estimation (34) for V, it allows us to state that

$$\lim_{n\to\infty} T_{st}\varphi_n(x) = T_{st}\varphi(x). \tag{35}$$

Using (35), the necessary properties of the operator T_{st} can be verified only for smooth φ. In particular, it is easy to conclude by using (35) that the operators T_{st} have property (31) also in the case where they act on functions φ from the class $\mathcal{B}(\mathbb{R}^n)$.

Now we will prove that the operator family (T_{st}) defined by formulas (29) and (30) has the property such that it transforms the nonnegative functions into nonnegative functions.

Lemma. *If properties A1), A2) and B1), B2) hold for coefficients of the operators $\mathcal{L}$ and $\mathcal{L}_0$ from (14) and (15), then, for every nonnegative function $\varphi \in \mathcal{B}(\mathbb{R}^n)$, the function $T_{st}\varphi$ is nonnegative.*

Proof. Taking (35) into account, it is enough to prove Lemma only in the case where a function φ belongs to the class $H^{2+\lambda}(\mathbb{R}^n)$ and is finite. From Theorem 1, it follows that, in this case, the function $T_{st}\varphi(x)$ satisfies Eq. (17) in the domain $(s,x) \in [0,t) \times \mathbb{R}^n_+$, initial condition (18), and boundary condition (19). Set some $0 < t \le T$ and denote, by γ, the minimum of the function $T_{st}\varphi$ in a domain $(s,x) \in [0,t] \times \overline{\mathbb{R}}^n_+$. Assume that $\gamma < 0$. Since $\varphi(x) \ge 0$ and $T_{st}\varphi \to 0$ when $|x| \to \infty$ and due to the maximum principle (see [13, Ch. II]), there exist $s_0 \in [0,t)$ and $x'_0 \in \mathbb{R}^{n-1}$ such that $T_{s_0t}\varphi_n(x'_0) = \gamma$.

Because the function $T_{st}\varphi(x)$ obviously is not a constant, there exists a neighborhood U of the point (s_0,x'_0) such that $T_{s_0t}\varphi(x'_0) > \gamma$ for $(s,x) \in U \cap \{[0,t) \times \mathbb{R}^n_+\}$. But then, as it follows from Theorem 14 [13, p. 69], the inequality $D_nT_{st}\varphi(x'_0) > 0$ holds at the point (s_0,x'_0). In addition, at the point (s_0,x'_0), the equalities $D_iT_{s_0t}\varphi(x'_0) = 0$, $i = 1,\dots,n-1$, $D_sT_{st}\varphi(x'_0)|_{s=s_0} = 0$ and the inequality

$$\sum_{k,l=1}^{n-1} \beta_{kl}(s_0,x'_0)D_{kl}T_{s_0t}\varphi(x'_0) \ge 0$$

obviously hold. Consequently, we obtain that, at the point (s_0,x'_0), the function $T_{st}\varphi(x)$ does not satisfy the boundary condition (19). So, our assumption that $\gamma < 0$ is false. Lemma is proved.

Noticing that $V(s,x',t,\varphi_0)\equiv 0$ for the function $\varphi_0(y)\equiv 1$ and, consequently,
$$T_{st}\varphi_0(x)\equiv 1,$$
we conclude that the operator family $(T_{st})_{0\le s<t\le T}$ determines some nonhomogeneous nonbreaking Feller process on $\overline{\mathbb{R}}^n_+$. If we will denote its transition probability by
$$P(s,x,t,dy),$$
then we can write the relation
$$T_{st}\varphi(x)=\int_{\mathbb{R}^n}P(s,x,t,dy)\varphi(y),\qquad 0\le s<t\le T,\quad x\in\overline{\mathbb{R}}^n_+.$$
Finally, we prove that trajectories of the constructed process are continuous. This statement is a corollary from the inequality
$$\sup_{x\in\overline{\mathbb{R}}^n_+}\int_{\mathbb{R}^n}|y-x|^4P(s,x,t,dy)\le C(t-s)^2,\quad s\in[0,t),$$
which can be verified by straight computations.

Hence, the following theorem holds.

Theorem 2. *Let conditions A1), A2) and B1),B2) hold for coefficients of the operators $\mathcal{L}$ from (14) and $\mathcal{L}_0$ from (15). Then there exists a multiplicative operator family $(T_{st})_{0\le s<t\le T}$ that is defined by formulas (30), (29) and describes a diffusion process in a closed domain $\overline{\mathbb{R}}^n_+$ such that, in inner points of the domain, it is controlled by the operator $\mathcal{L}$, and its behavior on the boundary is determined by the boundary condition (16).*

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