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LIMIT THEOREMS FOR BACKWARD STOCHASTIC EQUATIONS

Consider a weak convergence in the Meyer–Zheng topology of solutions of a backward stochastic equation in the form

$$Y^\epsilon(t) = E\left[g^\epsilon\Big(X^\epsilon(T)\Big) + \int_t^T f^\epsilon\Big(s, X^\epsilon(s), Y^\epsilon(s)\Big)ds\Big| F_t^{X^\epsilon}\right]$$

as $\epsilon \to 0$ for different classes of random processes $X^\epsilon(t)$ with the irregular dependence on the parameter ϵ. The equations for the limit process are obtained.

1. Introduction.

Backward stochastic differential equations of the form

$$Y(t) = g(X(T)) + \int_t^T f(s, X(s), Y(s))ds - \int_t^T Z(s)dW(s),$$
$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s),$$

where $W(s)$ is a Wiener process, have been introduced by E. Pardoux and S. Peng [14,15], who proved the existence and uniqueness of a $\mathcal{F}^W$ adapted solution. That is, the solutions of the equations are strong solutions. The aim of the authors was to describe a solution of a second-order quasilinear partial equation in probabilistic terms. Due to such stochastic representation of the solution of a quasilinear partial equation, the study of the limit behavior of backward stochastic differential equations allows us to develop a theory of the limit behavior of the corresponding partial equations. The limit behavior of stochastic systems when coefficients of the process $X(t)$ and a function f are of the form $h(\frac{x}{\epsilon})$ or $\frac{1}{\epsilon}h(\frac{x}{\epsilon})$, where ϵ is a small parameter and $h(x)$ is a periodic function, was studied in [3,12,13,16] as $\epsilon \to 0$.

Here, we continue this investigation in several directions. First, following [2], we write the equation for the processes Y^ϵ in another form. This will allow us to consider weak solutions of the processes $X^\epsilon(t)$. Second, the coefficients of the processes $X^\epsilon(t)$ may depend on the parameter ϵ in any way. This dependence can be irregular: we do not assume that the coefficients have limits as $\epsilon \to 0$, they may tend to infinity as $\epsilon \to 0$ or may have no limit at all. Third, the coefficients for the processes $Y^\epsilon(t)$ will also depend on a small parameter. The functions $g^\epsilon(x)$ will converge uniformly on compact sets, and the functions $f^\epsilon(t, x, y)$ will have limit in the space of summable functions.

The paper is organized as follows. In this section, we introduce our notation and assumptions. The limit result for the processes $Y^\epsilon(t)$ is proved in Section 2. The identification of the limit processes for different classes is realized in Sections 3–5. 5. For this, we will use the method developed in [12,13,16].

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Let R^p be a p-dimensional Euclidean space. By $D([0,T];R^p)$, we denote the space of cadlag (right-continuous with left-hand limits) functions $x(t), t \in [0,T]$. On this space, we consider the Skorokhod topology [1, part 3] and the Meyer–Zheng topology [11,2]. For weak convergence in these topologies, we use the notation $\stackrel{S}{\Longrightarrow}$ and $\stackrel{M-Z}{\Longrightarrow}$, respectively. Let $\Omega = D([0,T];R^d) \times D([0,T];R^m)$, $\mathcal{F}$ be the σ-algebra of Borel subsets of this set, $(\Omega, \mathcal{F}, P)$ be a probability space, and E be a symbol of expectation. If $\xi(t)$, $t \in [0,T]$, is a random process, then $\mathcal{F}_t^{\xi}$ is the smallest filtration generated by $\xi(s)$, $s \in [0,t]$. The notation $L_p([0,T] \times K)$ has a standard sense of the space of p-order integrable functions on $[0,T] \times K$ with the norm $\|\cdot\|_{L_p([0,T]\times K)}$. In this paper, we denote, by C, different constants independent of ϵ, and $I(A)$ is the indicator of an event A.

We consider solutions of backward stochastic equations (BSE) in the form

$$Y^\epsilon(t) = E\left[g^\epsilon\Big(X^\epsilon(T)\Big) + \int_t^T f^\epsilon\Big(s, X^\epsilon(s), Y^\epsilon(s)\Big)ds \Big| \mathcal{F}_t^{X^\epsilon}\right] \tag{1.1}$$

and investigate their weak convergence as $\epsilon \to 0$. Let the process $X^\epsilon(t)$ in (1.1) be a cadlag process with values in $D([0,T];R^d)$, and let the process $Y^\epsilon(t)$ be a cadlag process with values in $D([0,T];R^m)$. The measure corresponding to the process X^ϵ on $D([0,T];R^m)$ is denoted by μ^ϵ, and the measure corresponding to the process Y^ϵ on $D([0,T];R^m)$ is denoted by Q^ϵ. Following [2], we define the strong solution of (1.1) as a process $Y^\epsilon(t)$ such that, for any $\epsilon > 0$,

$$E|g^\epsilon(X(T))| + E\int_0^T \Big| f^\epsilon\Big(s, X^\epsilon(s), Y^\epsilon(s)\Big)\Big| ds < \infty,$$

and (1.1) is valid.

For the processes $X^\epsilon(t)$, we suppose that the following condition (I) is satisfied:

Condition (I)**:**

I_1. $\mu^\epsilon \stackrel{S}{\Longrightarrow} \mu$, and let the process $X(t)$ correspond to the measure μ.

I_2. Let $X^\epsilon(0) = x$, and let the moment estimates

$$E \sup_{t\in[0,T]} |X^\epsilon(t)|^2 + E \sup_{t\in[0,T]} |X(t)|^2 \le C$$

be valid.

I_3. Krylov's estimates for the processes $X^\epsilon(t)$ and $X(t)$

$$E\int_0^T \Big| h\Big(t, X^\epsilon(t)\Big)\Big| dt + E\int_0^T \Big| h\Big(t, X(t)\Big)\Big| dt \le C\|h\|_{L_{d+1}([0,T]\times R^d)}$$

are fulfilled.

We introduce the conditions for the coefficients in Eq. (1.1).

Condition (II). For measurable functions $g^\epsilon(x)$ and $f^\epsilon(t,x,y)$:

II_1. $|g^\epsilon(x)| \le C(1+|x|)$.

II_2. $|f^\epsilon(t,x,y)| \le C(1+|y|)$.

II_3. $|f^\epsilon(t,x,y_2) - f^\epsilon(t,x,y_1)| \le C|y_2 - y_1|$.

Condition (III): There exist a continuous function $g(x)$ and a measurable function $f(t,x,y)$ such that

III_1. For any compact $K \in R^d$, $\limsup_{x\in K} |g^\epsilon(x) - g(x)| = 0$.

III_2. For fixed $y \in R^m$ and any compact $K \in R^d$,

$\lim_{\epsilon\to 0} \|f^\epsilon(\cdot,\cdot,y) - f(\cdot,\cdot,y)\|_{L_{d+1}([0,T]\times K)} = 0.$

III_3. For the functions $g(x)$ and $f(t,x,y)$, condition (II) is valid, if the symbol ϵ is omitted.

It was proved in [2, Proposition 2.1] that, under conditions (II) and I_2 for any $\epsilon > 0$, there exists a strong solution of Eq. (1.1), and this solution is unique.

2. Limit for Y^ϵ

Consider the strong solutions of Eq. (1.1). The main result of this section is Theorem 1.

Theorem 1. *Suppose that conditions (I), (II), and (III) are satisfied. Then there exist a subsequence Q^{ϵ_k} of Q^ϵ and a probability law Q on $D([0,T];R^m)$ such that $Q^{\epsilon_k} \overset{M-Z}{\Longrightarrow} Q$, and, for corresponding process $Y(t)$, we have*

$$Y(t) = E\left[g\Big(X(T)\Big) + \int_t^T f\Big(s, X(s), Y(s)\Big)ds\Big|\mathcal{F}_t^{X,Y}\right]. \tag{2.1}$$

Proof. We will prove the theorem in several steps.

Step 1). As the constants in conditions (I) and (II) do not depend on ϵ, we can obtain the following estimate [2, proof of Proposition 2.1] (cf. Corollary 1 below):

$$E\sup_{t\in[0,T]}|Y^\epsilon(t)|^2 \le C. \tag{2.2}$$

Step 2). We verify that this sequence is relatively compact in the Meyer–Zheng topology. To prove this, we use a result from [11, Theorem 4]. For a subdivision $0 = t_0 < t_1 < ... < t_n = T$, we define

$$V_n(Y^\epsilon) = E|Y^\epsilon(T)| + \sum_{k=0}^{n-1} E\left|E\left[\Big(Y^\epsilon(t_{k+1}) - Y^\epsilon(t_k)\Big)\Big|\mathcal{F}_{t_k}^{Y^\epsilon}\right]\right|.$$

Using estimates (2.2) and the conditions of the theorem, we get

$$V_n(Y^\epsilon) = E|Y^\epsilon(T)| + \sum_{k=0}^{n-1} E\left|E\left[\int_{t_k}^{t_{k+1}} f^\epsilon(s, X^\epsilon(s), Y^\epsilon(s))ds\Big|\mathcal{F}_{t_k}^{Y^\epsilon}\right]\right| \le$$
$$\le E|Y^\epsilon(T)| + E\int_0^T \Big|f^\epsilon(s, X^\epsilon(s), Y^\epsilon(s))\Big|ds \le E|Y^\epsilon(T)| +$$
$$+ C\int_0^T E(1 + |Y^\epsilon(s)|)ds \le C.$$

Then there exists a subsequence of Q^ϵ which converges weakly to the law Q in the Meyer–Zheng topology on $D([0,T];R^m)$. We denote this subsequence by Q^ϵ again, and let $Y(t)$ be a process on $D([0,T];R^m)$ corresponding to the measure Q.

Step 3). Let $t \in [0,T]$, $G(t) \in D([0,T];R^m)$, $k \in [0,\infty)$, and let $G_k(t) = G(t)((k+1-|G(t)|)^+ \wedge 1)$, where $a^+ = a \vee 0$ and $\vee, \wedge$ are the symbols of max and min, respectively. It is relevant to remark that

$$|G_k(t)| \le |G(t)| \text{ and } |G(t) - G_k(t)| = |G(t)|I(|G(t)| > k).$$

For $l \ge k$, we denote $U_{l,k}(t) = |G_l(t) - G_k(t)|$. It is not difficult to check that the sequence $U_{l,k}(t)$ monotonically increases in l, and

$$\lim_{l\to\infty} U_{l,k}(t) = |G(t) - G_k(t)|.$$

Introduce the functional

$$N_{t,\delta}(G) = \delta^{-1} \int_t^{T\wedge(t+\delta)} G(s)ds.$$

The functionals $N_{t,\delta}(G_k)$ and $N_{t,\delta}(U_{l,k})$ are bounded and continuous in the Meyer–Zheng topology [2].

Step 4). In virtue of step 3),

$$E|Y^\epsilon(t) - Y_k^\epsilon(t)| \ = E|Y^\epsilon(t)|I_{(|Y^\epsilon(t)|>k)} \leq \Big(E|Y^\epsilon(t)|^2\Big)^{\frac{1}{2}} P\{|Y^\epsilon(t)| > k\}.$$

From this, estimate (2.2), and the Chebyshev inequality, we have

$$\sup_\epsilon \sup_{t\in[0,T]} E|Y^\epsilon(t) - Y_k^\epsilon(t)| \leq Ck^{-2} \tag{2.3}$$

and

$$\lim_{k\to\infty} \sup_\epsilon \sup_{t\in[0,T]} E|Y^\epsilon(t) - Y_k^\epsilon(t)| = 0. \tag{2.4}$$

Step 5). As the process $Y(t)$ belongs to $D(R^m)$, we use the Fatou's lemma and obtain

$$\begin{aligned} E|Y(t) - Y_k(t)| =& E \lim_{\delta\downarrow 0} \frac{1}{\delta} \int_t^{t+\delta} |Y(s) - Y_k(s)|ds \leq \\ &\leq \underline{\lim}_{\delta\downarrow 0} E\frac{1}{\delta} \int_t^{t+\delta} |Y(s) - Y_k(s)|ds. \end{aligned}$$

We continue the last inequality by the theorem on monotone convergence and the property of continuity of the functional $N_{t,\delta}(U_{l,k})$ from step 3). Then

$$\begin{aligned} E|Y(t) - Y_k(t)| \leq& \underline{\lim}_{\delta\downarrow 0} \lim_{l\uparrow\infty} E\frac{1}{\delta} \int_t^{t+\delta} |Y_l(s) - Y_k(s)|ds = \\ &= \underline{\lim}_{\delta\downarrow 0} \lim_{l\uparrow\infty} \lim_{\epsilon\to 0} E\frac{1}{\delta} \int_t^{t+\delta} |Y_l^\epsilon(s) - Y_k^\epsilon(s)|ds. \end{aligned}$$

From this and (2.3), we get

$$\lim_{k\to\infty} E|Y(t) - Y_k(t)| = 0. \tag{2.5}$$

Now we show that the limit process $Y(t)$ is uniformly integrable. From the inequality

$$|Y(t)| \leq |Y(t) - Y_k(t)| + |Y_k(t) - N_{t,\delta}(Y_k)| + |N_{t,\delta}(Y_k)|,$$

by using the continuity in the Meyer–Zheng topology of the functional $N_{t,\delta}(|Z_k|)$, we get

$$E|Y(t)| \leq E|Y(t) - Y_k(t)| + E|Y_k(t) - N_{t,\delta}(Y_k)| + \lim_{\epsilon\to 0} EN_{t,\delta}(|Y_k^\epsilon|).$$

By virtue of (2.2),

$$E|Y(t)| \leq E|Y(t) - Y_k(t)| + E\frac{1}{\delta}\left|\int_t^{t+\delta} (Y_k(t) - Y_k(s))ds\right| + C.$$

Approaching the limit as $\delta \to 0$ in this inequality and taking (2.5) into account, we have

$$E|Y(t)| \leq C. \tag{2.6}$$

Take (2.5) and step 3) into account, we get the uniform integrability of the process $Y(t)$:

$$\lim_{k\to\infty} \sup_{t\in[0,T]} E|Y(t)|I(|Y(t)| > k) = \lim_{k\to\infty} \sup_{t\in[0,T]} E|Y(t) - Y_k(t)| = 0. \tag{2.7}$$

We now prove that

$$E\left|E\left([Y^\epsilon_k(t)-N_{t,\delta}(Y^\epsilon_k)]\Big|\mathcal{F}^{X^\epsilon}_t\right)\right|\le C\left(\frac{1}{k^2}+\delta\right). \tag{2.8}$$

We have

$$\begin{aligned}E\left|E\left([Y^\epsilon_k(t)-N_{t,\delta}(Y^\epsilon_k)]\Big|\mathcal{F}^{X^\epsilon}_t\right)\right|&\le E|Y^\epsilon_k(t)-Y^\epsilon(t)|+\frac{1}{\delta}\int_t^{t+\delta}E|Y^\epsilon(s)-\\ &-Y^\epsilon_k(s)|ds+\frac{1}{\delta}\int_t^{t+\delta}E\left|E\left\{[Y^\epsilon(t)-Y^\epsilon(s)]\Big|\mathcal{F}^{X^\epsilon}_t\right\}\right|ds.\end{aligned} \tag{2.9}$$

Next, relation (1.1) yields

$$E\left\{[Y^\epsilon(t)-Y^\epsilon(s)]\Big|\mathcal{F}^{X^\epsilon}_t\right\}=\int_t^s E\left\{f^\epsilon(u,X^\epsilon(u),Y^\epsilon(u))\Big|\mathcal{F}^{X^\epsilon}_t\right\}du.$$

From this and (2.2), we get

$$\begin{aligned}E\left|E\left\{[Y^\epsilon(t)-Y^\epsilon(s)]\Big|\mathcal{F}^{X^\epsilon}_t\right\}\right|&\le CE\int_t^s E\left\{(1+|Y^\epsilon(u)|)\Big|\mathcal{F}^{X^\epsilon}_t\right\}du\le\\ &\le C\int_t^s E(1+|Y^\epsilon(u)|)du\le C(s-t).\end{aligned} \tag{2.10}$$

Inequality (2.8) follows from (2.9), (2.3), and (2.10).

Step 6). Let $\Phi_t(x,y)$ be a bounded continuous functional on $D([0,t];R^d)\times D([0,t];R^m)$ equipped with a product of the Skorokhod topology on the first factor and the Meyer–Zheng topology on the second factor. As follows from (1.1) for any such functional,

$$E\Phi_t(X^\epsilon,Y^\epsilon)[Y^\epsilon(t)-g^\epsilon(X^\epsilon(T))-\int_t^T f^\epsilon(s,X^\epsilon(s),Y^\epsilon(s))ds]=0.$$

We rewrite the left-hand side of the last equality as

$$\sum_{k=1}^3 J^\epsilon_k+E\Phi_t(X,Y)[Y(t)-g(X(T))-\int_t^T f(s,X(s),Y(s))ds]=0, \tag{2.11}$$

where

$$\begin{aligned}J^\epsilon_1&=E[\Phi_t(X^\epsilon,Y^\epsilon)Y^\epsilon(t)-\Phi_t(X,Y)Y(t)],\\ J^\epsilon_2&=E[\Phi_t(X,Y)g(X(T))-\Phi_t(X^\epsilon,Y^\epsilon)g^\epsilon(X^\epsilon(T))],\\ J^\epsilon_3&=E[\Phi_t(X,Y)\int_t^T f(s,X(s),Y(s))ds-\\ &\quad-\Phi_t(X^\epsilon,Y^\epsilon)\int_t^T f^\epsilon(s,X^\epsilon(s),Y^\epsilon(s))ds],\end{aligned}$$

and estimate each of J^ϵ_k. The expression for J^ϵ_1 can be represented in the form

$$J^\epsilon_1=\sum_{i=4}^6 J^\epsilon_i+J_7+J_8, \tag{2.12}$$

where

$$\begin{aligned}J^\epsilon_4&=E[\Phi_t(X^\epsilon,Y^\epsilon)[Y^\epsilon(t)-Y^\epsilon_k(t)],\\ J^\epsilon_5&=E\Phi_t(X^\epsilon,Y^\epsilon)[Y^\epsilon_k(t)-N_{t,\delta}(Y^\epsilon_k)],\\ J^\epsilon_6&=E\Phi_t(X^\epsilon,Y^\epsilon)N_{t,\delta}(Y^\epsilon_k)-E\Phi_t(X,Y)N_{t,\delta}(Y_k),\\ J_7&=E\Phi_t(X,Y)[N_{t,\delta}(Y_k)-Y_k(t)],\\ J_8&=E\Phi_t(X,Y)[Y_k(t)-Y(t)].\end{aligned}$$

Relations (2.4) and (2.5) yield

$$\lim_{k\to\infty} \sup_{\epsilon} |J_4^\epsilon| = 0 \tag{2.13}$$

and

$$\lim_{k\to\infty} |J_8| = 0. \tag{2.14}$$

From (2.8), we get

$$\lim_{k\to\infty} \lim_{\delta\downarrow 0} \sup_{\epsilon} |J_5^\epsilon| = 0. \tag{2.15}$$

In view of the weak convergence of (X^ϵ, Y^ϵ) to (X, Y),

$$\lim_{\epsilon\to 0} |J_6^\epsilon| = 0. \tag{2.16}$$

As with probability one, $\lim_{\delta\downarrow 0} N_{t,\delta}(Y_k) = Y_k(t)$, and $Y_k(t)$ are uniformly bounded on t, we have

$$\lim_{\delta\downarrow 0} |J_7| = 0. \tag{2.17}$$

Approaching the limit in (2.12) firstly as $\epsilon \to 0$, then as $\delta \to 0$, and then as $k \to \infty$, we get, by virtue (2.13)–(2.17):

$$\lim_{\epsilon\to 0} |J_1^\epsilon| = 0. \tag{2.18}$$

Before the estimation of J_2^ϵ, we introduce the continuous functions $r_N(x) : 0 \le r_N(x) \le 1$, $r_N(x) = 1$ if $|x| \le N$ and $r_N(x) = 0$ if $|x| \ge N + 1$ and define $g_N(x) = g(x)r_N(x)$. The expression J_2^ϵ can be presented as the sum

$$J_2^\epsilon = J_9^\epsilon + J_{10}^\epsilon + J_{11}^\epsilon + J_{12}, \tag{2.19}$$

where

$$\begin{aligned}
J_9^\epsilon &= E\Phi_t(X^\epsilon, Y^\epsilon)[g(X^\epsilon(T)) - g^\epsilon(X^\epsilon(T))],\\
J_{10}^\epsilon &= E\Phi_t(X^\epsilon, Y^\epsilon)[g_N(X^\epsilon(T)) - g(X^\epsilon(T))],\\
J_{11}^\epsilon &= E[\Phi_t(X, Y)g_N(X(T)) - \Phi_t(X^\epsilon, Y^\epsilon)g_N(X^\epsilon(T))],\\
J_{12} &= E\Phi_t(X, Y)[g(X(T)) - g_N(X(T))].
\end{aligned}$$

Using the estimate from I_2, it is not difficult to get that, for any K,

$$|J_9^\epsilon| \le CE|g(X^\epsilon(T) - g^\epsilon(X^\epsilon(T))|I(|X^\epsilon(T)| \le N) + \frac{C}{N^2}.$$

Taking the limits in the last formula firstly in $\epsilon \to 0$ and then in $N \to \infty$, we have, by condition III_1,

$$\lim_{\epsilon\to 0} |J_9^\epsilon| = 0. \tag{2.20}$$

As $|g(x) - g_N(x)| \le |g(x)|I(|x| > N)$, estimate I_2 and the Chebyshev inequality yield

$$\sup_{\epsilon} |J_{10}^\epsilon| \le \frac{C}{N^2}.$$

From this, we have

$$\lim_{N\to\infty} \sup_{\epsilon} |J_{10}^\epsilon| = 0. \tag{2.21}$$

Similarly,

$$\lim_{N\to\infty} |J_{12}| = 0. \tag{2.22}$$

From the weak convergence (X^ϵ, Y^ϵ) to (X, Y), we get

$$\lim_{\epsilon\to 0} |J_{11}^\epsilon| = 0. \tag{2.23}$$

From (2.19)–(2.23), we conclude

$$\lim_{\epsilon\to 0} J_2^\epsilon = 0. \tag{2.24}$$

To estimate J_3^ϵ, we will do some constructions. Let $R(a;A)=\{y:|y-a|\le A\}$ be a ball of radius A with its center at the point a. In $R(0;N)$, we introduce a δ-net $\{y_1,y_2,...,y_M\}$: $|y_{i+1}-y_i|\le\delta$ and a family of functions $q_i(y)$ with the properties:

$$a)\ \ q_i(y)\ge 0\ \text{ and }\ q_i(y)=0\ \text{ outside }\ R(y_i,\delta),$$
$$b)\ \ \sum_{i=1}^{M}q_i(y)=1,$$
$$c)\ \ q_i(y)\ \text{ are differentiable functions on }\ y\in R(0;N).$$

For the function $g(t,x,y)$, we define the function

$$g_{N,\delta}(t,x,y)=\sum_{i=1}^{M}q_i(y)g(t,x,y_i).$$

From this and condition II_3, we get, for any $t\in[0,T]$, $x\in R^d$, and $y\in R^m$,

$$\begin{aligned}|f^\epsilon(t,x,y)-f^\epsilon_{N,\delta}(t,x,y)|\le\sum_{i=1}^{M}q_i(y)|f^\epsilon(t,x,y)-f^\epsilon(t,x,y_i)|\le\\ \le C\delta.\end{aligned}\tag{2.25}$$

Similarly, we have

$$|f(t,x,y)-f_{N,\delta}(t,x,y)|\le C\delta.\tag{2.26}$$

Due to the convolution for the function $f(t,x,y)$, we can define a sequence of continuous functions $f^{(n)}(t,x,y)$ bounded for each n and such that

$$\lim_{n\to\infty}||f(\cdot,\cdot,y)-f^{(n)}(\cdot,\cdot,y)||_{L_{d+1}([0,T];R^d)}=0$$

for any $y\in R^m$. From this, we get

$$\lim_{n\to\infty}||f_{N,\delta}(\cdot,\cdot,y)-f^{(n)}_{N,\delta}(\cdot,\cdot,y)||_{L_{d+1}([0,T];R^d)}=0.\tag{2.27}$$

Then we rewrite the expression for J_3^ϵ in the form

$$J_3^\epsilon=J_{13}^\epsilon+J_{14}^\epsilon+J_{15}^\epsilon+J_{16}^\epsilon+J_{17}^\epsilon+J_{18}+J_{19},\tag{2.28}$$

where

$$\begin{aligned}J_{13}^\epsilon&=E\Phi_t(X^\epsilon,Y^\epsilon)\int_t^T[f^\epsilon_{N,\delta}(s,X^\epsilon(s),Y^\epsilon(s))-f^\epsilon(s,X^\epsilon(s),Y^\epsilon(s))]ds,\\ J_{14}^\epsilon&=E\Phi_t(X^\epsilon,Y^\epsilon)\int_t^T[f_{N,\delta}(s,X^\epsilon(s),Y^\epsilon(s))-f^\epsilon_{N,\delta}(s,X^\epsilon(s),Y^\epsilon(s))]\times\\ &\qquad\times r_K(X^\epsilon(s))ds,\\ J_{15}^\epsilon&=E\Phi_t(X^\epsilon,Y^\epsilon)\int_t^T[f_{N,\delta}(s,X^\epsilon(s),Y^\epsilon(s))-f^\epsilon_{N,\delta}(s,X^\epsilon(s),Y^\epsilon(s))]\times\\ &\qquad\times(1-r_K(X^\epsilon(s)))ds,\\ J_{16}^\epsilon&=E\Phi_t(X^\epsilon,Y^\epsilon)\int_t^T[f^{(n)}_{N,\delta}(s,X^\epsilon(s),Y^\epsilon(s))-f_{N,\delta}(s,X^\epsilon(s),Y^\epsilon(s))]ds,\end{aligned}$$

$$J_{17}^{\epsilon} = E\Phi_t(X,Y)\int_t^T [f_{N,\delta}^{(n)}(s,X(s),Y(s))ds -$$
$$- E\Phi_t(X^{\epsilon},Y^{\epsilon})\int_t^T f_{N,\delta}^{(n)}(s,X^{\epsilon}(s),Y^{\epsilon}(s))]ds,$$
$$J_{18} = E\Phi_t(X,Y)\int_t^T [f_{N,\delta}(s,X(s),Y(s)) - f_{N,\delta}^{(n)}(s,X(s),Y(s))]ds$$
$$J_{19} = E\Phi_t(X,Y)\int_t^T [f(s,X(s),Y(s)) - f_{N,\delta}(s,X(s),Y(s))]ds.$$

By (2.25) and (2.26,) we have

$$\sup_{\epsilon} |J_{13}^{\epsilon}| + |J_{19}| \le C\delta. \tag{2.29}$$

In view of Krylov's estimate I_3, we get

$$|J_{14}^{\epsilon}| \le C\sum_{i=1}^{M} ||f(\cdot,\cdot,y_i) - f^{\epsilon}(\cdot,\cdot,y_i)||_{L_{d+1}([0,T]\times K)}. \tag{2.30}$$

Similarly,

$$|J_{16}^{\epsilon}| \le C\sum_{i=1}^{M} ||f^{(n)}(\cdot,\cdot,y_i) - f(\cdot,\cdot,y_i)||_{L_{d+1}([0,T]\times R^d)} \tag{2.31}$$

and

$$|J_{18}| \le C\sum_{i=1}^{M} ||f(\cdot,\cdot,y_i) - f^{(n)}(\cdot,\cdot,y_i)||_{L_{d+1}([0,T]\times R^d)}. \tag{2.32}$$

From conditions of the theorem and from estimate I_2, we get

$$|J_{15}^{\epsilon}| \le CM\int_t^T E(1+|X^{\epsilon}(s)|)I(|X^{\epsilon}(s)|>K)ds \le \frac{CM}{K^2}. \tag{2.33}$$

As $X^{\epsilon} \overset{S}{\Longrightarrow} X$, $Y^{\epsilon} \overset{M-Z}{\Longrightarrow} Y$, and the functional $\Phi_t(x(\cdot),y(\cdot))\int_t^T f_{N,\delta}^{(n)}(x(s),y(s))ds$ is continuous in the corresponding topologies,

$$\lim_{\epsilon\to 0} |J_{17}^{\epsilon}| = 0. \tag{2.34}$$

Finally, we have

$$|J_3^{\epsilon}| \le C\delta + C\sum_{i=1}^{M} ||f^{\epsilon}(\cdot,\cdot,y_i) - f(\cdot,\cdot,y_i)||_{L_{d+1}([0,T]\times K)} +$$
$$+ C\sum_{i=1}^{M} ||f(\cdot,\cdot,y_i) - f^{(n)}(\cdot,\cdot,y_i)||_{L_{d+1}([0,T];R^d)} + \frac{CM}{K^2} + |J_{17}^{\epsilon}|.$$

Approaching the limit in this inequality firstly as $\epsilon \to 0$, then as $n \to \infty$, $K \to \infty$, and $\delta \to 0$, we have, by (2.29)–(2.34),

$$\lim_{\epsilon\to 0} J_3^{\epsilon} = 0. \tag{2.35}$$

From (2.11), (2.8), (2.24), and (2.35), we obtain

$$E\Phi_t(X,Y)[Y(t) - g(X(T)) - \int_t^T f(s,X(s),Y(s))ds] = 0. \tag{2.36}$$

From this, we get (2.1). The theorem is proved.

Corollary 1. *Under the conditions of Theorem 1,*

$$E \sup_{t\in[0,T]} |Y(t)|^2 \le C. \tag{2.37}$$

Proof. From (2.1), we have

$$\begin{aligned}&\sup_{t\in[0,T]} |Y(t)| \le \\ &\le \sup_{t\in[0,T]} E\biggl(|g(X(T))| + \int_t^T |f(s,X(s),Y(s))|ds \biggm| \mathcal{F}_t^{X,Y}\biggr) \le \\ &\le \sup_{t\in[0,T]} E\biggl(|g(X(T))| + \int_0^T |f(s,X(s),Y(s))|ds \biggm| \mathcal{F}_t^{X,Y}\biggr).\end{aligned}$$

As $E|g(X(T))| + E\int_0^T |f(s,X(s),Y(s))|ds \le C$, we can use Doob's inequality and obtain

$$\begin{aligned}E \sup_{t\in[0,T]} |Y(t)|^2 &\le 4E\biggl(|g(X(T)|^2 + |\int_0^T f(s,X(s),Y(s))ds|^2\biggr) \le \\ &\le C(1 + \int_0^T E \sup_{u\in[0,s]} |Y(u)|^2 ds\biggr).\end{aligned}$$

The Gronwall's lemma yields (2.37).

Corollary 2. *Under the conditions of Theorem 1, the process $Y(t)$ admits the decomposition in the following form*

$$Y(t) = g(X(T)) + \int_t^T f(s,X(s),Y(s))ds - M(T) + M(t), \tag{2.38}$$

where $(M(t),\mathcal{F}_t^{X,Y})$ is a square integrable martingale, and $EM(t) = 0$.

Proof. We define

$$\begin{aligned}M(t) = E\biggl[g\Bigl(X(T)\Bigr) + \int_0^T f\Bigl(s,X(s),Y(s)\Bigr)ds \biggm| \mathcal{F}_t^{X,Y}\biggr] - E\biggl[g\Bigl(X(T)\Bigr) + \\ + \int_0^T f\Bigl(s,X(s),Y(s)\Bigr)ds\biggr].\end{aligned}$$

The required assertion follows from (2.1) and Corollary 1.

3. Itô equation with coefficients bounded on ϵ.

In this section, we consider one-dimensional processes $X^\epsilon(t)$, because the conditions for weak convergence $\mu^\epsilon \stackrel{S}{\Longrightarrow} \mu$ have a very simple form in this case. The multidimensional case was considered in [6,7,10]. Let $X^\epsilon(t)$ in Eq. (1.1) be a solution of the one-dimensional stochastic equation

$$X^\epsilon(t) = x + \int_0^t [b_1^\epsilon(s,X^\epsilon(s)) + b_2^\epsilon(X^\epsilon(s))]ds + \int_0^t \sigma^\epsilon(s,X^\epsilon(s))dw, \tag{3.1}$$

where $w(t)$ is the standard Wiener process. We consider a weak solution of this equation and introduce the conditions for the coefficients of (3.1). Let the constants $0 < \lambda \le \Lambda < \infty$ be given. We say that a pair of measurable functions $(f(t,x), g(t,x)) \in \mathcal{L}(\lambda,\Lambda)$ if

$$|f(t,x)| + |g(t,x)| \le \Lambda, \;\; g(t,x) \ge \lambda. \tag{3.2}$$

For the coefficients of Eq. (3.1), we suppose that

$$(b_1^\epsilon + b_2^\epsilon, a^\epsilon) \in \mathcal{L}(\lambda, \Lambda), \tag{3.3}$$

where $a^\epsilon(x) := (\sigma^\epsilon(x))^2$.

Introduce the following condition.

Condition (IV). There exist the measurable functions $b_1(t,x)$, $b_2(x)$, $G^\epsilon(x)$, and $G(x)$ such that, for any compact $K \in R^1$,

IV_1. $\lim_{\epsilon\to 0} \|b_1^\epsilon - b_1\|_{L_2([0,T];K)} = 0$, $\lim_{\epsilon\to 0} \|a^\epsilon - G^\epsilon\|_{L_2([0,T];K)} = 0$,

IV_2. For any $x \in R^1$ $\lim_{\epsilon\to 0} \int_0^x \frac{b_2^\epsilon(y)}{G^\epsilon(y)} dy = \int_0^x b_2(y)dy$,
$\lim_{\epsilon\to 0} \int_0^x \frac{1}{G^\epsilon(y)} dy = \int_0^x G(y)dy$,

IV_3. $(b_1 + \frac{b_2}{G}, G) \in \mathcal{L}(\lambda, \Lambda)$.

In [7, Theorem 4], it was proved that if (3.3) and condition (IV) are valid, then $\mu^\epsilon \stackrel{S}{\Longrightarrow} \mu$, and the limit process $X(t)$ is a solution of the equation

$$X(t) = x + \int_0^t B(s, X(s))ds + \int_0^t \sigma(X(s))dw,$$

where $B(s,x) = b_1(s,x) + \frac{b_2(x)}{G(x)}$, $\sigma(x) = \frac{1}{\sqrt{G(x)}}$.

Theorem 2. *Let the processes $X^\epsilon(t)$, $Y^\epsilon(t)$ be defined by Eqs. (3.1) and (1.1). Suppose that (3.3) and conditions (II), (III), and (IV) are fulfilled. Then $Q^\epsilon \stackrel{M-Z}{\Longrightarrow} Q$, and there exist the Wiener process $(\bar{w}(t), \mathcal{F}_t^{X,Y})$ and a process $(Z(t), \mathcal{F}_t^{X,Y})$ such that $E\int_0^T |Z|^2(t)dt < \infty$ and*

$$X(t) = x + \int_0^t B(s, X(s))ds + \int_0^t \sigma(X(s))d\bar{w}, \tag{3.4}$$

$$Y(t) = g(X(T)) + \int_t^T f(s, X(s), Y(s))ds - \int_t^T Z(s)d\bar{w}(s). \tag{3.5}$$

Proof. To use Theorem 1, we must verify condition (I). The property I_1 is valid in view of the noted above [7, Theorem 4]. Estimate I_2 under condition (3.3) is a standard estimate from the theory of Itô stochastic equations. See, for example, [5. Corollary 2.5.12]. The estimate from I_3 is a result of [5, chapter 2]. Therefore, Theorem 1 may be employed. After the extraction of a suitable sequence, which we omit as an abuse notation, we have that $\mu^\epsilon \stackrel{S}{\Longrightarrow} \mu$ and $Q^\epsilon \stackrel{M-Z}{\Longrightarrow} Q$. Let $\Phi_t(x,y)$ be a functional as in step 6) in the proof of Theorem 1. Since $Y^\epsilon(t)$ is a strong ($\mathcal{F}_t^{X^\epsilon}$- measurable) solution of Eq. (1.1), it is proved in [6, proof of Theorem 1] that, for any infinitely differentiable function with a compact support $\phi(x)$,

$$\begin{aligned}\lim_{\epsilon\to 0} E\Phi_t(X^\epsilon, Y^\epsilon)\Big[\phi(X^\epsilon(s)) - \phi(X^\epsilon(t)) - \\ - \int_t^s \Big(\phi'(X^\epsilon(u))B(X^\epsilon(u)) + \frac{1}{2}\phi''(X^\epsilon(u))\sigma^2(X(u))\Big)du\Big] = 0.\end{aligned} \tag{3.6}$$

From this,

$$\begin{aligned}E\Phi_t(X,Y)\Big[\phi(X(s)) - \phi(X(t)) - \int_t^s \Big(\phi'(X(u))B(u, X(u)) + \\ + \frac{1}{2}\phi''(X(u))a(X(u))\Big)du\Big] = 0.\end{aligned}$$

Passing to the limit causes no difficulty if the functions $b(x)$, $a(x)$ are continuous functions. For only measurable functions, the passing to the limit proves by Krylov's estimate as for I_3^ϵ in the proof of Theorem 1. From [17, Theorem 4.5.1], we have that there exists a Wiener process $(\bar{w}(t), \mathcal{F}_t^{X,Y})$ such that (3.4) is valid.

Let $(\bar{Y}(t), Z(t))$ be a unique solution of BSE [2,13,14]

$$\bar{Y}(t) = g(X(T)) + \int_t^T f(s, X(s), \bar{Y}(s))ds - \int_t^T Z(s)d\bar{w}(s). \tag{3.7}$$

We denote $\bar{M}(t) = \int_0^t Z(s)d\bar{w}(s)$. From (2.38) and (3.7), we conclude that

$$Y(t) - \bar{Y}(t) = \int_t^T [f(s, X(s), Y(s)) - f(s, X(s), \bar{Y}(s))]ds+ \\ + (M(t) - \bar{M}(t)) + (\bar{M}(T) - M(T)).$$

From Itô's formula for semimartingales,

$$E(Y(t) - \bar{Y}(t))^2 + E[M - \bar{M}]_T - E[M - \bar{M}]_t = E\int_t^T [f(s, X(s), Y(s))- \\ - f(s, X(s), \bar{Y}(s))](Y(s) - \bar{Y}(s))ds \le C\int_t^T E|Y(s) - \bar{Y}(s)|^2 ds.$$

Hence, from Gronwall's lemma for all $t \in [0, T]$, the processes $Y(t) = \bar{Y}(t)$ and $M(t) = \bar{M}(t)$. By the unique solution of Eq. (3.5), we conclude that all sequence $Q^\epsilon \overset{M-Z}{\Longrightarrow} Q$. The theorem is proved.

4. Itô equation with drift unbounded on ϵ

Let $X^\epsilon(t)$ in Eq. (1.1) be a solution of the one-dimensional stochastic equation

$$X^\epsilon(t) = x + \int_0^t b^\epsilon(X^\epsilon(s))ds + \int_0^t \sigma^\epsilon(X^\epsilon(s))dw, \tag{4.1}$$

where $w(t)$ is a standard Wiener process. We consider a weak solution of this equation and introduce conditions for the coefficients of (4.1). For the constant $\lambda > 0$,

$$0 < \lambda \le a^\epsilon(x) := (\sigma^\epsilon(x))^2 \le C. \tag{4.2}$$

And, for any $x \in R^1$,

$$\left| \int_0^x \frac{b^\epsilon(y)}{a^\epsilon(y)} dy \right| \le C. \tag{4.3}$$

Under these restrictions, the coefficients $b^\epsilon(x)$ may not be bounded at certain points and tend to infinity as $\epsilon \to 0$ or may not have a limit at all. The limit process for the processes $X^\epsilon(t)$ may be also a solution of the Itô stochastic equation or may change its type and be a solution of the stochastic equation with a local time. In this section, we consider the case of the Itô equation for the limit process.

Denote

$$H^\epsilon(x) = \exp\left\{-2\int_0^x \frac{b^\epsilon(y)}{a^\epsilon(y)} dt\right\}, \quad h^\epsilon(x) = \int_0^x H^\epsilon(y)dy.$$

We set

$$\beta^\epsilon(x) = 2\int_0^x H^\epsilon(y)\left[\int_0^y \frac{b(z) - b^\epsilon(z)}{H^\epsilon(z)a^\epsilon(z)}dz + \beta\right]dy,$$

$$\alpha^\epsilon(x) = \int_0^x \frac{a(y) - a^\epsilon(y)[1 + (\beta^\epsilon(y))']^2}{H^\epsilon(y)a^\epsilon(y)}dy,$$

where the prime denotes a derivative, and introduce conditions (α) and (β).

Condition (β)**.** There exist a measurable bounded function $b(x)$ and a constant β such that, for any $x \in R^1$, $\lim_{\epsilon\to 0} \beta^\epsilon(x) = 0$.

Condition (α)**.** There exists a measurable bounded function $a(x) \geq \lambda$ such that, for any $x \in R^1$, $\lim_{\epsilon\to 0} \alpha^\epsilon(x) = 0$.

We note that the limit functions $b(x)$, $a(x)$ and the constant β are uniquely determined by conditions (β) and (α) [8, Lemma 3.2]. In [8, Theorem 1], it is proved that if (4.2) and (4.3) are satisfied, then conditions (α) and (β) are necessary and sufficient conditions for $\mu^\epsilon \overset{S}{\Longrightarrow} \mu$, and $X(t)$ is the unique weak solution of the stochastic equation

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sqrt{a(X(s))}dw. \tag{4.4}$$

Theorem 3. *Let processes $X^\epsilon(t)$, $Y^\epsilon(t)$ be defined by Eqs. (4.1) and (1.1). Suppose that (4.2), (4.3) and conditions (II), (III), (α), and (β) are fulfilled. Then $Q^\epsilon \overset{M-Z}{\Longrightarrow} Q$, and there exist the Wiener process $(\bar{w}(t), \mathcal{F}_t^{X,Y})$ and a process $(Z(t), \mathcal{F}_t^{X,Y})$ such that $E\int_0^T |Z|^2(t)dt < \infty$ and*

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sqrt{a(X(s))}d\bar{w}, \tag{4.5}$$

$$Y(t) = g(X(T)) + \int_t^T f(s, X(s), Y(s))ds - \int_t^T Z(s)d\bar{w}(s). \tag{4.6}$$

Proof. The proof of this theorem is completely analogous to that of Theorem 2, and we only indicate where the corresponding results were proved. Property I_1 is valid in view of the noted above [8, Theorem 1]. The estimate from I_2 follows from the same estimate in [8, Lemma 3.5]. The Krylov's estimate for the process $X^\epsilon(t)$ is a result of [8, Lemma 3.7]. The analog of formula (3.6) is formula (1.3) in [8]. Theorem is proved.

As an example of the use of Theorem 3, we consider the solutions of stochastic equations

$$X^\epsilon(t) = x + \frac{1}{\epsilon}\int_0^t b\left(\frac{X^\epsilon(s)}{\epsilon}\right)ds + \int_0^t \sigma\left(\frac{X^\epsilon(s)}{\epsilon}\right)dw. \tag{4.7}$$

In contrast to [3,12,14], we do not assume that $b(x)$, $a(x)$ are periodic functions. Let the following limits exist:

$$\lim_{|x|\to\infty} \frac{1}{x}\int_0^x \exp\left\{-2\int_0^y \frac{b(z)}{a(z)}dz\right\}dy = B_1 > 0, \tag{4.8}$$

$$\lim_{|x|\to\infty} \frac{1}{x}\int_0^x \frac{1}{a(y)}\exp\left\{2\int_0^y \frac{b(z)}{a(z)}dz\right\}dy = B_2 > 0. \tag{4.9}$$

It can easily be verified that conditions (β) and (α) are satisfied with the function $b(x) = 0$ and a constant $\beta : 2\beta + 1 = B_1^{-1}$, $a(x) = (B_1B_2)^{-1}$. In this case, (4.5) takes the form

$$X(t) = x + (B_1B_2)^{-\frac{1}{2}}\bar{w}(t).$$

Let us consider Eq.(4.7) under another assumptions,

$$\int_{-\infty}^{\infty} \left|\frac{b(x)}{a(x)}\right|dx < \infty, \quad \int_{-\infty}^{\infty} \frac{b(x)}{a(x)}dx = 0, \tag{4.10}$$

and let there exist

$$\lim_{|x|\to\infty} \frac{1}{x}\int_0^x \frac{dy}{a(y)} = A > 0. \tag{4.11}$$

In this case, conditions (β) and (α) are satisfied with the function $b(x) = 0$ and a constant β :

$$2\beta + 1 = \exp\left\{2\int_0^\infty \frac{b(y)}{a(y)}dy\right\},$$

$a(x) = A^{-1}$. In this case, (4.5) takes the form

$$X(t) = x + \frac{1}{\sqrt{A}}\bar{w}(t).$$

5. Limit process changes type

We now consider the stochastic equation (4.1) under conditions (4.2) and (4.3), but we introduce another suppositions for the coefficients of the equation. In this case, the limit process is the solution of a stochastic equation with local time. We will use notation from Section 4. Let $h_i > 0$, $i = 1, 2$ be some fixed constants. Suppose

$$\lim_{\epsilon\to 0} h^\epsilon(x) = h(x) = \begin{cases} h_1 x, & \text{if } x \le 0; \\ h_2 x, & \text{if } x \ge 0. \end{cases} \tag{5.1}$$

Under condition (4.3), the limit in (5.1) exists uniformly on compacts. As $h^\epsilon(x)$ is a monotonically increasing function on x, there exists an inverse function $\lambda^\epsilon(x) : \lambda^\epsilon(h^\epsilon(x)) = x$. Then $\lim_{\epsilon\to 0} \lambda^\epsilon(x) = \lambda(x)$ uniformly on compacts, and $\lambda(x)$ is an inverse function to the function $h(x)$. By $Dh(x)$, we denote symmetric derivatives of the function $h(x)$:

$$Dh(x) = \lim_{\delta\to 0} \frac{h(x+\delta) - h(x-\delta)}{2\delta}$$

and introduce the following condition: for any $x \in R^1$,

$$\lim_{\epsilon\to 0} \int_0^x \frac{1}{H^\epsilon(y)a^\epsilon(y)}dy = \int_0^t \frac{1}{a(y)Dh(y)}dy. \tag{5.2}$$

Let $X(t)$ be a weak solution of the equation

$$X(t) = x + \beta L^X(t, 0) + \int_0^t \sigma(X(s))dw. \tag{5.3}$$

It follows from [4, Theorem 4.35] that Eq. (5.3) has the unique weak solution.

In [9, Theorem], it is proved that if (4.2), (4.3,) and (5.1) are satisfied, then conditions (5.2) and

$$\beta = \frac{h_1 - h_2}{h_1 + h_2} \tag{5.4}$$

are necessary and sufficient conditions for $\mu^\epsilon \overset{S}{\Longrightarrow} \mu$, and $X(t)$ is the unique weak solution of the stochastic equation (5.3).

Theorem 4. *Let the processes $X^\epsilon(t)$, $Y^\epsilon(t)$ be defined by Eqs. (4.1) and (1.1). Suppose that (4.2), (4.3), (5.1), (5.2), (5.4), (II), and (III) are fulfilled. Moreover, the function $f(t, x, y)$ is continuous on x. Then $Q^\epsilon \overset{M-Z}{\Longrightarrow} Q$, and there exist the Wiener process $(\bar{w}(t), \mathcal{F}_t^{X,Y})$ and a process $(Z(t), \mathcal{F}_t^{X,Y})$ such that $E\int_0^T |Z|^2(t)dt < \infty$ and*

$$X(t) = x + \beta L^X(t, 0) + \int_0^t \sqrt{a(X(s))}d\bar{w} \tag{5.5}$$

$$Y(t) = g(X(T)) + \int_t^T f(s, X(s), Y(s))ds - \int_t^T Z(s)d\bar{w}(s). \tag{5.6}$$

Proof. We use Theorem 3 in the proof of Theorem 4. Let $\eta^\epsilon(t) = h^\epsilon(X^\epsilon(t))$. Then, by the Itô formula,

$$\eta^\epsilon(t) = \eta^\epsilon(0) + \int_0^t \hat{\sigma}^\epsilon(\eta^\epsilon(s))dw, \tag{5.7}$$

where $\hat{\sigma}^\epsilon(x) = H^\epsilon(\lambda^\epsilon(x))\sigma^\epsilon(\lambda^\epsilon(x))$. By using the one-to-one correspondence of the processes $X^\epsilon(t)$ and $\eta^\epsilon(t)$, we get $\mathcal{F}_t^{X^\epsilon} = \mathcal{F}_t^{\eta^\epsilon}$, and (1.1) yields

$$Y^\epsilon(t) = E\left[\hat{g}^\epsilon\Big(\eta^\epsilon(T)\Big) + \int_t^T \hat{f}^\epsilon\Big(s, \eta^\epsilon(s), Y^\epsilon(s)\Big)ds\Big|\mathcal{F}_t^{\eta^\epsilon}\right]. \tag{5.8}$$

In this formula, $\hat{g}^\epsilon(x) = g^\epsilon(\lambda^\epsilon(x))$ and $\hat{f}^\epsilon(t,x,y) = f^\epsilon(t, \lambda^\epsilon(x), y)$. The stochastic system (5.7), (5.8) has the same type as (4.1), (1.1). To prove the theorem, we verify the conditions of Theorem 3. According to the conditions of Theorem and the properties of the functions $h^\epsilon(x)$, $H^\epsilon(x)$, $\lambda^\epsilon(x)$ and $\lambda(x)$, we have that, uniformly on compacts, $\lim_{\epsilon\to 0} g^\epsilon(\lambda^\epsilon(x)) = g(\lambda(x))$ and, for any compact K,

$$\lim_{\epsilon\to 0} \|\hat{f}^\epsilon(\cdot,\cdot,y) - f(\cdot,\lambda(\cdot),y)\|_{L_2([0,T]\times K)} = 0.$$

Hence, condition (III) is valid. For process (5.7), condition (β) is automatically valid with the limit function $b(x) = 0$ and the constant $\beta = 0$. From (5.2), we get that condition (α) is valid too with the function $a(x) = a(\lambda(x))$. Consequently, by Theorem 3, there exists the Wiener process $(\bar{w}(t), \mathcal{F}_t^{\eta,Y})$, and $\mathcal{F}_t^{\eta,Y}$ is a measurable process $Z(t)$, $E\int_0^T Z^2(t)dt < \infty$, such that

$$\eta(t) = h(x) + \int_0^t \frac{\sqrt{a(\lambda(\eta(s)))}}{D(\lambda(\eta(s))}d\bar{w}, \tag{5.8}$$

$$Y(t) = g(\lambda(\eta(T))) + \int_t^T f(s, \lambda(\eta(s)), Y(s))ds - \int_t^T Z(s)d\bar{w}(s). \tag{5.9}$$

Observe that $X^\epsilon \overset{S}{\Longrightarrow} X = \lambda(\eta)$. Applying the Tanaka formula to the process $\eta(t)$ from (5.8) and using the function $\lambda(x)$, we get that the process $X(t)$ satisfies Eq. (5.5), and formula (5.9) coincides with formula (5.6). As above, we have $\mathcal{F}_t^{\eta,Y} = \mathcal{F}_t^{X,Y}$. The theorem is proved.

As a model example for this theorem, we again consider the stochastic equation (4.7) but change the second condition in (4.10). Suppose that

$$\int_{-\infty}^{\infty}\left|\frac{b(x)}{a(x)}\right|dx < \infty, \quad \int_{-\infty}^{0}\frac{b(x)}{a(x)}dx = K_1, \quad \int_{0}^{\infty}\frac{b(x)}{a(x)}dx = K_2,$$

and condition (4.11) is satisfied. Then the limit process $X(t)$ for the solution to Eq. (4.7) has the type

$$X(t) = x + th(K_1 + K_2)L^X(t,0) + \frac{1}{\sqrt{A}}\bar{w}(t).$$

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