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THE BROWNIAN MOTION PROCESS WITH GENERALIZED DIFFUSION MATRIX AND DRIFT VECTOR

Using the method of the classical potential theory, we have constructed a semigroup of operators that describes a multidimensional process of Brownian motion, for which the drift vector and the diffusion matrix are generalized functions.

1. Introduction and formulation of the problem.

On a Euclidean space $\mathbb{R}^d$, $d \geq 2$, let us consider two domains:

$\mathcal{D}_m = \{x : x = (x_1, \dots, x_d) \in \mathbb{R}^d, (-1)^m x_d > 0\}, m = 1, 2$. By $\overline{\mathcal{D}}_m$ and S, we denote the closure and the boundary of $\mathcal{D}_m$, that is, $S = \{x : x = (x', x_d) \in \mathbb{R}^d, x_d = 0\} = \mathbb{R}^{d-1}$, $\overline{\mathcal{D}}_m = \mathcal{D}_m \cup S$. Suppose that, in $\mathcal{D}_m$, a diffusion process is considered which is operated by a generating differential operator with constant coefficients

$$L = \frac{1}{2} \sum_{i,j=1}^{d} b_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \tag{1}$$

where $b = (b_{ij})$ is a symmetric and positive definite matrix.

We also suppose that the bounded continuous functions $q_1(x')$, $q_2(x')$, $\beta_{kl}(x')$, $\alpha_k(x')$, $k, l = 1, \dots, d-1$ are defined on S, which will be used for the description of the process at the points of the boundary of the domains $\mathcal{D}_1$, $\mathcal{D}_2$. We assume that $\beta(x') = \big(\beta_{kl}(x')\big)$ is a symmetric and nonnegative definite matrix.

We pose the problem to describe a general enough class of continuous Feller processes in $\mathbb{R}^d$, for which the generating differential operator at the points of the domains $\mathcal{D}_1$ and $\mathcal{D}_2$ coincides with the operator L, and their behavior at the points of the boundary S is defined with given conjugation condition of Wentzel [1]. This problem is also called the problem on the pasting of two diffusion processes (see [2,3]). For its solution, we use analytical methods. With such an approach, the required class of processes will be generated by a semigroup of operators we specify by means of a solution of the following conjugation problem for a linear parabolic equation with the second-order partial derivatives:

$$\frac{\partial u}{\partial t} = Lu, \quad (t, x) \in (0, \infty) \times \mathcal{D}_m, \quad m = 1, 2, \tag{2}$$

$$u(0, x) = \varphi(x), \quad x \in \mathbb{R}^d, \tag{3}$$

$$u(t, x', -0) = u(t, x', +0), \quad (t, x') \in (0, +\infty) \times \mathbb{R}^{d-1}, \tag{4}$$

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(5)
$$
\begin{aligned}
L_0 u \equiv \frac{1}{2}\sum_{k,l=1}^{d-1} \beta_{kl}(x')\frac{\partial^2 u(t,x',0)}{\partial x_k \partial x_l} \\
+ \sum_{k=1}^{d-1}\alpha_k(x')\frac{\partial u(t,x',0)}{\partial x_k} - q_1(x')\frac{\partial u(t,x',-0)}{\partial x_d} + q_2(x')\frac{\partial u(t,x',+0)}{\partial x_d} = 0, \\
(t,x') \in (0,\infty)\times\mathbb{R}^{d-1}.
\end{aligned}
$$

Note that equality (4) means that the required process will be a Feller one, and relation (5) corresponds to the general Wentzel boundary condition for the multi-dimensional diffusion processes.

We are interested in the classical solution of problem (2)-(5) that is determined by a function $u(t,x)$ continuous in the domain $(t,x) \in [0,\infty)\times\mathbb{R}^d$ bounded at infinity by the space variable x, has continuous derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i \partial x_j}$ $(i,j = 1,\ldots,d)$ at the points of the domains $(t,x) \in (0,\infty)\times\mathcal{D}_m$, $m = 1,2$, and satisfies Eq. (2) and the initial condition (3) in these domains and conditions (4) and (5) at the points of the boundary S. Moreover, as follows from (5), the derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i \partial x_j}$ $(i,j = 1,\ldots,d-1)$ have to exist and be continuous at all points of the domain $(t,x) \in (0,\infty)\times\mathbb{R}^d$. The existence of such a solution of problem (2)-(5) was first obtained by us using the method of boundary integral equations with the use of the common potential of a simple layer. In addition, we will prove that the Markov process constructed with the use of the solution of problem (2)-(5) can be interpreted as a generalized diffusion process in the sense of N.I. Portenko [2]. Recall that a similar problem was studied earlier by using analytical methods in [3], where it was used the construction of the special parabolic potential of a simple layer in an integral representation of the required semigroup. Moreover, the initial-boundary-value problem for the common second-order parabolic equation with the Wentzel boundary condition was considered in [4] and was analyzed by a normalization method. We mention also papers [5,6], where a problem of construction of the generalized diffusion was studied by methods of stochastic analysis.

2. Solution of the parabolic problem of conjugation using analytical methods.

We will construct the classical solution of problem (2)-(5) using the following assumptions for the parameters from condition (5):

a) $q_1, q_2, \alpha_k, \beta_{kl} \in H^\lambda(\mathbb{R}^{d-1})$, $\lambda \in (0,1)$, where $H^\lambda(\mathbb{R}^{d-1})$ is a Hölder space (see [7]), moreover $q_1(x') \geq 0$, $q_2(x') \geq 0$, $x' \in \mathbb{R}^{d-1}$, $\inf_{x'\in\mathbb{R}^{d-1}} \big(q_1(x') + q_2(x')\big) > 0$;

b) there exist positive constants β_1 and β_2 such that, for all $x' \in \mathbb{R}^{d-1}$ and for any real vector $\Theta' \in \mathbb{R}^{d-1}$,
$$\beta_1|\Theta'|^2 \leq (\beta(x')\Theta', \Theta') \leq \beta_2|\Theta'|^2\,.$$

By $g(t,x,y)$ $(t > 0,\ x,y \in \mathbb{R}^d)$, we denote a fundamental solution (f.s.) of Eq. (2) (see [7]). In this case, the function $g(t,x,y)$ is specified by the formula

$$g(t,x,y) = g(t,x-y) = (2\pi t)^{-d/2}(\det b)^{-1/2}\exp\left\{-\frac{1}{2t}\big(b^{-1}(y-x), y-x\big)\right\},$$

where b^{-1} is the matrix inverse to b, and $\big(b^{-1}(y-x), y-x\big)$ means the scalar product of the vectors $b^{-1}(y-x)$ and $(y-x)$ in $\mathbb{R}^d$.

Theorem 1. *Let the matrix b from (1) be symmetric and positive definite; elements of the symmetric matrix β and the functions α_k, $k = 1,\ldots,d-1$, q_1, q_2 from (5) satisfy conditions a), b), and the initial function φ from (3) is twice continuously differentiable* *and bounded together with its derivatives on $\mathbb{R}^d$. Then problem (2)-(5) has a unique classical solution, for which the estimation*

$$|u(t,x)| \le C\,||\varphi|| \tag{6}$$

holds at $(t,x) \in [0,T] \times \mathbb{R}^d$ ($T > 0$ – fixed), where

$$||\varphi|| = \sup_{x\in\mathbb{R}^d} |\varphi(x)| + \sup_{x\in\mathbb{R}^d} \sum_{i=1}^{d} \left|\frac{\partial\varphi(x)}{\partial x_i}\right| + \sup_{x\in\mathbb{R}^d} \sum_{i,j=1}^{d} \left|\frac{\partial^2\varphi(x)}{\partial x_i \partial x_j}\right| ,$$

and C is some constant finite for $T < \infty$.

Proof. We will find a solution of problem (2)-(5) in the form

$$u(t,x) = u_0(t,x) + u_1(t,x), \quad t > 0,\ x \in \mathcal{D}_m,\ m = 1,2, \tag{7}$$

where

$$\begin{aligned} u_0(t,x) &= \int_{\mathbb{R}^d} g(t,x,y)\varphi(y)dy, \\ u_1(t,x) &= \int_0^t d\tau \int_{\mathbb{R}^{d-1}} g(t-\tau,x,y')V(\tau,y')dy', \end{aligned}$$

$V(t,x')$ ($t > 0$, $x' \in \mathbb{R}^{d-1}$) is an unknown function. In the theory of parabolic equations, the functions u_0 and u_1 are called the Poisson potential and the potential of a simple layer respectively. For any bounded and measurable function V, the function u from (7) satisfies Eq. (2) and conditions (3) and (4). This follows from the conditions of the theorem and the properties of potentials (see [7]). In addition, the condition

$$u_0 \in C^{1,2}_{t,x}\big([0,\infty) \times \mathbb{R}^d\big) \tag{8}$$

and the estimation ($(t,x) \in [0,T] \times \mathbb{R}^d$)

$$\left|D_t^r\, D_x^p\, u_0(t,x)\right| \le C\,||\varphi||, \qquad 2r + p \le 2, \tag{9}$$

hold, where r and p are nonnegative and integer, D_t^r and D_x^p are, respectively, the symbols of partial derivatives with respect to t of order r and with respect to x of order p, and C is a constant.

Thus, for the solution u of the problem, we need to choose V such that Eq. (5), inequality (6), and the other properties of the defined classic solution hold.

We suppose *a priori* that the unknown density V is continuous in the domain $(t,x') \in [0,\infty) \times \mathbb{R}^{d-1}$. Also we suppose that V is bounded and continuously differentiable with respect to variable x' for $t > 0$, $x' \in \mathbb{R}^{d-1}$. In addition, $D_{x'}V(t,x')$ is a Hölder function of the same variable. To find V, we use the condition of conjugation (5). We separate the conormal derivative in the representation for $L_0 u$ and, after simple transformations, obtain the equation

$$\begin{aligned} L_0' u \equiv \frac{1}{2} \sum_{k,l=1}^{d-1} \beta_{kl}^{(0)}(x') \frac{\partial^2 u(t,x',0)}{\partial x_k \partial x_l} + \sum_{k=1}^{d-1} \alpha_k^{(0)}(x') \frac{\partial u(t,x',0)}{\partial x_k} - \\ - u(t,x',0) = \Theta^{(0)}(t,x'), \quad t > 0,\ x' \in \mathbb{R}^{d-1}, \end{aligned} \tag{10}$$

where

$$\beta^{(0)}_{kl}(x') = \frac{\sqrt{b_{dd}}}{q_1(x')+q_2(x')}\beta_{kl}(x'), \quad \alpha^{(0)}_k(x') = \frac{\sqrt{b_{dd}}}{q_1(x')+q_2(x')}\alpha_k(x') - \frac{q(x')}{\sqrt{b_{dd}}}b_{kd},$$
$$k,l = 1,\dots,d-1,$$
$$\Theta^{(0)}(t,x') = \frac{1}{2}\,\frac{1-q(x')}{\sqrt{b_{dd}}}\,\frac{\partial u(t,x',-0)}{\partial N(x')} - \frac{1}{2}\,\frac{1+q(x')}{\sqrt{b_{dd}}}\,\frac{\partial u(t,x',+0)}{\partial N(x')} - u(t,x',0),$$
$$q(x') = \frac{q_2(x')-q_1(x')}{q_1(x')+q_2(x')}, \quad |q(x')| \le 1,$$
$$N(x') = b\nu(x') \quad \Big(\nu(x') = (0,\dots,0,1) \in \mathbb{R}^d\Big) \text{ is the conormal vector.}$$

In view of (7) and the relation from a corollary of the theorem on a jump of the conormal derivative of the potential of a simple layer (see [2,7]), we can write the function $\Theta^{(0)}$ as

$$\begin{aligned}\Theta^{(0)}(t,x') = \frac{V(t,x')}{\sqrt{b_{dd}}} - \int_0^t d\tau \int_{\mathbb{R}^{d-1}} g(t-\tau,x',y')V(\tau,y')dy' \\ - \frac{q(x')}{\sqrt{b_{dd}}}\frac{\partial u_0(t,x',0)}{\partial N(x')} - u_0(t,x',0).\end{aligned} \tag{11}$$

Then we will consider equality (10) as an autonomous elliptical equation for $u(t,x',0)$ on $\mathbb{R}^{d-1}$.

At first, we note that the conditions of Theorem 1 guarantee the existence of the main f.s. $\Gamma(x',y')$ for the operator $L_0^{'}$ (see [8–10]) that can be described in our case by the formula

$$\Gamma(x',y') = \int_0^\infty e^{-s}\,G(s,x',y')ds,$$

where $G(s,x',y')$ $(s > 0,\ x',y' \in \mathbb{R}^{d-1})$ is a f.s. of the uniformly parabolic operator with Hölder coefficients

$$\frac{1}{2}\sum_{k,l=1}^{d-1}\beta^{(0)}_{kl}(x')\frac{\partial^2}{\partial x_k\partial x_l} + \sum_{k=1}^{d-1}\alpha^{(0)}_k(x')\frac{\partial}{\partial x_k} - \frac{\partial}{\partial s}.$$

The matrix with elements $\beta^{(0)}_{kl}(x')$, $k,l = 1,\dots,d-1$, and the vector, whose components are the functions $\alpha^{(0)}_k(x')$, $k = 1,\dots,d-1$, are denoted by $\beta^{(0)}(x')$ and $\alpha^{(0)}(x')$, respectively.

We note some known properties of the f.s. G (see [2,7]):

1) the function $G(s,x',y')$ is nonnegative, continuous in all variables, and is represented by the formula

$$G(s,x',y') = G_0(s,x',y') + G_1(s,x',y'), \quad s > 0,\ x',y' \in \mathbb{R}^{d-1}, \tag{12}$$

where

$$\begin{aligned}G_0(s,x',y') &= G_0^{(y')}(s,x'-y') \\ &= (2\pi s)^{-\frac{d-1}{2}}\big(\det\beta_0(y')\big)^{-1/2}\exp\left\{-\frac{1}{2s}\big(\beta_0^{-1}(y')(y-x),y-x\big)\right\},\end{aligned}$$

and $G_1(s,x',y')$ can be written with the use of an integral operator with kernel G_0 and density Φ_0 that is defined from some integral equation;

2) the functions G, G_0, G_1, as functions of the arguments t and x, are continuously differentiable with respect to t, twice continuously differentiable with respect to x' (the

function G_0 is infinitely continuously differentiable with respect to the mentioned variables), and satisfy the inequalities

$$\left|D_s^r D_{x'}^p G(s,x',y')\right| \le C s^{-\frac{(d-1)+2r+p}{2}} \exp\left\{-c\frac{|x'-y'|^2}{s}\right\}, \tag{13}$$

$$\left|D_s^r D_{x'}^p G_1(s,x',y')\right| \le C s^{-\frac{(d-1)+2r+p-\lambda}{2}} \exp\left\{-c\frac{|x'-y'|^2}{s}\right\}, \tag{14}$$

$$\begin{aligned} &\left|D_s^r D_{z'}^p G_0^{(y')}(s,z') - D_s^r D_{z'}^p G_0^{(\tilde{y}')}(s,z')\right| \\ &\qquad \le C\left|y'-\tilde{y}'\right|^{\gamma} s^{-\frac{(d-1)+2r+p}{2}} \exp\left\{-c\frac{|z'|^2}{s}\right\}, \qquad 0<\gamma\le\lambda, \end{aligned} \tag{15}$$

when ever $2r+p\le 2$, $s\in[0,T]$, $x',y',\tilde{y}',z'\in\mathbb{R}^{d-1}$ with positive constants C and c. We note also that inequality (15) is true for all nonnegative integers r and p, and the constant C in inequalities (13) and (14) depends, generally speaking, on T. However, in the case where the function G and its derivatives are estimated together with a coefficient of the form $e^{-\mu s}$, where μ is a positive number, we can always assume that, in inequalities (13),(14), the constant C does not depend on T. Such consequences are also true for other functions of such a type;

3)

$$\int_{\mathbb{R}^{d-1}} G(s,x',y')dy' = 1 \quad \text{for all} \quad s>0,\ x'\in\mathbb{R}^{d-1};$$

4)

$$\int_{\mathbb{R}^{d-1}} G(s,x',z')G(t,z',y')dz' = G(s+t,x',y') \quad \text{for} \quad s>0,\, t>0,\ x',y'\in\mathbb{R}^{d-1};$$

5) for all $s\ge 0$, $x'\in\mathbb{R}^{d-1}$, $\Theta'\in\mathbb{R}^{d-1}$, the equalities

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} G(s,x',y')(y'-x',\Theta')dy' &= \int_0^s d\tau \int_{\mathbb{R}^{d-1}} G(\tau,x',y')(\alpha_0(y'),\Theta')dy', \\ \int_{\mathbb{R}^{d-1}} G(s,x',y')(y'-x',\Theta')^2 dy' &= \int_0^s d\tau \int_{\mathbb{R}^{d-1}} G(\tau,x',y')(\beta_0(y')\Theta',\Theta')dy' + \\ &\quad + 2\int_0^s d\tau \int_{\mathbb{R}^{d-1}} G(\tau,x',y')(\alpha_0(y'),\Theta')(y'-x',\theta')dy' \end{aligned}$$

hold.

Let us consider the right-hand side of Eq. (10). We can assume that $\Theta_0(t,x')$ $(t\ge 0,\ x'\in\mathbb{R}^{d-1})$ from (11) is continuous in two variables and is continuously differentiable at $t>0$ with respect to the variable x' $(x'\in\mathbb{R}^{d-1})$ and bounded together with its derivative. This corollary can be obtained by using conditions of Theorem 1, an *a priori* assumption concerning V, and the properties of parabolic potentials. Then (see [9, Ch. III, §20]) the unique solution of Eq. (10) is represented by the formula

$$\begin{aligned} u(t,x',0) &= -\int_{\mathbb{R}^{d-1}} \Gamma(x',z')\Theta^{(0)}(t,z')dz' \\ &= -\int_0^\infty e^{-s}ds \int_{\mathbb{R}^{d-1}} G(s,x',z')\Theta^{(0)}(t,z')dz',\ t>0,\ x'\in\mathbb{R}^{d-1}. \end{aligned} \tag{16}$$

Equating the right-hand sides of relations (7) (where we need to put $x_d = 0$) and (16), we obtain the required equation for V:

(17)
$$\int_0^t d\tau \int_{\mathbb{R}^{d-1}} K_0(t-\tau, x', y')\ V(\tau, y')dy' + \int_0^\infty e^{-s}ds \int_{\mathbb{R}^{d-1}} G(s, x', z')\frac{V(t, z')}{\sqrt{b_{dd}}}dz' = \\ = \psi_0(t, x'), \quad t > 0,\ x' \in \mathbb{R}^{d-1},$$

where

$$K_0(t-\tau, x', y') = g(t-\tau, x', y') - \int_0^\infty e^{-s}ds \int_{\mathbb{R}^{d-1}} G(s, x', z')\ g(t-\tau, z', y')dz',$$
$$\psi_0(t, x') = \int_0^\infty e^{-s}ds \int_{\mathbb{R}^{d-1}} G(s, x', z') \left(\frac{q(z')}{\sqrt{b_{dd}}} \frac{\partial u_0(t, z', 0)}{\partial N(z')} + u_0(t, z', 0) \right) dz' \\ - u_0(t, x', 0).$$

Using properties of the f.s. G and the Poisson potential, we analyzed the function $\psi_0(t, x')$. We got that it is continuous, twice continuously differentiable with respect to x' at $t \geq 0$, $x' \in \mathbb{R}^{d-1}$, infinitely continuously differentiable with respect to t at $t > 0$, $x' \in \mathbb{R}^{d-1}$, and the following estimations hold for it:

$$\left|D^p_{x'}\psi(t, x')\right| \leq C||\varphi||, \qquad p \leq 2, \quad (t, x') \in [0, T] \times \mathbb{R}^{d-1}$$
$$\left|D^r_t\psi(t, x')\right| \leq C||\varphi||\, t^{-\frac{2r-1}{2}}, \qquad r \geq 1, \quad (t, x') \in (0, T] \times \mathbb{R}^{d-1}$$
$$\left|D^r_t D^p_{x'}\psi(t, x')\right| \leq C||\varphi||\, t^{-(r-1)-\frac{p}{2}}, \qquad r \geq 1,\ p = 1, 2, \quad (t, x') \in (0, T] \times \mathbb{R}^{d-1}.$$

As we can see, Eq. (17) is a first-kind integral equation of the Volterra–Fredholm type. With the purpose to normalize the equation, we consider an integro-differential operator $\mathcal{E}$ that acts by the rule

(18)
$$\mathcal{E}(t, x')\psi_0 \\ = \sqrt{2/\pi}\Bigg\{\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-1/2}d\tau \int_{\mathbb{R}^{d-1}} \psi_0(\tau, y')dy' \\ \times \Bigg[h(\hat{t}-\tau, x', y') \\ + \int_0^\infty (1 - \frac{u}{t-\tau})e^{-\frac{u^2}{2(t-\tau)}}du \int_{\mathbb{R}^{d-1}} h(\hat{t}-\tau, x', v')G(u, v', y')dv'\Bigg]\Bigg\}\Bigg|_{\hat{t}=t}, \\ t > 0, \qquad x' \in \mathbb{R}^{d-1},$$

where $h(t, x', y') \quad (t > 0,\ x', y' \in \mathbb{R}^{d-1})$ is an f.s. of a parabolic operator with constant coefficients

$$\frac{\partial}{\partial t} - \frac{1}{2}\sum_{i,j=1}^{d-1} \tilde{b}_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\,, \quad \tilde{b}_{ij} = b_{ij} - \frac{b_{id}\ b_{jd}}{b_{dd}}, \quad i, j = 1, \ldots, d-1.$$

Properties 1)-5) can be easily extended to the f.s. h with obvious changes.

Applying the operator $\mathcal{E}$ to both sides of Eq. (17) leads to the equivalent second-kind Volterra integral equation

$$(19) \quad V(t, x') = \int_0^t d\tau \int_{\mathbb{R}^{d-1}} K(t-\tau, x', y')V(\tau, y')dy' + \psi(t, x'), \quad t > 0,\ x' \in \mathbb{R}^{d-1},$$

where

$$\psi(t,x') = (b_{dd})^{1/2}\mathcal{E}(t,x')\psi_0,$$

$$\begin{aligned}
K(t-\tau,x',y') = \frac{1}{\pi}\bigg\{\frac{\partial}{\partial t}\int_\tau^t (t-s)^{-3/2}(s-\tau)^{-1/2}ds\int_0^\infty ue^{-\frac{u^2}{2(t-s)}}du \\
\times \int_{\mathbb{R}^{d-1}} h(\hat{t}-s,x',v')dv' \\
\times \int_{\mathbb{R}^{d-1}}\Big(G(u,v',z') - G_0^{(y')}(u,v'-z')\Big)h(s-\tau,z',y')dz'\bigg\}\bigg|_{\hat{t}=t} \\
-\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial t}\Big((t-\tau)^{-1/2}e^{-\frac{u^2}{2(t-\tau)}}\Big)du \\
\times \int_{\mathbb{R}^{d-1}} h(t-\tau,x',z')G_1(u,z',y')dz' \\
= K_1(t-\tau,x',y') - K_2(t-\tau,x',y'),
\end{aligned}$$

in addition, the kernel $K(t-\tau,x',y')$ at $0 \le \tau < t \le T,\ x',y' \in \mathbb{R}^{d-1}$ and some positive constants C and c allow the estimation

$$\Big|K(t-\tau,x',y')\Big| \le C(t-\tau)^{-\frac{3}{2}+\frac{\lambda}{4}}\int_0^\infty e^{-c\frac{u^2}{t-\tau}}(t-\tau+u)^{-\frac{d-1}{2}}\exp\Big\{-c\frac{|x'-y'|^2}{t-\tau+u}\Big\}du. \tag{20}$$

We prove the implementation of inequality (20) with an example of estimation of the function K_2 that is included in the formula for the kernel K. On the basis of estimations (13) and (14) applied to the f.s. h and G_1 and the obvious inequality

$$\left|\frac{\partial}{\partial t}(t-\tau)^{-1/2}e^{-\frac{u^2}{2(t-\tau)}}\right| \le C(t-\tau)^{-3/2}e^{-c_1\frac{u^2}{t-\tau}}, \tag{21}$$

we have $(0 \le \tau < t \le T,\ x',y' \in \mathbb{R}^{d-1})$

$$\begin{aligned}
\Big|K_2(t-\tau,x',y')\Big| \le C\int_0^\infty (t-\tau)^{-\frac{3}{2}}e^{-c\frac{u^2}{t-\tau}}du\times \\
\times \int_{\mathbb{R}^{d-1}}(t-\tau)^{-\frac{d-1}{2}}\exp\Big\{-c\frac{|x'-z'|^2}{t-\tau}\Big\}u^{-\frac{(d-1)-\lambda}{2}}\exp\Big\{-c\frac{|z'-y'|^2}{u}\Big\}dz'.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}}\exp\Big\{-c\frac{|x'-z'|^2}{t-\tau}\Big\}\exp\Big\{-c\frac{|z'-y'|^2}{u}\Big\}dz' \\
= \Big(\frac{\pi}{c}\Big)^{-\frac{d-1}{2}}\Big(\frac{(t-\tau)u}{t-\tau+u}\Big)^{\frac{d-1}{2}}\exp\Big\{-c\frac{|x'-y'|^2}{t-\tau+u}\Big\},
\end{aligned}$$

we have

$$\Big|K_2(t-\tau,x',y')\Big| \le C(t-\tau)^{-3/2}\int_0^\infty u^{\frac{\lambda}{2}}e^{-c\frac{u^2}{t-\tau}}(t-\tau+u)^{-\frac{d-1}{2}}\exp\Big\{-c\frac{|x'-y'|^2}{t-\tau+u}\Big\}du.$$

Using estimations (13)-(15), we can obtain the same inequality also for the function $K_1(t-\tau,x',y')$ from the kernel K. For this, we disclose a representation for the derivative of the function K_1 with respect to the variable t.

We explore the function $\psi(t,x')$ from (19). For this purpose, we use the representation

(22)
$$\psi(t,x') = \sqrt{\frac{2}{\pi}}\Big\{\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-1/2}d\tau\int_0^\infty e^{-\frac{u^2}{2(t-\tau)}}du\int_{\mathbb{R}^{d-1}} q(y')\frac{\partial u_0(\tau,y',0)}{\partial N(y')}dy' \times\int_{\mathbb{R}^{d-1}} h(\hat{t}-\tau,x',v')G(u,v',y')dv'\Big\}\Big|_{\hat{t}=t}$$
$$+\sqrt{\frac{2b_{dd}}{\pi}}\Big\{\frac{\partial}{\partial t}\int_0^t d\tau\int_0^\infty \frac{u}{(t-\tau)^{3/2}}\,e^{-\frac{u^2}{2(t-\tau)}}du\int_{\mathbb{R}^{d-1}}dy' \times\int_{\mathbb{R}^{d-1}} h(\hat{t}-\tau,x',v')G(u,v',y')\big(u_0(\tau,y',0)-u_0(\tau,v',0)\big)dv'\Big\}\Big|_{\hat{t}=t}$$
$$=\psi_1(t,x')+\psi_2(t,x'),$$

that is obtained after the substitution of the representation for ψ_0 from (17) in (18), by using elementary transformations and properties of the f.s. h and G.

We prove that the function $\psi(t,x')$ is continuous and continuously differentiable with respect to the variable $x'\in\mathbb{R}^{d-1}$ at $t>0$. Moreover, in every domain $(t,x')\in(0,T]\times\mathbb{R}^{d-1}$, the estimation

$$\Big|D^p_{x'}\psi(t,x')\Big| \le C\,||\varphi||\,t^{-\frac{p}{4}},\quad p=0,1, \tag{23}$$

holds.

At first, we estimate the function ψ_1 from (22). We represent it by the formula

$$\psi_1(t,x') = \sqrt{\frac{2}{\pi}}\int_0^t d\tau\int_0^\infty \frac{\partial}{\partial t}\left((t-\tau)^{-1/2}e^{-\frac{u^2}{2(t-\tau)}}\right)du\times \tag{24}$$
$$\times\Big[\int_{\mathbb{R}^{d-1}} q(y')\frac{\partial u_0(\tau,y',0)}{\partial N(y')}dy'\int_{\mathbb{R}^{d-1}} h(t-\tau,x',v')\big(G(u,v',y')-G(u,x',y')\big)dv'+$$
$$+\int_{\mathbb{R}^{d-1}} G(u,x',y')\left(q(y')\frac{\partial u_0(\tau,y',0)}{\partial N(y')}-q(x')\frac{\partial u_0(\tau,x',0)}{\partial N(x')}\right)dy'\Big]+q(x')\frac{\partial u_0(t,y',0)}{\partial N(x')}.$$

As a result of the estimation of all summands from the right-hand side of (24), we prove inequality (23) at $p=0$ for ψ_1. In order to obtain the same estimation for ψ_2 from (22) and consequently for ψ, we have to use the representation

(25)
$$\sqrt{\frac{\pi}{2b_{dd}}}\psi_2(t,x')$$
$$=\int_0^t d\tau\int_0^\infty \frac{\partial}{\partial t}\left(\frac{u}{(t-\tau)^{3/2}}\,e^{-\frac{u^2}{2(t-\tau)}}\right)du$$
$$\times\Big[\int_{\mathbb{R}^{d-1}}\big(\tilde{u}_0(\tau,y',x')-\tilde{u}_0(t,y',x')\big)dy'\int_{\mathbb{R}^{d-1}} h(t-\tau,x',v')G(u,v',y')dv'$$
$$+\int_{\mathbb{R}^{d-1}}\tilde{u}_0(t,y',x')dy'\int_{\mathbb{R}^{d-1}} h(t-\tau,x',v')$$
$$\times\Big(G(u,v',y')-G(u,x',y')-\big(\nabla'_{x'}G(u,x',y'),v'-x'\big)\Big)dv'$$
$$+\int_{\mathbb{R}^{d-1}} G(u,x',y')\Big(\tilde{u}_0(t,y',x')-\big(\nabla' u_0(t,x',0),y'-x'\big)\Big)dy'\Big]$$

$$+\int_0^\infty \frac{u}{t^{3/2}}\, e^{-\frac{u^2}{2t}} du \int_0^u ds \int_{\mathbb{R}^{d-1}} \big(\alpha_0(y'), \nabla' u_0(t,x',0)\big) dy'$$
$$+\frac{1}{2}\int_0^t (t-\tau)^{-3/2} d\tau$$
$$\times \int_{\mathbb{R}^{d-1}} h(t-\tau,x',v')\Big(\tilde{u}_0(\tau,v',y') - \big(\nabla' u_0(\tau,x',0), v'-x'\big)\Big) dv',$$

where $\tilde{u}_0(\tau,y',x') = u_0(\tau,y',0) - u_0(\tau,x',0)$, $\nabla' = \Big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_{d-1}}\Big)$. Here, as in the previous case, we skip the detailed expositions.

Using formula (22) and the scheme of establishing estimation (23) at $p=0$, we prove similarly that the function $\psi(t,x')$ is differentiable with respect to the space variable, inequality (23) is true at $p=1$ for it, and the derivative $D_{x'}\psi(t,x')$ has the Hölder property at x'.

We are going back to the integral equation (19). We note that its kernel, as a result of estimation (20), has an integrable singularity. Applying the method of progressive approximations to this equation, we find V. In addition, we prove that the obtained solution of Eq. (19) has the same properties as the right-hand side of this equation that is then the function $\psi(t,x')$.

Applying estimations (9) and (23) for the functions u_0 and V, respectively, and taking into account the Hölder property by the variable x' of the density of Eq. (19) guarantee us the existence of the desired solution of problem (2)-(5) and the implementation of inequality (6) for it.

Now we prove the statement of the theorem concerning the uniqueness of the found solution. It is enough only to notice that the solution of problem (2)-(5) constructed by formulas (7) and (19) for each domain $t>0$, $x\in\mathcal{D}_m$, $m=1,2,$ can be considered as a classical solution of the parabolic first-boundary problem

$$D_t u = Lu, \quad (t,x)\in(0,\infty)\times\mathcal{D}_m, \quad m=1,2\,,$$
$$u(0,x)=\varphi(x), \quad x\in\overline{\mathcal{D}}_m, \quad m=1,2\,,$$
$$u(t,x',0)=v(t,x'), \quad (t,x')\in[0,+\infty)\times\mathbb{R}^{d-1},$$

where the function v is defined, by using relation (16).

Theorem 1 is proved.

3. Construction of process.

By $\mathcal{B}(\mathbb{R}^d)$, we denote the Banach space of all real bounded measurable functions on $\mathbb{R}^d$ with the norm $\|\varphi\| = \sup_{x\in\mathbb{R}^d} |\varphi(x)|$. We prove that the solution of problem (2)-(5) constructed by formulas (7) and (19) determines a family of linear operators $(T_t)_{t>0}$ that acts in the space $\mathcal{B}(\mathbb{R}^d)$. Consequently, let $\varphi\in\mathcal{B}(\mathbb{R}^d)$. Then the existence of the first summand in (7) follows from the obvious inequality

$$\big|u_0(t,x)\big| \le C||\varphi|| \tag{26}$$

that is true with some constant C for $t>0$, $x\in\mathbb{R}^d$. Now we consider the integral equation (19) and estimate its right-hand side that is the function $\psi(t,x')$. For this, it is possible to use representations (22), (24), and (25) again. By estimating, we need to take into account that, under the condition $\varphi\in\mathcal{B}(\mathbb{R}^d)$, the derivatives of the second or higher orders with respect to x and those of the first or higher orders with respect to t of the function $u_0(t,x)$ have a nonintegrable singularity by t. This follows from the inequality ($t\in(0,T]$, $x\in\mathbb{R}^d$)

$$\big|D_t^r\, D_x^p\, u_0(t,x)\big| \le C\,||\varphi||\, t^{-\frac{2r+p}{2}}, \tag{27}$$

where r and p are positive and integer, and C is a constant.

We estimate, for example, the last summand from (25). We denote it by I and write it in the form

$$I = \frac{1}{2}\int_0^{t/2} (t-\tau)^{-3/2} d\tau \int_{\mathbb{R}^{d-1}} h(t-\tau, x', v')\Big(\tilde{u}_0(\tau, v', y') - \big(\nabla' u_0(\tau, x', 0), v'-x'\big)\Big) dv' +$$
$$+ \frac{1}{2}\int_{t/2}^{t} (t-\tau)^{-3/2} d\tau \int_{\mathbb{R}^{d-1}} h(t-\tau, x', v')\Big(\tilde{u}_0(\tau, v', y') - \big(\nabla' u_0(\tau, x', 0), v'-x'\big)\Big) dv'.$$

Estimating the right-hand side of the last equality with the use of (13) and (27), we obtain ($t \in (0,T],\ x' \in \mathbb{R}^{d-1}$)

$$|I| \le C\|\varphi\| \bigg[\int_0^{t/2} (t-\tau)^{-3/2}\tau^{-1/2} d\tau \int_{\mathbb{R}^{d-1}} (t-\tau)^{-\frac{d-1}{2}} \exp\left\{-c\frac{|x'-v'|^2}{t-\tau}\right\} \big|v'-x'\big| dv'$$
$$+ \int_{t/2}^{t} (t-\tau)^{-1/2}\tau^{-1} d\tau \int_{\mathbb{R}^{d-1}} (t-\tau)^{-\frac{d-1}{2}} \exp\left\{-c\frac{|x'-v'|^2}{t-\tau}\right\} \big|v'-x'\big|^2 dv'\bigg]$$
$$\le C\|\varphi\|\, t^{-1/2}.$$

Similar inequalities are true for other summands included in the right-hand sides of formulas (24) and (25). Uniting these inequalities, we obtain that the estimation

$$\big|\psi(t,x')\big| \le C\,||\varphi||\, t^{-1/2},$$

is true at $t \in (0,T],\ x' \in \mathbb{R}^{d-1}$.

The last inequality and (20) lead to the estimation for the solution of Eq. (19):

$$\big|V(t,x')\big| \le C\,||\varphi||\, t^{-1/2}. \tag{28}$$

Here, $t \in (0,T],\ x' \in \mathbb{R}^{d-1}$, and C is some constant.

Estimation (28) yields the existence of the potential of a simple layer in (7) and the implementation of inequality (26) for it and, consequently, for the function $T_t\varphi(x)$.

As was done in work [3], we prove that the family of operators $(T_t)_{t>0}$ constructed by formulas (7) and (19) creates a semigroup that determines some homogeneous non-precipice Feller process on $\mathbb{R}^d$. We denote its transition probability by $P(t,x,dy)$, so

$$T_t\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)\, P(t,x,dy).$$

At the end of our researches, we prove, by direct calculations, that the transition probability $P(t,x,dy)$ has the properties:

a)

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |y-x|^6\, P(t,x,dy) \le C\, t^{3/2}\,, \quad t \in (0,T], \tag{29}$$

where C is some positive constant;

b) for any finitary continuous function $\varphi(x)$, $x \in \mathbb{R}^d$, and all $\Theta \in \mathbb{R}^d$,

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} \varphi(x)\left[\frac{1}{t}\int_{\mathbb{R}^d} (y-x,\Theta) P(t,x,dy)\right] dx = \int_{\mathbb{R}^{d-1}} \varphi(x',0)\big(\bar{\alpha}(x'),\Theta\big) dx',$$
$$\lim_{t\downarrow 0} \int\limits_{\mathbb{R}^d} \varphi(x)\left[\frac{1}{t}\int\limits_{\mathbb{R}^d} (y-x,\Theta)^2 P(t,x,dy)\right] dx = (b\Theta,\Theta)\int_{\mathbb{R}^d} \varphi(x) dx +$$
$$+ \int_{\mathbb{R}^{d-1}} \varphi(x',0)\big(\bar{\beta}(x')\Theta,\Theta\big)\, dx', \tag{30}$$

where

$$\bar{\alpha}(x') = \frac{b_{dd}}{q_1(x') + q_2(x')}\Big(\alpha_1(x'), \dots, \alpha_{d-1}(x'), q_2(x') - q_1(x')\Big) \in \mathbb{R}^d,$$

$$\bar{\beta}(x') = \Big(\bar{\beta}_{ij}(x')\Big)_{i,j=1}^{d},$$

$$\overline{\beta}_{ij}(x') = \begin{cases} \frac{b_{dd}}{q_1(x')+q_2(x')}\ \beta_{ij}(x') & \text{if } i,j = 1, \dots, d-1\,, \\ 0 & \text{if } i = d \text{ or } j = d. \end{cases}$$

From inequality (29), it follows that the trajectories of the constructed process are continuous, and relations (30) show that this process can be interpreted as a generalized diffusion in the sense of N.I. Portenko [2]. Here, the drift vector and its diffusion matrix equal, respectively,

$$\overline{\alpha}(x')\,\delta_S(x) \quad \text{and} \quad b + \overline{\beta}(x')\,\delta_S(x)\,,$$

where $\delta_S(x)$ is a generalized function on $\mathbb{R}^d$ concentrated on the surface S.

Therefore, we proved such a theorem.

Theorem 2. *Let the conditions of Theorem 1 be true for the coefficients of the operators L and L_0 from (1) and (5). Then the solution of problem (2)-(5) determines a semigroup of operators that describes a generalized diffusive process in $\mathbb{R}^d$ with the characteristic expressed by formulas (30).*

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