Theory of Stochastic Processes
Vol. 14 (30), no. 2, 2008, pp. 28–34

UDC 519.21

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LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS IN THE DUAL OF A MULTI-HILBERTIAN SPACE

We prove the existence and uniqueness of strong solutions for linear stochastic differential equations in the space dual to a multi–Hilbertian space driven by a finite dimensional Brownian motion under relaxed assumptions on the coefficients. As an application, we consider equtions in $\mathcal{S}^{'}$ with coefficients which are differential operators violating the typical growth and monotonicity conditions.

1. Assumptions

We consider a countably Hilbertian space (Φ, τ), whose topology τ is determined by a family of separable Hilbertian seminorms $\|\cdot\|_p$, $p \in R$ (for a detailed exposition, see [4]).

For any $p \in R_+$, we identify $\phi \in \Phi$ with $[\phi]_p \in \Phi/\ker\|\cdot\|_p$ and denote the completion of Φ in $\|\cdot\|_p$ by H_p. Then H_p is a real separable Hilbert space containing Φ as its dense subspace, and the embedding $(\Phi,\tau) \hookrightarrow (H_p, \|\cdot\|_p)$ is continuous. Assume that, for $q \le p$, the canonical embedding $(H_p, \|\cdot\|_p) \hookrightarrow (H_q, \|\cdot\|_q)$ is continuous, i.e., $\|\cdot\|_p$ dominates $\|\cdot\|_q$, denoted by $\|\cdot\|_q \prec \|\cdot\|_p$.

In applications, the strong dual $\Phi^{'}$ of Φ is realized through Hilbert spaces H_{-p} isomorphic to $H_p^{'}$, as $\Phi^{'} = \bigcup_{p\in R_+} H_{-p}$, where

$$\Phi \subset H_p \subset H_0 \subset H_{-p} \subset \Phi^{'},$$

and all the inclusions are continuous. The Hilbert spaces H_p and H_{-p} are dual, in the pairing

$$_{H_p}\langle h^p, h^{-p}\rangle_{H_{-p}}, \quad h^p \in H_p,\ h^{-p} \in H_{-p},$$

being an extension of the duality between Φ and $\Phi^{'}$.

Assume there exists a total set $\{\phi_j\}_{j=1}^{\infty}$ in Φ, which is a common orthogonal system for all Hilbert spaces H_p, $p \in R$, and denote, by $\{h_j^p\} = \|\phi_j\|_p^{-1}\phi_j$, the ONB in H_p derived from ϕ_j. We set $_{\Phi}\langle \phi_n, \phi_n\rangle_{\Phi'} = \|\phi_n\|_0^2 = 1$. For $f \in \Phi$, the scalar product in H_p, $p \in R$, can be calculated as $\langle f, h_n^p\rangle_p = \langle f, \phi_n\rangle_0 \|\phi_n\|_p$.

For linear topological vector spaces A and B, we denote, by $L(A,B)$, the space of continuous linear operators from A to B. For a bounded linear operator $T \in L\left(R^d, H_p\right)$, its Hilbert–Schmidt norm is calculated as $\|T\|_{HS(p)} = \left(\sum_{i=1}^d \|Te_i\|_p^2\right)^{1/2}$, where $\{e_i\}_{i=1}^d$ is the canonical basis in R^d.

We will study a stochastic process with values in Φ and $\Phi^{'}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$ be a filtered probability space satisfying the *usual conditions*: $\mathcal{F}_0$ contains all $A \in \mathcal{F}$, such that $P(A) = 0$, and $\mathcal{F}_t = \bigcap_{s>t}\mathcal{F}_s$. Measurability will be understood with respect to the Borel σ–fields $\mathcal{B}_\Phi$, $\mathcal{B}_{\Phi'}$ (respectively) and this filtered probability space. Since Φ is

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2000 *AMS Mathematics Subject Classification.* Primary 60H15.

Key words and phrases. Infinite dimensional stochastic differential equations, multi-Hilbertian spaces, existence, uniqueness, monotonicity.

a countable multi–Hilbertian space, the Borel σ–fields on $\Phi^{'}$ generated by strongly open sets and by weakly open sets coincide.

For $0 \le t \le T$, consider the functions

$$L : [0,T] \times \Omega \to L(\Phi', \Phi'), \quad A : [0,T] \times \Omega \to L\big(\Phi', L(R^d, \Phi')\big)$$

We introduce the following conditions on L and A. Below, let $q \le p$.

1. (Invariance [INV(Φ)]) Φ is invariant for L and A, i.e. $L(t,\omega) : \Phi \to \Phi$ and $A(t,\omega) : \Phi \to L(R^d, \Phi)$.
2. (Measurability [MR($\Phi^{'}$)]) For any progressively measurable Φ–valued process $\{X_t\}_{t\le T}$ and any $x \in R^d$, $\{L(t,\omega)X_t(\omega)\}_{t\le T}$ and $\{A(t,\omega)X_t(\omega)x\}_{t\le T}$ are $\Phi^{'}$–valued progressively measurable processes.
3. (Measurability [MR(p,q)]) For any progressively measurable H_p–valued process $\{X_t\}_{t\le T}$ and any $x \in R^d$, $\{L(t,\omega)X_t(\omega)\}_{t\le T}$ and $\{A(t,\omega)X_t(\omega)x\}_{t\le T}$ are H_q–valued progressively measurable processes.
4. (Boundedness [B(p,q)]) $L : [0,T] \times \Omega \to L(H_p, H_q)$ and $A : [0,T] \times \Omega \to L(H_p, L(R^d, H_q))$ and L and A are uniformly bounded, i.e.
$$\|L(t,\omega)u\|_q + \|A(t,\omega)u\|_{HS(q)} \le \theta \|u\|_p$$
$\forall u \in H_p$, $0 \le t \le T$ and $\omega \in \Omega$, with θ depending only on p and q.
5. (Monotonicity [M(p)])
$$2\langle u, L(t,\omega)u\rangle_p + \|A(t,\omega)u\|^2_{HS(p)} \le \theta \|u\|^2_p$$
$\forall u \in \Phi$, $0 \le t \le T$ and $\omega \in \Omega$, with θ depending only on p.
6. (Monotonicity [M(p,q)]) $L : [0,T] \times \Omega \to L(H_p, H_q)$ and $A : [0,T] \times \Omega \to L(H_p, L(R^d, H_q))$, and
$$2\langle u, L(t,\omega)u\rangle_q + \|A(t,\omega)u\|^2_{HS(q)} \le \theta \|u\|^2_q$$
$\forall u \in H_p$, $0 \le t \le T$ and $\omega \in \Omega$, with θ depending only on p and q.

Condition [B(p,q)] is very weak, since the growth of $A(t,\omega)$ in H_q is bounded by the norm of the argument in H_p, and $\|\cdot\|_p \succ \|\cdot\|_q$. This weakness in the growth condition is the major difficulty in proving the existence result. Note, for example, that one part of the linear growth condition in Kallianpur et al. [5] is stated within the same space. However, operators as basic as differentiation in $\mathcal{S}^{'}$ fail to satisfy such growth condition.

2. Existence and Uniqueness of the Solution

Let $\{B_t,\ t \ge 0\}$ be a given d-dimensional standard Brownian motion with respect to $\{\mathcal{F}_t\}_{t\ge 0}$. Let H be a Hilbert space. We denote, by $\int_0^t \Psi(s)\, dB_s$, the stochastic integral of an $L(R^d, H)$–valued process $\Psi(t)$, w.r.t. B_t. Note that $\int_0^t \Psi(s)\, dB_s = \sum_{i=1}^d \int_0^t \Psi(s) e_i dB^i_s$, where e_i is the standard ONB in R^d. The integrals on the RHS are the integrals of the H–valued processes $\Psi(t)e_i$ with respect to the real-valued processes B^i_t.

We consider the following stochastic differential equation in $\Phi^{'}$:

$$\begin{cases} dX_t = L(t)X_t dt + A(t)X_t dB_t \\ X_0 = \phi. \end{cases} \tag{2.1}$$

The initial condition ϕ is a $\Phi^{'}$–valued $\mathcal{F}_0$–measurable random variable.

Definition 1. *Let $q \le p \in R$ and $\phi(\omega) \in H_p$ for all $\omega \in \Omega$. Assume that the coefficients of Eq. (2.1) satisfy conditions [MR(p,q)] and [B(p,q)]. An H_p-valued $\mathcal{F}_t$–progressively measurable stochastic process $\{X_t\}_{0\le t\le T}$ defined on a filtered probability* *space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\le T}, P)$ *is a strong solution of Eq. (2.1) in* H_q *if* $E\int_0^T \|X_t\|_p^2\,dt < \infty$ *and the following equation holds in* H_q*:*

$$X_t = \phi + \int_0^t L(s)X_s ds + \int_0^t A(s)X_s dB_s \quad \textit{for almost all } (t,\omega). \tag{2.2}$$

Conditions [MR(p,q)], [B(p,q)], and progressive measurability assumed in Definition 1 guarantee that the integrals in Eq. (2.2) are well-defined $\mathcal{F}_t$-adapted continuous H_q-valued processes. Thus, the strong solution has a continuous version in H_q (and, hence, a progressively measurable version in H_q).

We use techniques similar to those found in [6], [7], and [9]. The next lemma discusses properties of a solution to an SDE, whose coefficients satisfy the monotonicity condition.

Lemma 1. *(Part 1) Assume that the coefficients L and A of Eq. (2.1) satisfy conditions [INV(Φ)], [MR(Φ$'$)], [M(r)]. Let* $\phi(\omega) \in \Phi$ *for all* ω *and* $E\|\phi\|_r^2 < \infty$*. If* $\{X_t\}$ *is a* Φ*–valued process satisfying Eq. (2.2) in* H_r*, for each* $t \ge 0$*, a.s., in the usual sense of an SDE in a Hilbert space (in particular* X_t *is continuous in* H_r*,* $P(\int_0^T \|L(s)X_s\|_r\,ds < \infty) = 1$*, and* $P(\int_0^T \|A(s)X_s\|_{HS(r)}^2\,ds < \infty) = 1$*), then*

$$\sup_{t\le T} E\|X_t\|_r^2 \le CE\|\phi\|_r^2. \tag{2.3}$$

(Part 2) Let $r \ge p \ge q$*. Assume that the coefficients L and A of Eq. (2.1) satisfy conditions [MR(r,p)], [M(r,p)], [M(p,q)], [B(p,q)], and that* $E\|\phi\|_p^2 < \infty$*. Let* $\{X_t\}_{0\le t\le T}$ *be an* H_r*–valued process satisfying Eq. (2.1) in* H_p*. Let* $\{Y_t\}_{0\le t\le T}$ *be the continuous version of* $\{X_t\}_{0\le t\le T}$ *in* H_p *defined by the RHS of (2.2). Then*

$$E\sup_{t\le T}\|Y_t\|_q^2 \le CE\|\phi\|_p^2. \tag{2.4}$$

Proof. (Part 1) Using Itô's formula for $\|\cdot\|_r^2$ and condition [M(r)], we obtain

$$\|X_t\|_r^2 \le \|\phi\|_r^2 + \int_0^t \theta\|X_s\|_r^2\,ds + 2\int_0^t \sum_{j=1}^d \big\langle X_s, A(s)X_s(e_j)\big\rangle_r\,dB_s^j. \tag{2.5}$$

Let $\{\tau_n\}_{n=1}^\infty$ be stopping times localizing the local martingale represented by the stochastic integral above, then

$$E\|X_{t\wedge\tau_n}\|_r^2 \le E\|\phi\|_r^2 + \int_0^t E\theta\|X_{s\wedge\tau_n}\|_r^2\,ds.$$

Using Gronwall's lemma and the fact that $\tau_n \to \infty$, we obtain (2.3).
(Part 2) By repeating the proof of (2.3) with the condition [M(r,p)] replacing [M(r)], we arrive at

$$\sup_{t\le T} E\|Y_t\|_p^2 \le CE\|\phi\|_p^2$$

for the H_p-continuous version Y_t of the H_r–valued solution X_t. Since $H_p \hookrightarrow H_q$, and $\|\cdot\|_q \prec \|\cdot\|_p$, Y_t is an H_p–valued process satisfying Eq. (2.2) in H_q. Thus, in (2.5), we can replace the r–norm with the q–norm, by using condition [M(p,q)]. Consider the stochastic integral in (2.5). It follows from Burkholder's inequality, assumption [B(p,q)], and the bound for $E\|Y_t\|_p^2$ that

$$\begin{aligned} &E\sup_{t\le T}\Big|\int_0^{t\wedge\tau_n} \sum_{j=1}^d \big\langle Y_s, A(s)Y_s(e_j)\big\rangle_q\,dB_s^j\Big| \\ &\qquad \le CE\Big(\int_0^T\Big(\sum_{j=1}^d \|Y_{s\wedge\tau_n}\|_q\|A(s\wedge\tau_n)Y_{s\wedge\tau_n}(e_j)\|_q\Big)^2 ds\Big)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\le CE\bigg(\big(\sup_{t\le T} \|Y_{t\wedge\tau_n}\|_q^2\big)^{\frac12} \Big(\int_0^T \|Y_s\|_p^2\, ds\Big)^{\frac12}\bigg) \\
&\le \frac{C}{2}\Big(\varepsilon E \sup_{t\le T} \|Y_{t\wedge\tau_n}\|_q^2 + \frac{1}{\varepsilon} E\int_0^T \|Y_s\|_p^2\, ds\Big) \\
&\le \frac{C}{2}\Big(\varepsilon E \sup_{t\le T} \|Y_{t\wedge\tau_n}\|_q^2 + \frac{1}{\varepsilon} E\|\phi\|_p^2\Big)
\end{aligned}
$$

for any $\varepsilon > 0$. Because $\|\cdot\|_q \prec \|\cdot\|_p$, we have

$$
\begin{aligned}
E\sup_{t\le T}\|Y_{t\wedge\tau_n}\|_q^2 &\le E\|\phi\|_q^2 + E\int_0^T \theta\|Y_{t\wedge\tau_n}\|_q^2\, ds + \frac{C}{2}\Big(\varepsilon E\sup_{t\le T}\|Y_{t\wedge\tau_n}\|_q^2 + \frac{1}{\varepsilon}E\|\phi\|_p^2\Big) \\
&\le CE\|\phi\|_p^2 + \frac12 E\sup_{t\le T}\|Y_{t\wedge\tau_n}\|_q^2,
\end{aligned}
$$

since $\varepsilon > 0$ is arbitrary. The constant C depends only on q, p, and T and can change its value from line to line. Thus

$$E\sup_{t\le T}\|Y_{t\wedge\tau_n}\|_q^2 \le CE\|\phi\|_p^2,$$

and (2.4) follows by Fatou's lemma.

We will use the same symbol X_t to denote the H_r–valued solution satisfying (2.1) in H_p and its H_p-continuous version. We now state our main result.

Theorem 1. *Let the coefficients A and L of Eq. (2.1) satisfy conditions [INV(Φ)], [MR(Φ')], [MR(r,p)], [B(r,p)], and [M(r)], for some $r \ge p$. Assume that $E\|\phi\|_r^2 < \infty$. Then equation (2.1) has an H_r–valued strong solution X_t in H_p. If in the above assumptions [M(p)] holds instead of [M(r)], then the solution is unique.*

If, in addition, there exists $q \le p$, such that A and L satisfy conditions [M(p,q)] and [B(p,q)], then X_t viewed as a continuous H_p–valued strong solution of Eq. (2.1) satisfying Eq. (2.2) in H_q, is continuous with respect to the initial condition, i.e. for the initial conditions $\phi_n \to \phi$ in $L^2(\Omega, H_p)$, the corresponding solutions $X_n(t)$ and X_t satisfy

$$X_n \to X \ \text{ in } L^2(\Omega, C([0,T], H_q)).$$

Proof. Uniqueness follows from the argument provided in Krylov and Rozovskii [6].

Let $p \le r$ and $X_t^1,\ X_t^2 \in C([0,T], H_p)$ be (continuous versions of) two H_r–valued strong solutions of Eq. (2.2) in H_p. We denote $Y_t = X_t^1 - X_t^2$ and apply Itô's formula to $\|Y_t\|_p^2$, to obtain

$$\|Y_t\|_p^2 = \int_0^t \Big\{2\langle L(s)Y_s, Y_s\rangle_p + \|A(s)Y_s\|_{HS(p)}^2\Big\}\, ds + M_t,$$

where M_t is a local L^2–martingale. We apply Itô's formula again and obtain

$$
\begin{aligned}
e^{-\mu t}\|Y_t\|_p^2 = -\mu\int_0^t \|Y_s\|_p^2 e^{-\mu s}\, ds &+ \int_0^t \Big\{2\langle L(s)Y_s, Y_s\rangle_p + \|A(s)Y_s\|_{HS(p)}^2\Big\} e^{-\mu s}\, ds \\
&+ \int_0^t e^{-\mu s}\, dM_s.
\end{aligned}
$$

Since conditions [M(p)] and [B(r,p)] imply [M(r,p)], taking $\mu > \theta$ in the latter condition gives

$$e^{-\mu t}\|Y_t\|_p^2 \le \int_0^t e^{-\mu s}\, dM_s.$$

Using Doob's inequality for the non–negative continuous local martingale

$$N_t = \int_0^t e^{-\mu s}\, dM_s,$$

we have $\sup_{0\le t\le T}\{N_t\}=0$, P–a.s., and the pathwise uniqueness follows.

To prove the existence, we let P_n to be an orthogonal projection of H_p on an n–dimensional subspace of Φ, spanned by $\{h_1^p,\dots,h_n^p\}$, $P_n u=\sum_{k=1}^n\langle u,h_k^p\rangle_p h_k^p$. For $r\ge p$, P_n is a bounded operator from H_p to H_r. In addition, P_n is an n–dimensional orthogonal projection on H_r, since, for $u\in H_r$, we have

$$P_n(u)=\sum_{k=1}^n\langle u,h_k^p\rangle_p h_k^p=\sum_{k=1}^n\langle u,h_k^r\rangle_r\langle h_k^r,h_k^p\rangle_p h_k^p=\sum_{k=1}^n\langle u,h_k^r\rangle_r h_k^r.$$

Using condition [INV(Φ)], consider the coefficients $P_nL:[0,T]\times\Omega\to L(P_nH_r,P_nH_r)$ and $P_nA:[0,T]\times\Omega\to L(P_nH_r,L(R^d,P_nH_r))$, and a finite dimensional SDE

$$X_n(t)=P_n\phi+\int_0^t P_nL(s)X_n(s)\,ds+\int_0^t P_nA(s)X_n(s)\,dB_s. \tag{2.6}$$

By [B(r,p)] and linearity, it is easy to see that the coefficients of this equation are Lipschitz–continuous, so that, by the finite dimensional result (e.g., Theorem 3, Chapter II, vol. 3, in Gikhman and Skorokhod [3]), there exists a strong solution $X_n(t)$ in P_nH_r. We verify that the coefficients P_nL and P_nA satisfy condition [M(r)] for $u\in P_nH_r\subset\Phi$,

$$2\langle P_nL(s)u,u\rangle_r+\|P_nA(s)u\|^2_{HS(r)}\le 2\langle L(s)u,u\rangle_r+\|P_n\|^2\|A(s)u\|^2_{HS(r)}\le\theta\|u\|_r^2,$$

due to the assumptions [INV(Φ)] and [M(r)], on L and A. Thus, by (2.3),

$$\sup_n\sup_{t\le T}E\|X_n(t)\|_r^2\le CE\|\phi\|_r^2.$$

Hence, the sequence X_n is bounded in $L^2\big(\Omega\times[0,T],H_r\big)$, and we can select a subsequence, denoted again by X_n, which converges weakly to an element X in $L^2\big(\Omega\times[0,T],H_r\big)$. We can choose the limit X such that it has a progressively measurable modification $\{X_t\}_{0\le t\le T}$, since the limit in $L^2\big(\Omega\times[0,T]\big)$ of the sequence $\{\langle h_i^r,X_n(t)\rangle_r\}_{n=1}^\infty$ viz. $\langle h_i^r,X_t\rangle_r$ is progressively measurable for each i.

We now prove that the process $\{X_t\}_{0\le t\le T}$ satisfies SDE (2.2) in H_p by showing that, in (2.6), we can replace X_n with X on the RHS and with P_nX on the LHS.

Let $\eta(s,\omega)=\ \eta_1(s)\eta_2(\omega)h_i^p$, where η_1 and η_2 are real–valued bounded and measurable. Note that, for $u\in H_p$, $\langle h_i^p,u\rangle_p=\langle h_i^p,h_i^r\rangle_p\langle h_i^r,u\rangle_r$. So, using the weak convergence of X_n to X in $L^2\big(\Omega\times[0,T],H_r\big)$, we obtain

$$E\int_0^T\big\langle\eta(s),X_n(s)\big\rangle_p\,ds\to E\int_0^T\big\langle\eta(s),X_s\big\rangle_p\,ds.$$

Note that, by condition [B(r,p)] and the boundedness of X_n in $L^2\big(\Omega\times[0,T],H_r\big)$, we have

$$E\left|\eta_2\int_0^s\big\langle h_i^p,L(u)X_n(u)\big\rangle_p\,du\right|\ \le\ C\ \text{and}\ \ E\left|\eta_2\int_0^s\langle h_i^p,(A(u)X_n(u))\,e_j\rangle_p\,du\right|\ \le\ C,$$

where the constant C is independent of n and s.

By the weak convergence of X_n to X in $L^2\big(\Omega\times[0,T],H_r\big)$, it follows that

$$\begin{aligned}E\eta_2\int_0^s\big\langle h_i^p,L(u)X_n(u)\big\rangle_p\,du&=E\eta_2\int_0^s\big\langle L^*(u)h_i^p,X_n(u)\big\rangle_r\,du\\ \to E\eta_2\int_0^s\big\langle L^*(u)h_i^p,X_u\big\rangle_r\,du&=E\eta_2\int_0^s\big\langle h_i^p,L(u)X_u\big\rangle_p\,du.\end{aligned}$$

Now, by the Lebesgue DCT,

$$\lim_{n\to\infty} E\int_0^T \eta_1(s)\eta_2(\omega)\int_0^s \left\langle h_i^p, P_nL(u)X_n(u)\right\rangle_p du\, ds$$
$$= E\int_0^T \eta_1(s)\eta_2(\omega)\int_0^s \left\langle h_i^p, L(u)X_u\right\rangle_p du\, ds.$$

Let $A^j(u): H_r \to H_p$ be defined by

$$A^j(u)h_k^r = \left(A(u)h_k^r\right)(e_j).$$

Repeating the above arguments with the operator A^j replacing L proves that, for all i, j,

$$\lim_{n\to\infty} E\eta_2\int_0^T \eta_1(u)\langle h_i^p, (A(u)X_n(u))e_j\rangle_p\, du = E\eta_2\int_0^T \eta_1(u)\langle h_i^p, (A(u)X_u)e_j\rangle_p\, du.$$

Thus, $\langle h_i^p, (A(u)X_n(u))e_j\rangle_p \to \langle h_i^p, (A(u)X_u)e_j\rangle_p$ weakly in $L^2(\Omega\times[0,T])$. By Doob's inequality, with a one–dimensional Brownian motion β_t and a stochastically integrable predictable process $\xi(t)$, we have

$$E\int_0^T \left|\int_0^s \xi(u)\, d\beta_u\right|^2 ds \le TE\left(\sup_{0\le s\le T}\left|\int_0^s \xi(u)\, d\beta_u\right|^2\right) \le TE\int_0^T |\xi(s)|^2\, ds,$$

which implies that the stochastic integral is a continuous linear operator from $L^2(\Omega\times[0,T],\mathcal{P})$ to $L^2(\Omega\times[0,T],\mathcal{F}_T\otimes\mathcal{B}[0,T])$ (here, $\mathcal{P}$ is the predictable σ–field, and $\mathcal{B}$ is the Borel σ–field). By Theorem 15, [DS], Ch. V, §4, it is also continuous in the weak topologies, so that

$$\lim_{n\to\infty} E\int_0^T \eta_1(s)\eta_2(\omega)\sum_{j=1}^d\int_0^s \left\langle h_i^p, (P_nA(u)X_n(u))e_j\right\rangle_p dB_u^j\, ds$$
$$= E\int_0^T \eta_1(s)\eta_2(\omega)\sum_{j=1}^d\int_0^s \left\langle h_i^p, (A(u)X_u)e_j\right\rangle_p dB_u^j\, ds.$$

To complete the proof, we multiply Eq. (2.6) by $\eta(s)$ and integrate w.r.t. $dP\times dt$. Then, by letting $n\to\infty$, we get, for a.e. (ω, t), $dP\times dt$,

$$\langle h_i^p, X_t\rangle_p = \langle h_i^p, \phi\rangle_p + \int_0^t \left\langle h_i^p, L(u)X_s\right\rangle_p ds + \sum_{j=1}^d\int_0^t \left\langle h_i^p, (A(u)X_s)e_j\right\rangle_p dB_s^j.$$

The process X_t has values in H_r, with $X\in L^2\big(\Omega\times[0,T],H_r\big)\subset L^2\big(\Omega\times[0,T],H_p\big)$, and satisfies Eq. (2.2) in H_p a.e. $dP\times dt$. Thus, X_t is a strong H_r–valued solution of Eq. (2.1) in H_p.

The continuity of $\{X_t\}_{t\le T}$ with respect to the initial condition follows from (2.4).

Example. The space $\mathcal{S}$ of smooth rapidly decreasing functions on R^d with the topology given by L. Schwartz is nuclear. Let S_p be the completion of $\mathcal{S}$ with respect to the Hilbertian norms $\|f\|_p^2 = \sum_{|k|=0}^\infty (2|k|+d)^{2p}\langle f, h_k\rangle_{L^2(R^d)}$, $f, g\in\mathcal{S}$, where $\{h_k\}_{k=1}^\infty$ is an ONB in $L^2\left(R^d, dx\right)$ given by Hermite functions. Then $\mathcal{S}' = \bigcup_{p>0} S_{-p}$. Let $\{\sigma_{ij}(t)\}_{t\ge 0}$ and $\{b_i(t)\}_{t\ge0}$ be bounded progressively measurable processes. Define, for $\varphi\in\mathcal{S}'$,

$$L(t,\omega)\varphi := \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^T)_{ij}(t,\omega)\ \partial_{ij}^2\varphi - \sum_{i=1}^d b_i(t,\omega)\ \partial_i\varphi$$

$$A_i(t,\omega)\varphi := \sum_{j=1}^d \sigma_{ji}(t,\omega)\ \partial_j\varphi,$$

and let $A(t,\omega)\varphi \equiv (A_1\varphi(t,\omega),\dots A_d\varphi(t,\omega))$. Then A and L satisfy the conditions for existence and uniqueness of the solution in Theorem 1 (for details, see Gawarecki et al. [2]). Specifically, condition [M(r)] holds true for any $r \in R$, and condition [M(p,q)] is satisfied for $q \le p-1$. It is easy to verify using the recurrence properties of Hermite polynomials that condition [B(r,p)] is valid for any $p \le r-1$. Hence, setting $r \ge p+1$, and $q \le p-1$, for any $p \in R$, and $\phi \in L^2(\Omega, S_r)$, Eq. (2.1) has a unique continuous S_r–valued strong solution in S_p which is continuous in $L^2(\Omega, C([0,T], S_q))$ with respect to $\phi_n \to \phi$ in $L^2(\Omega, S_p)$.

Consider a special case where $A\varphi = (-\partial_1\varphi,\dots,-\partial_d\varphi)$ and $L\varphi = \frac{1}{2}\sum_{i=1}^d \partial_i^2\varphi$. The unique solution of Eq. (2.1) with the initial condition δ_x is δ_{B_t}, where $P(B_0 = x) = 1$. This follows from the Itô formula in [8],

$$\rho_{B_t}\phi = \rho_{B_0}\phi - \sum_{i=1}^d \int_0^t \partial_i\left(\rho_{B_s}\phi\right) dB_s^i + \frac{1}{2}\sum_{i=1}^d \int_0^t \partial_i^2\left(\rho_{B_s}\phi\right) ds.$$

Here, for $x \in R^d$, ρ_x denotes the translation operator on R^d. If $\phi \in \mathcal{S}'$, then $\langle f, \rho_x\phi\rangle := \langle \rho_{-x}f, \phi\rangle = \langle f(\cdot + x), \phi\rangle$ for $f \in \mathcal{S}$. For each t, $\rho_{B_t}\phi$ denotes the $\mathcal{S}'$–valued random variable $\omega \to \rho_{B_t(\omega)}\phi$. Then $\{\rho_{B_t}\phi\}_{t\ge 0}$ is an $\mathcal{S}_{-p}$–valued stochastic process for some $p > 0$, as shown in [8]. Taking $\phi = \delta_0$ gives $\rho_{B_t}\phi = \delta_{B_t}$.

However, it is easy to verify that the coefficients A and L do not satisfy the coercivity inequality in [6], and they violate the linear growth condition in [5].

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