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CERTAIN PROPERTIES OF TRIANGULAR TRANSFORMATIONS OF MEASURES

We study the convergence of triangular mappings on $\mathbb{R}^n$, i.e., mappings T such that the ith coordinate function T_i depends only on the variables $x_1, \dots, x_i$. We show that, under broad assumptions, the inverse mapping to a canonical triangular transformation is canonical triangular as well. An example is constructed showing that the convergence in variation of measures is not sufficient for the convergence almost everywhere of the associated canonical triangular transformations. Finally, we show that the weak convergence of absolutely continuous convex measures to an absolutely continuous measure yields the convergence in variation. As a corollary, this implies the convergence in measure of the associated canonical triangular transformations.

Introduction

We study triangular transformations of measures which have been investigated in works [1], [2], [3], [4], [5], [6], and [8]. A mapping $T = (T_1, \dots, T_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ is called triangular if T_1 is a function of x_1, T_2 is a function of (x_1, x_2), T_3 is a function of (x_1, x_2, x_3), and so on: T_i is a function of $(x_1, x_2, \dots, x_i)$. Similarly one defines triangular mappings on $\mathbb{R}^\infty$, the countable product of real lines. A triangular mapping T is called increasing if every component T_i is increasing with respect to the variable x_i. The same terminology is used for the mappings defined on subsets of $\mathbb{R}^n$ or $\mathbb{R}^\infty$.

Such mappings were employed in paper [8], where triangular transformations of uniform distributions on convex sets were constructed (see also [2]). The existence of a triangular transformation of the standard Gaussian measure γ on $\mathbb{R}^n$ into an arbitrary absolutely continuous probability measure $\nu = f \cdot \gamma$ was established in [12]. In [6], a positive solution was given to the problem on the representability of an arbitrary probability measure ν that is absolutely continuous with respect to a Gaussian measure γ on an infinite dimensional space as the image of γ under a mapping of the form $T(x) = x + F(x)$, where F takes values in the Cameron–Martin space of the measure γ. For further related results, see [3], [4], [6], and [7].

Recall that, for every pair of probability measures μ and ν on $\mathbb{R}^n$, where μ is absolutely continuous, there exists a Borel increasing triangular mapping $T_{\mu,\nu}$ that transforms μ into ν. This mapping is defined on some Borel set of full μ-measure (which may be smaller than $\mathbb{R}^n$), and every kth component of $T_{\mu,\nu}$, as a function of the variables $x_1, \dots, x_k$, is defined on a Borel set in $\mathbb{R}^k$, whose intersections with the straight lines parallel to the kth coordinate line are intervals (possibly, unbounded). Such a mapping is unique up to a redefinition on a set of μ-measure zero provided that ν possesses nonatomic conditional measures on the coordinate lines and has a nonatomic projection on the first coordinate line (e.g., is absolutely continuous). In [5], a *canonical* triangular mapping

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was introduced, i.e., a version of $T_{\mu,\nu}$ that is defined as follows by induction on n. For $n=1$, we set

$$F_\mu(t) := \mu\big((-\infty,t)\big), \quad t\in\mathbb{R}^1, \quad G_\mu(u) := \inf\big\{s\colon F_\mu(s)\ge u\big\}, \quad u\in(0,1),$$

$$T_{\mu,\nu} := G_\nu\circ F_\mu.$$

If the function G_ν has a finite limit as $u\to 0$ or $u\to 1$, then we define $G_\nu(0)$ or $G_\nu(1)$ as the corresponding limit. If the function F_μ assumes some of the values 0 and 1 (the sets $F_\mu^{-1}(0)$ and $F_\mu^{-1}(1)$ are either empty or rays) and the function G_ν has no finite limit at the corresponding point, then the mapping $T_{\mu,\nu}$ is defined on some interval (bounded or unbounded) of full μ-measure. The mapping F_μ takes μ to a Lebesgue measure λ on $(0,1)$, and G_ν takes λ to ν. The construction continues inductively by using the one-dimensional conditional densities on the last coordinate line. Suppose that, for some $n\ge 1$, the existence of canonical triangular mappings is already established. The projections of the measures μ and ν on $\mathbb{R}^n$ are denoted by μ_n and ν_n. The corresponding conditional measures on the last coordinate line are denoted by μ_x and ν_x, $x\in\mathbb{R}^n$, and the density of μ_x is denoted by ϱ_μ^x (details can be found in [3]). By the inductive assumption, there exists a canonical Borel triangular mapping $T=(T_1,\dots,T_n)\colon\mathbb{R}^n\to\mathbb{R}^n$ taking μ_n to ν_n (the domain of definition of T may be a proper Borel subset of $\mathbb{R}^n$ of full μ_n-measure). For $T_{\mu,\nu}$, we take the mapping $T_{\mu,\nu}=(T_1,\dots,T_{n+1})$, where the last component is defined as follows: for any fixed $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, the function $t\mapsto T_{n+1}(x_1,\dots,x_n,t)$ is the canonical transformation of the measure with density ϱ_μ^x to the measure $\nu_{T(x)}$.

It was shown in [6] that the canonical triangular transformation depends continuously on the transformed measure with respect to the variation norm. More precisely, the following theorem was proved.

Theorem 1. *Suppose a sequence of absolutely continuous probability measures ν_j on $\mathbb{R}^n$ converges in variation to a measure ν, and let μ be a probability measure on $\mathbb{R}^n$ equivalent to a Lebesgue measure. Then the sequence of canonical triangular mappings T_{μ,ν_j} converges in measure μ to the mapping $T_{\mu,\nu}$.*

This theorem was generalized in paper [1] for the case of $\mathbb{R}^\infty$. Suppose that two sequences of Borel probability measures μ_j and ν_j on $\mathbb{R}^\infty$ converge in variation to μ and ν, respectively, and that the projections of the measures μ_j and μ on the spaces $\mathbb{R}^n$ and the corresponding conditional measures have no atoms. Then the sequence of canonical triangular mappings T_{μ_j,ν_j} (extended to Borel mappings on all of $\mathbb{R}^\infty$) converges in measure μ to the mapping $T_{\mu,\nu}$. Hence, there is a subsequence of T_{μ_j,ν_j} convergent to $T_{\mu,\nu}$ almost everywhere with respect to μ. Moreover, this statement holds in the case of the countable product of Souslin spaces (see [1]).

However, in the general case, we cannot replace the convergence in measure in the aforementioned assertions by the convergence almost everywhere. A counter-example is constructed in §2. In addition, the case of convex measures is considered. We show that the weak convergence of absolutely continuous convex measures to an absolutely continuous convex measure is sufficient for the convergence in measure of the associated canonical triangular transformations.

1. Convergence of triangular transformations

We note that if the measures μ and ν possess atomless projections on the first coordinate line and atomless conditional measures on the other coordinate lines, then the canonical triangular mapping $T_{\mu,\nu}$ is injective on a Borel set of full μ-measure. It is easy to justify this claim by induction. Now we consider the case where canonical triangular mappings are defined on all of $\mathbb{R}^n$.

Proposition 1. *Let μ and ν be Borel probability measures such that μ is equivalent to the Lebesgue measure, and let the projection of ν on the first coordinate line and the conditional measures on the other coordinate lines have no atoms. Then the inverse mapping to the canonical triangular transformation $T_{\mu,\nu}$ is canonical triangular as well.*

Proof. Note that, in this case, $T_{\mu,\nu}$ is defined on all of $\mathbb{R}^n$ due to the construction of a canonical triangular mapping. The mapping $T_{\mu,\nu}$ is injective since its first component is injective on the real line, the second component $T_2(x_1, x_2)$ is injective as a function of x_2 with fixed x_1, and so on. So the inverse mapping $F = (F_1, \dots, F_n)$ has a triangular form, and every its component $F_n(y_1, y_2, \dots, y_n)$ is increasing with respect to the variable y_n. Since there holds the uniqueness property in the class of ν-equivalent increasing triangular mappings (see [6]), the transformation F is canonical.

Remark 1. Let μ and ν be Borel probability measures such that their projections on the first coordinate line and the conditional measures on the other coordinate lines have no atoms, and let one of the following additional conditions be fulfilled:

(i) the projection of μ on the first coordinate line and the conditional measures on the other coordinate lines are not concentrated on rays,

(ii) the measure ν is concentrated on a bounded set.

Then $T_{\mu,\nu}$ is defined on all of $\mathbb{R}^n$, and the statement of Proposition 1 is true if we consider $T_{\mu,\nu}$ on a Borel set of full μ-measure, where it is injective.

Proposition 2. *There exists a sequence of Borel probability measures ν_j on $[0,1]\times[0,1]$ with strictly positive densities ρ_j that converges in variation to the Lebesgue measure λ, but the sequence of the canonical triangular mappings $T_{\nu_j,\lambda}$ does not converge a.e. to $T_{\lambda,\lambda}$.*

Proof. Let us set
$$\rho_j(x, y) = 1 + \theta_j(x)\psi(y),$$
where the sequence of functions θ_j converges in measure to 0, but does not converge at any point, $0 \leqslant \theta_j \leqslant 1$, and ψ is defined by the formula
$$\psi(y) = \begin{cases} 1, & \text{y} \in [0,1/2] \\ -1, & \text{y} \in (1/2,1]. \end{cases}$$
The trivial estimate $|\rho_j(x, y) - 1| \leqslant \theta_j(x)$ yields the convergence in variation of ν_j to λ. Since the projections of ν_j on [0,1] coincide with the Lebesgue measure on [0,1], we obtain the following formula for the canonical triangular mapping $T_{\nu_j,\lambda}$:
$$T_{\nu_j,\lambda} = (x, T_j(x, y)),$$
where $T_j(x, y)$ is the distribution function of the conditional measure ν_j^x, i.e.,
$$T_j(x, y) = \int_0^y 1 + \theta_j(x)\psi(t)dt = y + \theta_j(x)\int_0^y \psi(t)dt.$$
We observe that, in the case $y \neq 0$, there is no convergence at any point.

A similar example can be constructed in the case of the mappings T_{λ,ν_j}.

Recall that an absolutely continuous probability measure ν on $\mathbb{R}^n$ is called convex if it has a density $\rho_\nu = \exp(-V)$, where V is a convex function (which may be infinite outside its proper domain).

Theorem 2. *Suppose a sequence of absolutely continuous convex probability measures ν_j on $\mathbb{R}^n$ converges weakly to an absolutely continuous convex measure ν. Then ν_j converges in variation to ν.*

Proof. Let $\rho_{\nu_j} = \exp(-V_j)$ and $\rho_\nu = \exp(-V)$ denote the densities of ν_j and ν, respectively. Recall the following fact (see [11]). Let f_j be a sequence of convex functions on an open convex set C, and let C' be a dense subset of C. Suppose that the sequence $\{f_j(x)\}$ is bounded at every point $x \in C'$ (no uniformity is assumed). Then, passing to a subsequence, we may assume that $\{f_j\}$ converges uniformly to a convex function f on every compact subset of C. So it is sufficient to prove that, in every open ball, we can find a point x_0 such that the sequence $\{V_j(x_0)\}$ is bounded. In fact, in that case, we obtain that every subsequence in ρ_{ν_j} contains a further subsequence convergent uniformly on every compact set. Hence, we may conclude that $\{\nu_j\}$ converges in variation to ν.

Let us consider an arbitrary closed ball B in the set $\{V < \infty\}$. Since each ρ_{ν_j} is a probability density, Fatou's theorem yields

$$\int_B \liminf_{j\longrightarrow\infty} e^{-V_j(x)}dx \leqslant 1.$$

Now let us consider the compact convex sets

$$W_j := \{x \in B:\ \rho_j(x) \geq \alpha\} = \left\{x \in B:\ V_j(x) \leq \ln\frac{1}{\alpha}\right\}, \quad \alpha := \frac{\nu(B)}{2\lambda(B)}.$$

Since $\{\nu_j\}$ converges weakly to ν, there is j_0 such that, for all $j \geq j_0$, the sets W_j are not empty (otherwise, we would get $\nu(B) \leq \alpha\lambda(B) = \nu(B)/2$). It is known that there is a subsequence j_k such that the sets W_{j_k} converge to some compact convex W with respect to the Hausdorff distance (see [10]), i.e., given $\varepsilon > 0$, there is k_0 such that, for all $k \geq k_0$, the sets W_{j_k} and W belong to the ε-neighborhoods of each other. If W is of positive measure, then it contains a ball B', on which one has $V_{j_k}(x) \leq \ln\alpha$, which along with the above estimate based on Fatou's theorem proves our claim. Let W be of measure zero. We can do the same for every $2^{-m}\alpha$ in place of α. If, for some natural m, the corresponding limit convex set is of positive measure, then our previous reasoning applies. Otherwise, we have $\rho_j \to 0$ in measure on B and a uniform bound $\rho_j \leq \alpha$ on a smaller ball B' in B, which yields the convergence in variation to zero on B', a contradiction.

The next assertion is seen directly from Theorems 1 and 2.

Theorem 3. *Suppose a sequence of absolutely continuous probability convex measures ν_j on $\mathbb{R}^n$ converges weakly to an absolutely continuous convex measure ν, and let μ be a probability measure on $\mathbb{R}^n$ equivalent to the Lebesgue measure. Then, the sequence of the canonical triangular mappings T_{μ,ν_j} converges in measure μ to the mapping $T_{\mu,\nu}$.*

Apparently, unlike Proposition 2, in the situation of Theorem 3, one has even the convergence almost everywhere of the whole sequence T_{μ,ν_j}.

Finally, it should be noted that, for general absolutely continuous measures, the weak convergence or even the setwise convergence is not enough for the convergence of the associated canonical triangular transformations in measure.

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