Theory of Stochastic Processes
Vol. 14 (30), no. 1, 2008, pp. 60–68

UDC 519.21

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ASYMPTOTIC PROPERTIES OF L_P-ESTIMATORS

Some sufficient conditions for consistency and asymptotic normality of a non-linear regression parameter L_p-estimator are presented for a continuous time regression model with Gaussian stationary noise possessing the long-range dependence or weak dependence property.

Introduction

Consider a regression model

$$X(t) = g(t,\theta) + \varepsilon(t),\ t \geq 0,$$

where $g:\ [0,\infty) \times \Theta^c \to \mathbb{R}^1$ is a continuous function, Θ^c is a closure in $\mathbb{R}^m$ of an open bounded convex set Θ, $\theta \in \Theta$. It is supposed that

A$_1$**.** $\varepsilon(t),\ t \in \mathbb{R}^1$ is a real measurable mean-square continuous stationary Gaussian process defined on the complete probability space $(\Omega, \mathcal{F}, P)$, $E\varepsilon(0) = 0$.

Definition. *Any random variable (r.v.)* $\widehat{\theta}_T$ *having a property*

$$Q_{pT}(\hat{\theta}_T) = \inf_{\tau \in \Theta^c} Q_{pT}(\tau),\ Q_{pT}(\tau) = \int_0^T |X(t) - g(t,\tau)|^p dt,\ 1 \leq p < \infty$$

is said to be an L_p*-estimator of the unknown* $\theta \in \Theta$.

It follows from [1–3] that our assumptions provide the existence of the L_p-estimator.

L_p-estimators belong to a wide class of M-estimators [4] and use the loss function $\rho(x) = |x|^p$. Least squares estimators ($p = 2$) and least moduli estimators ($p = 1$) are the most studied L_p-estimators [5,6]. The discription of the asymptotic properties of L_p-estimators for $p \in (1,2)$ is a challenging theoretical problem. For linear and nonlinear regression models with discrete time and independent identically distributed observation errors, the consistency and asymptotic normality of l_p-estimators were considered in [4, 6–10].

1. Consistency of L_p-estimators

Suppose $g(t,\cdot) \in C^1(\Theta^c);\ g_i(t,\theta) = \dfrac{\partial}{\partial\theta_i} g(t,\theta)$;

$$d_{iT}^2(\theta) = \int_0^T g_i^2(t,\theta)dt,\ i = 1,\ldots,m;\ d_T^2(\theta) = \mathrm{diag}\left(d_{iT}^2(\theta)\right)_{i=1}^m;$$

$$\varliminf_{T\to\infty} T^{-1} d_{iT}^2(\theta) > 0,\ i = 1,\ldots,m.$$

2000 *AMS Mathematics Subject Classification.* Primary 62J02; Secondary 62J99.
Key words and phrases. L_p-estimator, regression model.

Let $U_T(\theta)=T^{-\frac{1}{2}}d_T(\theta)(\Theta-\theta)$; $\widehat{u}_T=T^{-\frac{1}{2}}d_T(\theta)(\widehat{\theta}_T-\theta)$; $f(t,u)=g(t,\theta+T^{\frac{1}{2}}d_T^{-1}(\theta)u)$;

$$f_i(t,u)=g_i(t,\theta+T^{\frac{1}{2}}d_T^{-1}(\theta)u),\ \Phi_{pT}(u_1,u_2)=\int_0^T|f(t,u_1)-f(t,u_2)|^p dt,$$

$$\widetilde{Q}_{pT}(u)=Q_{pT}(\theta+T^{\frac{1}{2}}d_T^{-1}(\theta)u),\ u\in U_T^c(\theta);$$
$$v(r)=\{u\in\mathbb{R}^m:\ \|u\|<r\},\ \mu_p=E|\varepsilon(0)|^p.$$

$\mathbf{B}_1$. For any $R>0$, there exist $k^i(R)<+\infty,\ i=1,\dots,m$ such that

$$\sup_{u\in U_T^c(\theta)\cap v^c(R)}\ \sup_{t\in[0,T]}|g_i(t,\theta+T^{\frac{1}{2}}d_T^{-1}(\theta)u)|d_{iT}^{-1}(\theta)\le k^i(R)T^{-1/2}.$$

$\mathbf{C}_1$ (contrast condition). For any $r>0$, there exists $\Delta(r)>0$ such that

$$\inf_{u\in U_T^c(\theta)\setminus v(r)}T^{-\frac{1}{p}}E\widetilde{Q}_{pT}^{\frac{1}{p}}(u)\ge T^{-\frac{1}{p}}E\widetilde{Q}_{pT}^{\frac{1}{p}}(0)+\Delta(r),\tag{1}$$

and $\Delta(R_0)=\rho_0\mu_p^{\frac{1}{p}}+\Delta_0$ for some $R_0>0$, where $\rho_0>2$ and $\Delta_0>0$ are some numbers.

$\mathbf{A}_2$. $\varepsilon(t),\ t\in\mathbb{R}^1$, is a strongly dependent process, namely: $B(t)=E\varepsilon(t)\varepsilon(0)=\dfrac{L(|t|)}{|t|^\alpha},\ 0<\alpha<1$, where $L(t),\ t\in[0,\infty)$ is a function slowly varying at infinity, $B(0)=1$.

$\mathbf{A}_3$. $B\in L_1(\mathbb{R}^1),\ B(0)=1$.

Theorem 1. *For any $r>0$ as $T\to\infty$:*
1) under assumptions $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$, and $\mathbf{C}_1$,

$$P\{\|\widehat{u}_T\|\ge r\}=O(B(T));\tag{2}$$

2) under assumptions $\mathbf{A}_1$, $\mathbf{A}_3$, $\mathbf{B}_1$, and $\mathbf{C}_1$,

$$P\{\|\widehat{u}_T\|\ge r\}=O(T^{-1}).\tag{3}$$

We will give an outline of the proof of statement (2). The proof of (3) is similar. Let

$$h_T(\theta,u)=\widetilde{Q}_{pT}^{\frac{1}{p}}(u)-E\widetilde{Q}_{pT}^{\frac{1}{p}}(u).$$

By the definition of L_p-estimator,

$$\widetilde{Q}_{pT}^{\frac{1}{p}}(\widehat{u}_T)\le h_T(\theta,0)+E\widetilde{Q}_{pT}^{\frac{1}{p}}(0)\ \text{ a.s.}$$

Therefore, by condition $\mathbf{C}_1$ for $\gamma\in(0,1)$, one has

$$\begin{aligned}P\left\{\|\widehat{u}_T\|\ge r\right\}&=P\left\{\|\widehat{u}_T\|\ge r,\ \widetilde{Q}_{pT}^{\frac{1}{p}}(\widehat{u}_T)\le h_T(\theta,0)+E\widetilde{Q}_{pT}^{\frac{1}{p}}(0)\right\}\le\\&\le P\left\{\inf_{u\in U_T^c(\theta)\setminus v(r)}T^{-\frac{1}{p}}\widetilde{Q}_{pT}^{\frac{1}{p}}(u)\le h_T(\theta,0)+E\widetilde{Q}_{pT}^{\frac{1}{p}}(0)\right\}\le\\&\le P\left\{-\inf_{u\in U_T^c(\theta)\setminus v(r)}T^{-\frac{1}{p}}h_T(\theta,u)+T^{-\frac{1}{p}}h_T(\theta,0)\ge\Delta(r)\right\}\le\\&\le P\left\{\sup_{u\in U_T^c(\theta)\setminus v(r)}T^{-\frac{1}{p}}|h_T(\theta,u)|\ge\gamma\Delta(r)\right\}+\\&\quad+P\left\{T^{-\frac{1}{p}}h_T(\theta,0)\ge(1-\gamma)\Delta(r)\right\}=\\&=P_1+P_2.\end{aligned}\tag{4}$$

To estimate P_2,, we set

$$\xi(t) = |\varepsilon(t)|^p - \mu_p,\ \eta_T = T^{-1}\int_0^T \xi(t)dt.$$

Using the expansion of the function $|x|^p$ in the Hilbert space $L_2(R^1,\ \varphi(x)dx),\ \varphi(x) = (2\pi)^{-\frac{1}{2}}e^{-\frac{x^2}{2}}$, in Hermite polynomials, one can obtain the inequality (see, for example, [5, 11])

$$E\eta_T^2 \le D\xi(0)\frac{1}{T^2}\int_0^T\int_0^T B^2(t-s)dtds. \tag{5}$$

Applying the standard argument [11, 12], it can be shown from $\mathbf{A}_2$ and (5) that $\eta_T \underset{T\to\infty}{\to} 0$ a.s. If so, then

$$\zeta_T = T^{-\frac{1}{p}}\left(\int_0^T |\varepsilon(t)|^p dt\right)^{\frac{1}{p}} \underset{T\to\infty}{\to} \mu_p^{\frac{1}{p}} \text{ a.s.} \tag{6}$$

On the other hand, $E\zeta_T^p = \mu_p$ for any T. Therefore ([13], p. 105),

$$E\zeta_T = ET^{-\frac{1}{p}}\widetilde{Q}_{pT}^{\frac{1}{p}}(0) \underset{T\to\infty}{\to} \mu_p^{\frac{1}{p}}, \tag{7}$$

and, for $T > T_0$ and some $0 < C_0 < (1-\gamma)\Delta(r)$,

$$P_2 = \{\zeta_T \ge (1-\gamma)\Delta(r) + E\zeta_T\} \le \left\{\zeta_T \ge (1-\gamma)\Delta(r) + \mu_p^{\frac{1}{p}} - C_0\right\} =$$

$$= \left\{\eta_T \ge \left(\mu_p^{\frac{1}{p}} + (1-\gamma)\Delta(r) - C_0\right)^p - \mu_p\right\} = O(B^2(T)), \tag{8}$$

as follows from (5).

To estimate P_1, one obtains, by the triangle inequality,

$$\Phi_{pT}^{\frac{1}{p}}(0,u) - \widetilde{Q}_{pT}^{\frac{1}{p}}(0) \le \widetilde{Q}_{pT}^{\frac{1}{p}}(u) \le \Phi_{pT}^{\frac{1}{p}}(0,u) + \widetilde{Q}_{pT}^{\frac{1}{p}}(0), \tag{9}$$

and, taking the expectations,

$$-E\widetilde{Q}_{pT}^{\frac{1}{p}}(0) - \Phi_{pT}^{\frac{1}{p}}(0,u) \le -E\widetilde{Q}_{pT}^{\frac{1}{p}}(u) \le E\widetilde{Q}_{pT}^{\frac{1}{p}}(0) - \Phi_{pT}^{\frac{1}{p}}(0,u). \tag{10}$$

The addition of inequalities (9) and (10) leads to the majorant

$$|h(\theta,u)| \le \widetilde{Q}_{pT}^{\frac{1}{p}}(0) + E\widetilde{Q}_{pT}^{\frac{1}{p}}(0).$$

Therefore,

$$P_1 \le P\left\{\zeta_t + E\zeta_T \ge \gamma\Delta(r)\right\}. \tag{11}$$

Having taken in (11) $r = R_0$ from condition $\mathbf{C}_1$ and $\gamma = \dfrac{2}{\rho_0}$, we arrive at the inequality

$$P_1 \le P\left\{\zeta_T \ge \left(\mu_p^{\frac{1}{p}} - E\zeta_T\right) + \mu_p^{\frac{1}{p}} + \frac{2\Delta_0}{\rho_0}\right\}. \tag{12}$$

Relation (6) shows that, for $T > T_0$,

$$P_1 \le P\left\{\zeta_T \ge \mu_p^{\frac{1}{p}} + \frac{\Delta_0}{\rho_0}\right\} = P\left\{\eta_T \ge \left(\mu_p^{\frac{1}{p}} + \frac{\Delta_0}{\rho_0}\right)^p - \mu_p\right\} = O(B^2(T)). \tag{13}$$

Taking bound (8) for $r = R_0$ and bound (13) into account, one has, for any $r \in (0, R_0)$,

$$\begin{aligned} P\left\{\|\widehat{u}_T\| \ge r\right\} &\le P\left\{R_0 \ge \|\widehat{u}_T\| \ge r\right\} + P\left\{\|\widehat{u}_T\| \ge R_0\right\} \\ &= P\left\{R_0 \ge \|\widehat{u}_T\| \ge r\right\} + O(B^2(T)). \end{aligned} \tag{14}$$

As far as

$$\inf_{u\in U_T^c(\theta)\cap(v^c(R_0)\setminus v(r))} T^{-\frac{1}{p}} E\widetilde{Q}_{pT}^{\frac{1}{p}}(u) \geq \inf_{u\in U_T^c(\theta)\setminus v(r))} T^{-\frac{1}{p}} E\widetilde{Q}_{pT}^{\frac{1}{p}}(u), \tag{15}$$

condition $\mathbf{C}_1$ is fulfilled also for the left-hand side of inequality (15). So, as previously, we obtain an inequality similar to (4) for $\gamma' \in (0,1)$:

$$\begin{aligned} P\left\{R_0 \geq \|\widehat{u}_T\| \geq r\right\} \leq P\left\{-\inf_{u\in U_T^c(\theta)\cap(v^c(R_0)\setminus v(r))} T^{-\frac{1}{p}} h_T(\theta,u) \geq \gamma'\Delta(r)\right\} + \\ + P\left\{T^{-\frac{1}{p}} h_T(\theta,0) \geq (1-\gamma')\Delta(r)\right\} \leq P_3 + O(B^2(T)), \end{aligned} \tag{16}$$

$$P_3 = P\left\{\sup_{u\in U_T^c(\theta)\cap v^c(R_0)} T^{-\frac{1}{p}} |h_T(\theta,u)| \geq \gamma'\Delta(r)\right\}.$$

For any $\varepsilon > 0,\ R > 0$, condition $\mathbf{B}_1$ yields the existence of $\delta = \delta(\varepsilon, R) > 0$ such that

$$\sup_{u_1,u_2\in U_T^c(u)\cap v^c(R),\ \|u_1-u_2\|<\delta} T^{-1}\Phi_{pT}(u_1,u_2) < \varepsilon. \tag{17}$$

Let $F^{(1)},\ldots,F^{(l)}$ be closed sets of diameters less than δ that corresponds to the number $R = R_0$ and $\varepsilon = \left(\frac{c_1\Delta(r)\gamma'}{2}\right)^p$ from inequality (17), and let $c_1 \in (0,1)$ be some number, $\bigcup_{i=1}^{l} F^{(i)} = v^c(R_0)$. If the points $u_i \in F^{(i)} \cap U_T^c(\theta),\ i = 1,\ldots,l_0,\ l_0 \leq l$ are fixed, then

$$P_3 \leq \sum_{i=1}^{l_0} P\left\{\sup_{u',u''\in F^{(i)}\cap U_T^c(\theta)} T^{-\frac{1}{p}}\left|h_T(\theta,u') - h_T(\theta,u'')\right| + T^{-\frac{1}{p}}\left|h_T(\theta,u_i)\right| \geq \gamma'\Delta(r)\right\}. \tag{18}$$

For $u', u'' \in F^{(i)}$, one has, by inequality (17),

$$\begin{aligned} &T^{-\frac{1}{p}}\left|h_T(\theta,u') - h_T(\theta,u'')\right| \leq \\ &\quad\leq T^{-\frac{1}{p}}\left|\widetilde{Q}_{pT}^{\frac{1}{p}}(u') - \widetilde{Q}_{pT}^{\frac{1}{p}}(u'')\right| + T^{-\frac{1}{p}} E\left|\widetilde{Q}_{pT}^{\frac{1}{p}}(u') - \widetilde{Q}_{pT}^{\frac{1}{p}}(u'')\right| \leq \\ &\quad\leq 2T^{-\frac{1}{p}}\Phi_{pT}^{\frac{1}{p}}(u',u'') < c_1\gamma'\Delta(r) \end{aligned}$$

and

$$P_3 \leq \sum_{i=1}^{l_0} P\left\{T^{-\frac{1}{p}}|h_T(\theta,u_i)| \geq (1-c_1)\gamma'\Delta(R)\right\}. \tag{19}$$

For any $u \in v^c(R_0)$, one obtains further

$$|h_T(\theta,u)| \leq \left|\widetilde{Q}_{pT}^{\frac{1}{p}}(u) - \left(E\widetilde{Q}_{pT}(u)\right)^{\frac{1}{p}}\right| + \left(E\widetilde{Q}_{pT}(u)\right)^{\frac{1}{p}} - E\widetilde{Q}_{pT}^{\frac{1}{p}}(u) = a_1(u) + a_2(u). \tag{20}$$

Taking the expectation of both parts of the inequality

$$\left|E\widetilde{Q}_{pT}^{\frac{1}{p}}(u) - \widetilde{Q}_{pT}(u)\right|^{\frac{1}{p}} \geq \left(E\widetilde{Q}_{pT}(u)\right)^{\frac{1}{p}} - \widetilde{Q}_{pT}^{\frac{1}{p}}(u), \tag{21}$$

we derive the bound

$$T^{-\frac{1}{p}} a_2(u) \leq T^{-\frac{1}{p}} E\left|E\widetilde{Q}_{pT}^{\frac{1}{p}}(u) - \widetilde{Q}_{pT}(u)\right|^{\frac{1}{p}} \leq \left(T^{-2}D\widetilde{Q}_{pT}(u)\right)^{\frac{1}{2p}}. \tag{22}$$

Let us use the notation

$$\Delta f(t,u) = f(t,0) - f(t,u),\ \xi(t,u) = |\varepsilon(t) + \Delta f(t,u)|^p.$$

Then $\mathbf{B}_1$ yields

$$\sup_{u \in U_T^c(\theta) \cap v^c(R_0)} \sup_{t \in [0,T]} |\Delta f(t,u)| \le R_0 \|k(R_0)\|, \tag{23}$$

$k(R_0) = \left(k^1(R_0), \ldots, k^q(R_0)\right)$, and consequently,

$$E\xi^2(t,u) \le 2^{2p-1}\left(\mu_{2p} + (R_0\|k(R_0)\|)^{2p}\right) = c_2 < \infty.$$

Therefore,

$$\operatorname{cov}(\xi(t,u), \xi(s,u)) = \sum_{m=1}^{\infty} \frac{C_m(t,u)C_m(s,u)}{m!} B^m(t-s) \tag{24}$$

with

$$C_m(t,u) = \int_{-\infty}^{\infty} |x + \Delta f(t,u)|^p H_m(x)\varphi(x)dx,$$

where $H_m(x),\ m \ge 1$, are Hermite polynomials.

With regard for the relation

$$\sum_{m=1}^{\infty} \frac{C_m^2(t)}{m!} = D\xi(t,u) \le c_2, \tag{25}$$

we arrive at the bound [11]

$$T^{-2} D\widetilde{Q}_{pT}(u) = T^{-2}\int_0^T\int_0^T \operatorname{cov}(\xi(t,u),\xi(s,u))\,dtds \le$$
$$\le \sum_{m=1}^{\infty}\frac{1}{m!}\left(T^{-2}\int_0^T\int_0^T C_m^2(t,u)B^m(t-s)dtds\right) \le$$
$$\le c_2 T^{-2}\int_0^T\int_0^T B(t-s)dtds = O(B(T)), \tag{26}$$

and

$$T^{-\frac{1}{p}} a_2(u) = O(B^{\frac{1}{2p}}(T)). \tag{27}$$

On the other hand,

$$T^{-\frac{1}{p}} a_1(u) \le T^{-\frac{1}{p}} \left|\widetilde{Q}_{pT}(u) - E\widetilde{Q}_{pT}(u)\right|^{\frac{1}{p}}. \tag{28}$$

Due to (26)-(28) for any number $0 < c_3 < (1-c_1)\gamma'\Delta(r)$ and $u \in v^c(R_0)$ for $T > T_0$,

$$P\left\{T^{-\frac{1}{p}}|h_T(\theta,u)| \ge (1-c_1)\gamma'\Delta(r)\right\} \le P\left\{T^{-1}\left|\widetilde{Q}_{pT}(u) - E\widetilde{Q}_{pT}(u)\right| \ge c_3^p\right\} \le$$
$$c_3^{-2p} T^{-2} D\widetilde{Q}_{pT}(u) = O(B(T)) \tag{29}$$

hence

$$P_3 = O(B(T)). \tag{30}$$

Relations (16) and (30) yield (2). ■

Sometimes, it is sufficient to check a simpler modification of condition $\mathbf{C}_1$. For example, if

$$\sup_{t \ge 0} \sup_{\tau_1,\tau_2 \in \Theta^c} |g(t,\tau_1) - g(t,\tau_2)| \le g_0 < \infty, \tag{31}$$

then, to obtain (2) and (3) instead of (1), one can use the contrast inequality

$$\inf_{u\in U_T^c(\theta)\setminus v(r)} T^{-\frac{1}{p}}\left(E\widetilde{Q}_{pT}(u)\right)^{\frac{1}{p}} \geq \mu_p^{\frac{1}{p}} + \Delta(r). \tag{32}$$

Assuming

$$d_{iT}(\theta) \asymp T^{\frac{1}{2}},\ i=1,\dots,m,$$

one can take the normalization

$$T^{-\frac{1}{2}}d_T(\theta) = \mathbb{I}_m$$

without loss of generality. Then $U_T(\theta) = \Theta - \theta$, $\widetilde{Q}_{pT}(u) = Q_{pT}(\theta+u)$ and so on.

Instead of the differentiability of g and assumption $\mathbf{B}_1$, we suppose

$\mathbf{B}_2$. Inequality (31) is valid, and for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that

$$\sup_{\tau_1,\tau_2\in\Theta^c:\ \|\tau_1-\tau_2\|<\delta} \frac{1}{T}\int_0^T |g(t,\tau_1) - g(t,\tau_2)|^p dt < \varepsilon.$$

Instead of $\mathbf{C}_1$, we assume

$\mathbf{C}_2$ (contrast condition). For any $r > 0$, there exists $\Delta(r) > 0$ such that

$$\inf_{u\in(\Theta-\theta)\setminus v(r)} T^{-1}\int_0^T (g(t,\theta+u) - g(t,\theta))^2 dt \geq \Delta(r).$$

Theorem 2. *If Θ is a bounded set, then under assumptions $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_2$, and $\mathbf{C}_2$ for any $r > 0$,*

$$P\{\|\hat{\theta}_T - \theta\| \geq r\} = O(B(T)) \text{ as } T\to\infty.$$

A similar statement can be formulated for the process $\varepsilon(t)$, $t\in\mathbb{R}^1$ with integrated covariance function.

To prove the theorem, one has to check contrast conditions $\mathbf{C}_1$ or (32). They can be written now in the form of the following assumption:

For any $r > 0$, there exists $\Delta^*(r) > 0$ such that

$$\inf_{\tau\in\Theta^c:\ \|\tau-\theta\|\geq r} T^{-1}EQ_{pT}(\tau) \geq \mu_p + \Delta^*(r).$$

Write

$$g_0(t) = |g(t,\theta) - g(t,\tau)|.$$

The validity of $\mathbf{C}_1$ follows from the inequalities

$$T^{-1}EQ_{pT}(\tau) - \mu_p \geq \frac{p}{2}T^{-1}\int_0^T g_0^2(t)\int_{g_0(t)}^\infty x^p\varphi(x)dx dt \geq \frac{p}{2}G_0\Delta(r) = \Delta^*(r) > 0, \tag{33}$$

where $\|\tau-\theta\| \geq r$, $\Delta(r)$ is taken from $\mathbf{C}_2$,

$$G_0 = \int_{g_0}^\infty x^p\varphi(x)dx,\ \varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},$$

and g_0 is defined in (31).

In fact, inequality (33) is true for any bounded, even continuously differentiable density function on $\mathbb{R}^1$ which is non-decreasing on $(-\infty, 0]$, and $\mu_p < \infty$ [6].

Suppose

$$g(t,\theta) = \sum_{i=1}^m g_i(t)\theta_i. \tag{34}$$

Then $d_{iT}^2 = \int_0^T g_i^2(t)dt,\ i=1,\dots,m,\ d_T = \mathrm{diag}(d_{iT})$. Condition $\mathbf{B}_1$ is transformed into

$\mathbf{B}_3$. For some $k^i < +\infty,\ i = 1, \dots, m$,

$$\max_{t\in[0,T]} |g_i(t)| d_{iT}^{-1} \leq k^i T^{-1/2}.$$

Set

$$J_T^{il} = d_{iT}^{-1} d_{lT}^{-1} \int_0^T g_i(t) g_l(t) dt,\ i, l = 1, \dots, m;$$

$J_T = \left(J_T^{il}\right)_{i,l=1}^m$, *and*$\lambda_{\min}(J_T)$ is the least eigenvalue of a positive definite matrix J_T.

$\mathbf{B}_4$. $\lambda_{\min}(J_T) \geq \lambda_* > 0$.

Theorem 3. *Let the regression function g be of the form* (34) *and satisfy assumptions* $\mathbf{B}_3$ *and* $\mathbf{B}_4$. *Then, for any* $r > 0$ *as* $T \to \infty$*:*

1) $P\{\|\widehat{u}_T\| \geq r\} = O(B(T))$, *if the process* $\varepsilon(t),\ t \in \mathbb{R}^1$, *is subjected to* $\mathbf{A}_1,\ \mathbf{A}_2$*;*

2) $P\{\|\widehat{u}_T\| \geq r\} = O(T^{-1})$, *if the process* $\varepsilon(t),\ t \in \mathbb{R}^1$, *is subjected to* $\mathbf{A}_1,\ \mathbf{A}_3$.

Outline the proof of 1). By the triangle inequality,

$$T^{-\frac{1}{p}} E\widetilde{Q}_{pT}^{\frac{1}{p}}(u) \geq T^{-\frac{1}{p}} \Phi_{pT}^{\frac{1}{p}}(u, 0) - T^{-\frac{1}{p}} E\widetilde{Q}_{pT}^{\frac{1}{p}}(0). \tag{35}$$

Using (7), we conclude that condition $\mathbf{C}_1$ will be fulfilled if

(**i**) there exists $R_0 > 0$ such that, for $\|u\| \geq R_0$ and $T > T_0$,

$$T^{-\frac{1}{p}} \Phi_{pT}^{\frac{1}{p}}(u, 0) \geq 2\mu_p^{\frac{1}{p}} + \Delta(R_0), \tag{36}$$

where $\Delta(R_0)$ has the same property as that in $\mathbf{C}_1$;

(ii) for any $0 < r < R_0$ and $r \leq \|u\| < R_0$,

$$T^{-\frac{1}{p}} E\widetilde{Q}_{pT}^{\frac{1}{p}}(u) \geq \mu_p^{\frac{1}{p}} + \Delta(r, R_0) \tag{37}$$

for some $\Delta(r, R_0) > 0$.

To check (36), we will use the representation

$$T^{-1}\Phi_{pT}(u, 0) = T^{-1} \int_0^T \frac{\left|\sum_{i=1}^m g_i(t) T^{\frac{1}{2}} d_{iT}^{-1} u_i\right|^2}{\left|\sum_{i=1}^m g_i(t) T^{\frac{1}{2}} d_{iT}^{-1} u_i\right|^{2-p}} dt. \tag{38}$$

It follows from $\mathbf{B}_3$ that

$$\left|\sum_{i=1}^m g_i(t) T^{\frac{1}{2}} d_{iT}^{-1} u_i\right|^{2-p} \leq \left(\max_{1\leq i\leq m} k^i\right)^{2-p} m^{\frac{2-p}{2}} \|u\|^{2-p}. \tag{39}$$

On the other hand, we have, by $\mathbf{B}_4$,

$$T^{-1} \int_0^T \left|\sum_{i=1}^m g_i(t) T^{\frac{1}{2}} d_{iT}^{-1} u_i\right|^2 dt = \sum_{i,l=1}^m J_T^{il} u_i u_l \geq \lambda_* \|u\|^2, \tag{40}$$

and, therefore,

$$T^{-\frac{1}{p}} \Phi_{pT}^{\frac{1}{p}}(u, 0) \geq c_4 \|u\|, \tag{41}$$

where

$$c_4 = \lambda_*^{\frac{1}{p}} \left(\max_{1\leq i\leq m} k^i\right)^{\frac{p-2}{p}} \cdot m^{\frac{p-2}{2p}}.$$

It is clear from (41) that inequality (36) can be satisfied by the proper choice of $\|u\|$.

As follows from (7) and (27), condition (37) will be fulfilled for $R_0 > \|u\| \geq r_0$, if

$$T^{-\frac{1}{p}} \left(E\widetilde{Q}_{pT}(u)\right)^{\frac{1}{p}} \geq \mu_p^{\frac{1}{p}} + \Delta_1(r, R_0) \tag{42}$$

or

$$T^{-1}E\widetilde{Q}_{pT}(u) \geq \mu_p + \Delta_2(r, R_0), \tag{43}$$

where $\Delta_1(r, R_0)$ and $\Delta_2(r, R_0)$ are some positive constants.

Similarly to (8),

$$T^{-1}E\widetilde{Q}_{pT}(u) - \mu_p \geq \frac{p}{2}T^{-1}\int\limits_0^T \Delta^2 f(t,u) \int\limits_{|\Delta f(t,u)|}^{\infty} x^p\varphi(x)dxdt. \tag{44}$$

If $\|u\| < R_0$, then we have, by inequality (23),

$$\int\limits_{|\Delta f(t,u)|}^{\infty} x^p\varphi(x)dx \geq \int\limits_{R_0\|k(R_0)\|}^{\infty} x^p\varphi(x)dx = G_0 > 0. \tag{45}$$

Thus, (44), (45), and (40) yield

$$T^{-1}E\widetilde{Q}_{pT}(u) - \mu_p \geq \frac{p}{2}G_0\lambda_* r^2 = \Delta_2(r, R_0) > 0. \tag{46}$$

2. Asymptotic uniqueness of the solution to a system of normal equations

If $\rho(x) = |x|^p$, then $\rho'(x) = \psi(x) = p|x|^{p-1}\text{sgn}x$, $\rho'' = \psi' = p(p-1)|x|^{p-2}$, $x \neq 0$, and $\psi'(0) = +\infty$.

The L_p-estimator $\hat{\theta}_T$ is a solution to the system of "normal" equations

$$\text{grad}\left(\gamma T^{-1}Q_{pT}(\tau)\right) = 0,\ \gamma = \left(E\psi'(\varepsilon(0))\right)^{-1} > 0 \tag{47}$$

or

$$\text{grad}\left(\gamma T^{-1}\widetilde{Q}_{pT}(u)\right) = 0,\ u = T^{-\frac{1}{2}}d_T(\theta)(\tau - \theta). \tag{48}$$

Assume $\Theta \subset \mathbb{R}^m$ to be an open bounded set and $g(t,\cdot) \in C^2(\Theta^c)$. Write

$$g_{il}(t,\theta) = \frac{\partial^2}{\partial\tau_i\partial\tau_l}g(t,\theta),\ d^2_{il,T}(\theta) = \int_0^T g^2_{il}(t,\theta)dt,\ i,l = 1,\dots,m.$$

$\mathbf{B}_5$:

1) $\sup\limits_{t\in[0,T]}\ \sup\limits_{\tau\in\Theta^c} |g_i(t,\tau)|d^{-1}_{iT}(\theta) \leq k^i T^{-\frac{1}{2}}$;

2) $\sup\limits_{t\in[0,T]}\ \sup\limits_{\tau\in\Theta^c} |g_{il}(t,\tau)|d^{-1}_{il,T}(\theta) \leq k^{il} T^{-\frac{1}{2}}$;

3) $\sup\limits_{\tau\in\Theta^c} d_{il,T}(\tau)d^{-1}_{iT}(\theta)d^{-1}_{lT}(\theta) \leq \tilde{k}^{il} T^{-\frac{1}{2}}$;

4) $Td^{-2}_{iT}(\theta)d^{-2}_{lT}(\theta)\int\limits_0^T \left(g_{il}(t,\theta + T^{\frac{1}{2}}d^{-1}_T(\theta)u) - g_{il}(t,\theta)\right)^2 dt \leq k_{il}\|u\|^2,\ i,l = 1,\dots,m.$

Theorem 4. *Suppose $p \in (\frac{3}{2}, 2)$. Then, under assumptions $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_4$, $\mathbf{B}_5$, and $\mathbf{C}_1$, the system of equations (47) (or (48)) has a unique solution with probability $1 - O(B(T))$ as $T \to \infty$.*

The idea of the proof consists in the comparison of two matrices

$$H_T(u) = \text{Hessian}\left(\gamma T^{-1}\widetilde{Q}_{pT}(u)\right) \text{ and } J_T(\theta).$$

Using the inequality for symmetric matrices [14]

$$|\lambda_{\min}(H_T(u)) - \lambda_{\min}(J_T(\theta))| \le m \cdot \max_{1\le i,l\le m} \left|H_T^{il}(u) - J_T^{il}(\theta)\right|,$$

one can prove that $H_T(u)$ is a positive definite matrix in some neighborhood of zero with probability $1 - O(B(T))$ as $T \to \infty$.

3. Asymptotic normality of L_p-estimators

Assume further that there exist the limits $\Lambda(\theta) = \lim\limits_{T\to\infty} J_T^{-1}(\theta)$ and

$$\sigma(\theta) = \lim_{T\to\infty} D_T^{-1}(\theta) \left(\int_0^1 \int_0^1 \frac{\nabla g(tT,\theta)\nabla^* g(sT,\theta)}{|t-s|^\alpha} \right) D_T^{-1}(\theta),$$

$D_T^2(\theta) = T^{-1} d_T^2(\theta)$.

It follows from Theorem 4 that one can apply the Brouwer fixed-point theorem to prove

Theorem 5. *Under assumptions of Theorem 4, the normalized L_p-estimator*

$$B^{-\frac{1}{2}}(T)T^{-\frac{1}{2}}d_T(\theta)(\hat\theta_T - \theta)$$

is asymptotically normal $N(0, \Lambda(\theta)\Sigma(\theta)\Lambda(\theta))$ r.v.

The details of the proof can be found in [11].

The results similar to Theorems 4 and 5 can be obtained for the process $\varepsilon(t),\ t \in \mathbb{R}^1$ satisfying the weak dependence condition.

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