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ON THE EQUIVALENCE OF INTEGRAL NORMS ON THE SPACE OF MEASURABLE POLYNOMIALS WITH RESPECT TO A CONVEX MEASURE

We prove that, for a convex product-measure μ on a locally convex space, for any set A of positive measure, on the space of measurable polynomials of degree d, all $L^p(\mu)$-norms coincide with the norms obtained by restricting μ to A.

It is well known that if γ is a Gaussian measure on a locally convex space X, then all L^p-norms on $P_d(\gamma)$, the space of measurable polynomials of degree at most d, are mutually equivalent. In the case where X is a Hilbert space, A.A. Dorogovtsev [1] has shown that the $L^2(\gamma)$-norm on $P_d(\gamma)$ is equivalent to the $L^2(\gamma|B)$-norm, where $\gamma|B$ is the restriction of γ to a unit ball B of X.

In the recent paper [2], the author has reinforced that result and has shown that, in the case of any locally convex space and an arbitrary measurable set A with $\gamma(A) > 0$, all $L^p(\gamma|A)$-norms are equivalent to the $L^p(\gamma)$-norms. In particular, they are mutually equivalent.

The main result of this paper shows that it is valid also for convex measures satisfying certain additional conditions.

Let us recall some definitions (see, e.g., [3],[4]).

Definition 1. *A Borel probability measure μ on R^n is called a convex (or log-concave) measure if there exists an affine subspace E with $\mu(E) = 1$, on which μ is given by a density ϱ with respect to the Lebesgue measure on E such that, for all $x, y \in E$ and $\lambda \in [0, 1]$, the following inequality holds:*

$$\varrho(\lambda x + (1 - \lambda)y) \geq \varrho(x)^{\lambda} \varrho(y)^{1-\lambda}.$$

Definition 2. *Let X be a locally convex space equipped with the σ-algebra $\sigma(X)$ generated by the dual space X^*. A probability measure μ on $\sigma(X)$ is called convex (or log-concave) if, for all $l_1, \dots, l_n \in X^*$, its image under the mapping $x \mapsto (l_1(x), \dots, l_n(x))$ to R^n is a convex measure on R^n.*

Recently S.G. Bobkov [5] has obtained the following important result for convex measures.

Set $C := 22/\ln 2$. Let ν be a convex probability measure on the space R^k and f be a polynomial of degree at most d on R^k. Then, for all $p \in [1, \infty)$, the following inequality holds:

$$\|f\|_{L^p(\nu)} \leq p^{Cd} \|f\|_{L^1(\nu)}. \qquad (1)$$

In particular, on the space $P_d(R^k)$ of all polynomials of degree at most d on R^k, all $L^p(\nu)$-norms are equivalent with constants which are independent of k and depend only on d and p.

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Suppose we are given a sequence (X_k, B_k, μ_k), $k \in N$, of probability spaces. The measure $\mu = \otimes_{k=1}^{\infty} \mu_k$ is called a product-measure, where we deal with the product of the measures μ_k defined on the product of the spaces $X = \prod_{k=1}^{\infty} X_k$ which is equipped with the σ-algebra $B := \bigotimes_{k=1}^{\infty} B_k$.

Let X be a locally convex space, and let $P_{d,fin}(X)$ be the class of all finite-dimensional polynomials on X of the form

$$f(x) = P(l_1(x), \dots, l_k(x)),$$

where P is a polynomial of degree at most d on R^k and $l_1, \dots, l_k$ are continuous linear functionals on X. Let ν be a probability measure on $\sigma(X)$. Let us denote, by $P_d(\nu)$, the closure of the set $P_{d,fin}(X)$ in the space of all ν-measurable functions with respect to a metric corresponding to the convergence in measure ν; e.g., one can take the metric

$$\varrho(f, g) := \int_X \frac{|f - g|}{1 + |f - g|} \, d\mu.$$

Lemma. *Let μ be a convex probability measure on a locally convex space X. Then the following assertions are true.*

(i) *For every $p \in [1, \infty)$, one has $P_d(\mu) \subset L^p(\mu)$.*

(ii) *On the space $P_d(\mu)$, all norms from all spaces $L^p(\mu)$, where $p \in [1, +\infty)$, are equivalent and $P_d(\mu)$ is complete with respect to each of these norms.*

(iii) *If a sequence $\{f_j\} \subset P_d(\mu)$ converges in measure μ, then it converges in every $L^p(\mu)$, $p \in [1, +\infty)$.*

Proof. It is known that, in the finite-dimensional case, every convex measure has all moments (see [3]). So $P_{d,fin}(X) \subset L^p(\mu)$ for all $p < \infty$. Suppose that a sequence of polynomials $\varphi_j \in P_{d,fin}(X)$ converges in measure to φ. Due to the above-mentioned result of Bobkov, we have the estimates

$$\|\psi\|_{L^p(\mu)} \le C(d, p)\|\psi\|_{L^1(\mu)}, \quad \psi \in P_{d,fin}(X), \tag{2}$$

where the number $C(d, p)$ depends only on d and p. In particular, we have these estimates for $p = 2$ and $\psi = \varphi_j$. According to Example 2.8.10 in [4], the norms $\|\varphi_j\|_{L^1(\mu)}$ are uniformly bounded. Indeed, otherwise passing to a subsequence, we may assume that $\{\varphi_j\}$ converges almost everywhere. Then the aforementioned example applies. The boundedness in $L^p(\mu)$ along with the convergence in measure yield the convergence in $L^r(\mu)$ for $r < p$. Since this is true for all $p < \infty$, the sequence $\{\varphi_j\}$ converges to φ in all $L^p(\mu)$. Thus, we obtain not only the inclusion $\varphi \in L^p(\mu)$ but also estimate (2) for all $\psi \in P_d(\mu)$. If we apply the same reasoning to the whole class $P_d(\mu)$, we obtain all assertions of the lemma. In particular, the equivalence of all L^p-norms follows from (2) and the inequality $\|f\|_{L^1(\mu)} \le \|f\|_{L^p(\mu)}$. The completeness of $P_d(\mu)$ with respect to L^p-norms follows from what has been said.

Theorem. *Suppose we are given a sequence of finite-dimensional spaces $X_k = R^{n_k}$ equipped with their Borel σ-algebras B_k. For every k, let μ_k be a convex probability measure on X_k. Let us consider the space $X = \prod_{k=1}^{\infty} X_k$ equipped with the product-measure $\mu = \bigotimes_{k=1}^{\infty} \mu_k$. Let us fix a set $M \subseteqq X$ with $\mu(M) > 0$ and a positive integer d. Then, the following assertions are true.*

(i) *If a sequence of functions in $P_d(\mu)$ converges in the measure μ on M, then it converges in measure μ on all of X and in all $L^p(\mu)$, $p < \infty$, too.*

(ii) *For every $p \in [1, +\infty)$, the norm of $L^p(\mu)$ on the space $P_d(\mu)$ is equivalent to the norm of $L^p(\mu|_M)$. Therefore, whenever $1 \le p, q < \infty$, the norm of $L^p(\mu)$ on $P_d(\mu)$ is equivalent to the norm of $L^q(\mu|_M)$.*

Proof. Let us introduce the following two norms on the space $P_d(\mu)$:

$$\|f\|_1 := \left(\int_M |f|^p \, \mu(dt)\right)^{1/p}, \quad \|f\|_2 := \left(\int_X |f|^p \, \mu(dt)\right)^{1/p}.$$

We have seen that the space $P_d(\mu)$ is a Banach one with respect to the norm $\|\cdot\|_2$. Let us show that the space $P_d(\mu)$ is a Banach one with respect to the norm $\|\cdot\|_1$ as well. Then assertion (ii) will follow by Banach's theorem on equivalent norms. In addition, we will show that the convergence of a sequence from $P_d(\mu)$ in measure μ on the whole space X follows from its convergence in measure μ on M, which will yield assertion (i) by the lemma.

So far it is not even obvious that $\|\cdot\|_1$ is not only a semi-norm but a norm (i.e., it is not obvious that if a function from $P_d(\mu)$ vanishes almost everywhere on M, then it vanishes almost everywhere on X).

Let a sequence $\{f_j\}$ converge in measure μ on M. For proving its convergence in measure μ on all of X, it is sufficient to check that every subsequence in it contains a further subsequence convergent almost everywhere on X. Hence, passing to a subsequence, we may assume that the sequence $\{f_j\}$ converges almost everywhere on M. For simplification of notation, we assume that the measures μ_k are absolutely continuous (otherwise we could take their affine supports). Furthermore, it is sufficient to consider polynomials f_j from the class $P_{d,fin}(X)$, because we can replace the initial sequence $\{f_j\}$ by a sequence of finite-dimensional polynomials with the same limit in measure on M, as every element in $P_d(\mu)$ is the limit of a sequence of finite-dimensional polynomials which converges in measure (and in all $L^p(\mu)$, too).

We aim at proving the convergence of the sequence $\{f_j\}$ almost everywhere on the whole space X. Then the application of the above lemma will complete our proof. We apply a modification of the reasoning from [2] and [6] (see §5.10 in [6]).

Set

$$E := \Big\{x \in X \colon \exists \lim_{j\to\infty} f_j(x)\Big\}.$$

Then $E \in B$ and $\mu(E) > 0$ since $M \subset E$. In order to prove the equality $\mu(E) = 1$, we apply Kolmogorov's zero-one law. To this end, as is known (see Theorem 10.10.17 in [4]), it is sufficient to satisfy the following condition: if $x \in E$, then $y \in E$ for every $y \in X$ with $y_k = x_k$ for all sufficiently large k. We shall achieve this condition on some subset E_1 of the set E such that E_1 is also of positive measure. Since we assume that all measures μ_k are absolutely continuous, for every fixed n, due to Fubini's theorem, for almost every $x \in M$, the section

$$M_n^z := \Big\{z \in X_n \colon (x_1, \dots, x_{n-1}, z, x_{n+1}, \dots) \in M\Big\}$$

has a positive Lebesgue measure in X_n. This implies that almost every point in M has this property for all $n \in N$. Hence, the measurable set

$$E_1 := \Big\{x \in E \colon \lambda_n(E_n^x) > 0 \;\forall n \in N\Big\}$$

has a positive μ measure, where λ_n is the Lebesgue measure on X_n and

$$E_n^z := \Big\{z \in X_n \colon (x_1, \dots, x_{n-1}, z, x_{n+1}, \dots) \in E\Big\}.$$

If a sequence of polynomials of degree d on X_n converges on a set of positive Lebesgue measures, then it converges at every point in X_n. Therefore, for every $x \in E_1$, the section E_n^x coincides with the whole space X_n for every n. Thus, if $x \in E_1$, then $x + u \in E_1$ for every u of the form $u = (u_1, \dots, u_n, 0, 0, \dots)$. Due to the zero-one law, one has

$\mu(E_1) = 1$, whence it follows that $\mu(E) = 1$. Hence, $\{f_j\}$ converges almost everywhere on all of X. Along with the lemma, this proves assertion (i).

Now we can easily complete the proof of assertion (ii). Suppose we are given a sequence $\{f_j\} \subset P_d(\mu)$ that is fundamental with respect to the $L^p(\mu|_M)$-norm. It converges on M in measure μ. Hence, as shown above, it converges in $L^p(\mu)$ to some function $g \in P_d(\mu)$. Clearly, the sequence $\{f_j\}$ converges to g in $L^p(\mu|_M)$ too. The proof is completed.

Remark. It would be interesting to extend this theorem to more general cases of convex measures.

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