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BOUNDS FOR A SUM OF RANDOM VARIABLES UNDER A MIXTURE OF NORMALS

In two papers: Dhaene et al. (2002). Insurance: Mathematics and Economics 31, pp.3-33 and pp. 133-161, the approximation for sums of random variables (rv's) was derived for the case where the distribution of the components is lognormal and known, but the stochastic dependence structure is unknown or too cumbersome to work with. In finance and actuarial science a lot of attention is paid to a regime switching model. In this paper we give the approximation for sums under a mixture of normals and consider approximate evaluation of provision under switching regime.

1. Introduction

In an insurance context, one is often interested in the distribution function of a sum of random variables (rv's). Such a sum appears when considering the aggregate claims of an insurance portfolio over a certain reference period. It also appears when considering discounted payments related to a single policy or a portfolio, at different future points in time. The assumption of mutual independence of the components of the sum is very convenient from a computational point of view, but sometimes not a realistic one. In the papers Dhaene et al. (2002a) and (2002b) the approximation for sum of rv's was derived for the case where the distributions of the components are lognormal and known, but the stochastic dependence structure is unknown or too cumbersome to work with. In this paper we consider the case of a switching regime which can represent a change in the economic environment, see Yang (2006). The distribution of the components is a mixture of lognormal distributions.

The paper is organized as follows. In Sections 2 and 3 we give upper and lower bounds for a sum under a mixture of arbitrary distributions. In

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Sections 4 and 5 we derive bounds for provision under the switching regime. In Section 6 we give some numerical illustration of lower and upper bounds, and Section 7 concludes.

We consider the problem similar to Section 4.1 from Dhaene et al. (2002b). We want to bound the sum

$$S := \sum_{i=1}^{n} \alpha_i e^{-(Y_1+\ldots+Y_i)},$$

where $\alpha_i \in \mathbb{R}$, $i = \overline{1,n}$.

In Dhaene et al. (2002b) it was assumed that the vector $(Y_1, \ldots, Y_n)$ has a multivariate normal distribution. The random variable (r.v.) S is then a linear combination of dependent lognormal rv's. The computation of upper and lower bounds in Dhaene et al. (2002b) is based on the concept of comonotonicity.

In this paper we give the approximation for sums under a mixture of arbitrary distributions and consider approximate evaluation of provision under the switching regime. Then we calculate the bounds for a mixture of normals in linear and Markovian ways.

2. Upper bound for a sum

Let $X_1, \ldots, X_n$ be rv's, and Λ be some r.v. with a given cdf, such that we know the conditional cdfs of the r.v. X_i, given $\Lambda = \lambda$, for all $i = \overline{1,n}$ and possible values of λ. Denote by $F^{-1}_{X_i|\Lambda}(U)$ the r.v. $f_i(U, \Lambda)$, where U is uniform $(0, 1)$ and $f_i(u, \lambda) = F^{-1}_{X_i|\Lambda=\lambda}(u)$, F^{-1} stands for a generalized inverse of a cdf F.

Definition 1. *Consider two random variables X and Y. Then X is said to procede Y in the convex order sense, notation $X \leq_{cx} Y$, if and only if,*

$$E[X] = E[Y], \qquad \textit{and} \qquad E[(X-d)_+] \leq E[(Y-d)_+], \qquad \textit{for all } d \in \mathbb{R},$$

where $(x-d)_+ = \max(x-d, 0)$.

It can be proven that $X \leq_{cx} Y$ if, and only if, $E[g(X)] \leq E[g(Y)]$ for all convex functions g, provided the expectations exist.

Theorem 9 from Dhaene et al. (2002a) states that if U is uniform $(0, 1)$ and independent of Λ, then

$$\sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} F^{-1}_{X_i|\Lambda}(U). \tag{1}$$

Assume the following.

(i) $\Lambda = \Phi(X_1, \ldots, X_n)$, where Φ is a nonrandom function.

(ii) A joint distribution of $(X_1, \dots, X_n)$ equals $\sum_{j=1}^N p_j \mu_j^X$, where $0 < p_j < 1$, $j = \overline{1, N}$, $\sum_{j=1}^N p_j = 1$, and μ_j are probability measures on $(\mathbb{R}, \mathbf{B}(\mathbb{R}))$, $\mathbf{B}(\mathbb{R})$ being a Borel $\sigma-$field on real line.

Here p_j have a sense of prior probabilities, and μ_j^X is a conditional distribution provided $(X_1, \dots, X_n)$ belongs to a class A_j; $j = \overline{1, N}$. Due to condition (i), a joint distribution of $(X_1, \dots, X_n, \Lambda)$ equals

$$\sum_{j=1}^N p_j \mu_j^{X\Lambda}, \tag{2}$$

where $\mu_j^{X\Lambda}$ is a conditional distribution of $(X_1, \dots, X_n, \Lambda)$ provided $(X_1, \dots \dots, X_n)$ belongs to the class A_j; $j = \overline{1, N}$.

Now, we find a distribution of X_i given $\Lambda = \lambda$. A joint distribution of (X_i, Λ) equals

$$\sum_{j=1}^N p_j \mu_j^{X_i\Lambda}, \tag{3}$$

where $\mu_j^{X_i\Lambda}$ is a conditional distribution of (X_i, Λ) provided $(X_1, \dots, X_n)$ belongs to the class A_j; $j = \overline{1, N}$. Suppose that

$$d(\mu_j^{X_i\Lambda}) = \rho_j^{X_i\Lambda}(x_i, \lambda) dx_i d\lambda, \tag{4}$$

i.e., the measure $\mu_j^{X_i\Lambda}$ has a density. Then a conditional density of X_i given $\Lambda = \lambda$ equals

$$\rho_{X_i|\Lambda=\lambda}(x_i) = \frac{\sum_{j=1}^N p_j \rho_j^{X_i\Lambda}(x_i, \lambda)}{\int_{\mathbb{R}} \sum_{j=1}^N p_j \rho_j^{X_i\Lambda}(x_i, \lambda) dx_i},$$

$$\rho_{X_i|\Lambda=\lambda}(x_i) = \sum_{j=1}^N q_j(\lambda) \rho^j_{X_i|\Lambda=\lambda}(x_i). \tag{5}$$

Thus the conditional density of X_i given $\Lambda = \lambda$ is a mixture of partial conditional densities, with the posterior probabilities $q_j(\lambda)$ instead of the prior probabilities p_j,

$$q_j(\lambda) = \frac{p_j}{\int_{\mathbb{R}} \sum_{j=1}^N p_j \rho_j^{X_i\Lambda}(x_i, \lambda) dx_i} \int_{\mathbb{R}} \rho_j^{X_i\Lambda}(x_i, \lambda) dx_i, \tag{6}$$

$$\rho^j_{X_i|\Lambda=\lambda}(x_i) = \frac{\rho_j^{X_i\Lambda}(x_i, \lambda)}{\int_{\mathbb{R}} \rho_j^{X_i\Lambda}(x_i, \lambda) dx_i}. \tag{7}$$

At the end of Section 3 we will explain that $q_j(\lambda)$ does not depend of i. The cdf $F_{X_i|\Lambda=\lambda}$ can be computed based on (5)-(7),

$$F_{X_i|\Lambda=\lambda}(z) = \int_{-\infty}^{z} \rho_{X_i|\Lambda=\lambda}(x_i)dx_i, \qquad z \in \mathbb{R}.$$

This can be applied, e.g., when under the class A_j,

$$(\log X_1, \ldots, \log X_n) \sim N(m_j, S_j), \tag{8}$$

and

$$\Lambda = \sum_{i=1}^{n} \beta_i \log X_i, \qquad \beta_i \in \mathbb{R}, \qquad i = \overline{1, n}. \tag{9}$$

Then $q_j(\lambda)$ and $\rho^j_{X_i|\Lambda=\lambda}(x_i)$ can be computed directly.

3. Lower bound for a sum

Theorem 10 from Dhaene et al. (2002a) states that for any r.v. Λ,

$$\sum_{i=1}^{n} E[X_i|\Lambda] \leq_{cx} \sum_{i=1}^{n} X_i. \tag{10}$$

We assume (i) and (ii). From (5) to (7) we obtain that the conditional density of X_i given Λ equals

$$\rho_{X_i|\Lambda}(x_i) = \sum_{j=1}^{n} q_j(\Lambda)\rho^j_{X_i|\Lambda}(x_i).$$

Here

$$q_j(\Lambda) = q_j(\lambda)|_{\lambda=\Lambda},$$

$$\rho^j_{X_i|\Lambda}(x_i) = \frac{\rho_j^{X_i\Lambda}(x_i, \Lambda)}{\int_{\mathbb{R}} \rho_j^{X_i\Lambda}(x_i, \Lambda)dx_i}.$$

Then

$$E[X_i|\Lambda] = \int_{\mathbb{R}} x_i \rho_{X_i|\Lambda}(x_i)dx_i = \sum_{j=1}^{n} q_j(\Lambda) \int_{\mathbb{R}} x_i \rho^j_{X_i|\Lambda}(x_i)dx_i,$$

$$E[X_i|\Lambda] = \sum_{j=1}^{N} q_j(\Lambda)E_j[X_i|\Lambda]. \tag{11}$$

Here $E_j[X_i|\Lambda]$ is a conditional expectation of X_i given Λ, provided $(X_1, \ldots, X_n)$ belongs to the class A_j. Now, (10) and (11) imply that

$$\sum_{i=1}^{n}\sum_{j=1}^{N} q_j(\Lambda)E_j[X_i|\Lambda] \leq_{cx} \sum_{i=1}^{n} X_i. \tag{12}$$

Formula (6) can be rewritten as

$$q_j(\lambda) = \frac{p_j\rho_j^{\Lambda}(\lambda)}{\sum_{j=1}^{N} p_j\rho_j^{\Lambda}(\lambda)}. \tag{13}$$

Here $\rho_j^{\Lambda}(\lambda)$ is a density of Λ provided $(X_1,\ldots,X_n)$ belongs to the class A_j. Thus the posterior probability $q_j(\lambda)$ does not depend on i. Relation (12) is simplified as

$$\sum_{j=1}^{N} q_j(\Lambda)\sum_{i=1}^{n} E_j[X_i|\Lambda] \leq_{cx} \sum_{i=1}^{n} X_i. \tag{14}$$

4. Approximate evaluation of provisions under switching regime: lower bound

Let $Y_1,\ldots,Y_n$ be an i.i.d. sequence. We deal with the sum

$$S := \sum_{i=1}^{n} \alpha_i e^{-(Y_1+\ldots+Y_i)}, \tag{15}$$

where $\alpha_i \in \mathbb{R}$, $i = \overline{1,n}$.

Let $Y_{(i)} := Y_1 + \ldots + Y_i$ and $Y^{(i)} := Y_{i+1} + \ldots + Y_n$. Theorem 1 from Dhaene et al. (2002b) states the following.

Theorem 1 *Let S be given in (15), where the random vector $(Y_1,\ldots,Y_n)$ has a multivariate normal distribution. Consider the conditional r.v. $\Lambda' = \sum_{i=1}^{n}\beta_i Y_i$. Then the lower bound S^l and upper bound S^u are given by*

$$S^l = \sum_{i=1}^{n} \alpha_i \exp\left[-E[Y_{(i)}] - r_i\sigma_{Y_{(i)}}\Phi^{-1}(V) + (1-r_i^2)\sigma^2_{Y_{(i)}}/2\right],$$

$$S^u = \sum_{i=1}^{n} \alpha_i \exp\left[-E[Y_{(i)}] - r_i\sigma_{Y_{(i)}}\Phi^{-1}(V) + sign(\alpha_i)\sqrt{1-r_i^2}\,\sigma_{Y_{(i)}}\Phi^{-1}(U)\right],$$

where U and V are mutually independent uniform $(0,1)$ rv's, Φ is the cdf of the $N(0,1)$ distribution and r_i is defined by

$$r_i = r(Y_{(i)},\Lambda') = \frac{cov[Y_{(i)},\Lambda']}{\sigma_{Y_{(i)}}\sigma_{\Lambda'}}.$$

In this paper we consider the sum from (15) for a mixture of normal distributions in both linear and Markovian way.

4.1. Mixture of N independent normals in linear way

Let the distribution of Y_1 be a mixture of N independent normals:

$$\sum_{i=1}^{N} \pi_i N(\mu_i, \sigma_i^2), \tag{16}$$

where $\pi_i > 0$, $i = \overline{1, N}$, $\sum_{i=1}^{N} \pi_i = 1$, $(\mu_i, \sigma_i^2) \neq (\mu_j, \sigma_j^2)$, $i \neq j$, $\sigma_i > 0$, $i = \overline{1, N}$. The joint distribution of $Y_1, \ldots, Y_n$ is

$$\left(\sum_{i=1}^{N} \pi_i N(\mu_i, \sigma_i^2) \right)^n,$$

where the power corresponds to a product of measures. We consider the conditioning r.v.

$$\Lambda := \sum_{i=1}^{n} Y_i.$$

We have by (10)

$$\sum_{i=1}^{n} \alpha_i E[e^{-(Y_1+\ldots+Y_i)} | \Lambda] \leq_{cx} S. \tag{17}$$

Consider a joint distribution of $Y_{(i)} := Y_1 + \ldots + Y_i$ and $\Lambda = Y_{(i)} + Y^{(i)}$, where $Y^{(i)} := Y_{i+1} + \ldots + Y_n$.

A joint distribution of $Z_i := (Y_{(i)}, Y^{(i)})$ equals $L(Y_{(i)}) \times L(Y^{(i)})$, where $L(\cdot)$ stands for the probability law. Now,

$$L(Y_{(i)}) = \sum_{k_1+\ldots+k_N=i} \binom{i}{k_1 \ldots k_N} \pi_1^{k_1} \cdot \ldots \cdot \pi_N^{k_N} N\left(\sum_{j=1}^{N} k_j \mu_j, \sum_{j=1}^{N} k_j \sigma_j^2 \right), \tag{18}$$

$$L(Y^{(i)}) = L(Y_{(n-i)}) = \sum_{l_1+\ldots+l_N=n-i} \binom{n-i}{l_1 \ldots l_N} \pi_1^{l_1} \cdot \ldots \cdot \pi_N^{l_N} \times$$
$$\times \; N\left(\sum_{j=1}^{N} l_j \mu_j, \sum_{j=1}^{N} l_j \sigma_j^2 \right), \tag{19}$$

where

$$\binom{i}{k_1 \dots k_N} = \frac{i!}{k_1! \cdot \ldots \cdot k_N!}.$$

Let $U_1 \sim N(m_1, \tau_1^2)$, $U_2 \sim N(m_2, \tau_2^2)$, U_1 and U_2 be independent. Then

$$(U_1, U_1 + U_2) \sim N(m_1, m_1 + m_2, \tau_1^2, \tau_1^2 + \tau_2^2, \rho),$$

$$\rho = \frac{E(U_1 - m_1)(U_1 + U_2 - m_1 - m_2)}{\tau_1\sqrt{\tau_1^2 + \tau_2^2}} = \frac{E(U_1 - m_1)^2}{\tau_1\sqrt{\tau_1^2 + \tau_2^2}} = \frac{\tau_1}{\sqrt{\tau_1^2 + \tau_2^2}}.$$

Therefore we have, using (18) and (19):

$$(Y_{(i)}, \Lambda) \sim \sum_{k_1+\ldots+k_N=i} \quad \sum_{l_1+\ldots+l_N=n-i} \binom{i}{k_1 \dots k_N}\binom{n-i}{l_1 \dots l_N} \times$$
$$\times \quad \pi_1^{k_1+l_1} \cdot \ldots \cdot \pi_N^{k_N+l_N} \cdot N\Bigg(\sum_{j=1}^N k_j\mu_j, \sum_{j=1}^N (k_j + l_j)\mu_j,$$
$$\sum_{j=1}^N k_j\sigma_j^2, \sum_{j=1}^N (k_j + l_j)\sigma_j^2, \sqrt{\frac{\sum_{j=1}^N k_j\sigma_j^2}{\sum_{j=1}^N (k_j + l_j)\sigma_j^2}}\Bigg). \qquad (20)$$

In particular,

$$\Lambda = Y_{(n)} \sim \sum_{k_1+\ldots+k_N=n} \binom{n}{k_1 \dots k_N} \pi_1^{k_1} \cdot \ldots \cdot \pi_N^{k_N} N\left(\sum_{j=1}^N k_j\mu_j, \sum_{j=1}^N k_j\sigma_j^2\right),$$

but this can be rewritten based on (20):

$$\Lambda \sim \sum_{k_1+\ldots+k_N=i} \quad \sum_{l_1+\ldots+l_N=n-i} \binom{i}{k_1 \dots k_N}\binom{n-i}{l_1 \dots l_N} \pi_1^{k_1+l_1} \cdot \ldots$$
$$\ldots \cdot \pi_N^{k_N+l_N} N\left(\sum_{j=1}^N (k_j + l_j)\mu_j, \sum_{j=1}^N (k_j + l_j)\sigma_j^2\right). \qquad (21)$$

Now, we use (11). In our case the prior probabilities for joint distribution of $(Y_{(i)}, \Lambda)$ are

$$p_{k_1 \dots k_N l_1 \dots l_N} = \binom{i}{k_1 \dots k_N}\binom{n-i}{l_1 \dots l_N} \pi_1^{k_1+l_1} \cdot \ldots \cdot \pi_N^{k_N+l_N}.$$

And the posterior probabilities given Λ are, see (13),

$$q_{k_1\ldots k_N l_1\ldots l_N}(\Lambda) = \frac{p_{k_1\ldots k_N l_1\ldots l_N} \cdot \rho^{\Lambda}_{k_1\ldots k_N l_1\ldots l_N}(\Lambda)}{\sum\limits_{k_1+\ldots+k_N=i} \sum\limits_{l_1+\ldots+l_N=n-i} p_{k_1\ldots k_N l_1\ldots l_N} \cdot \rho^{\Lambda}_{k_1\ldots k_N l_1\ldots l_N}(\Lambda)}, \quad (22)$$

where according to (21), $\rho^{\Lambda}_{k_1\ldots k_N l_1\ldots l_N}(\Lambda)$ is a density at point Λ of $N\left(\sum_{j=1}^{N}(k_j+l_j)\mu_j, \sum_{j=1}^{N}(k_j+l_j)\sigma_j^2\right)$.

Next we need

$$E_{k_1\ldots k_N l_1\ldots l_N}(e^{-Y_{(i)}}|\Lambda). \quad (23)$$

The joint distribution $(Y_{(i)}, \Lambda)$ under the class $A_{k_1\ldots k_N l_1\ldots l_N}$ has the following distribution, cf. (20):

$$N\left(\sum_{j=1}^{N} k_j\mu_j, \sum_{j=1}^{N}(k_j+l_j)\mu_j, \sum_{j=1}^{N} k_j\sigma_j^2, \sum_{j=1}^{N}(k_j+l_j)\sigma_j^2, \sqrt{\frac{\sum_{j=1}^{N} k_j\sigma_j^2}{\sum_{j=1}^{N}(k_j+l_j)\sigma_j^2}}\right). \quad (24)$$

We use the next well-known Regression Theorem.

Theorem 2 (Regression Theorem.) *Let $\xi \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then a conditional density equals $f_{\xi_1|\xi_2}(x|z) \sim N(m(z), \sigma_1^2(1-\rho^2))$, where $m(z) = E(\xi_1|\xi_2 = z) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(z-\mu_2)$.*

We have for a conditional density, if the joint distribution of $Y_{(i)}$ and Λ equals (24), that

$$f_{Y_{(i)}|\Lambda}(y|\lambda) \sim N(m(\lambda), \widetilde{\sigma_1}^2(1-\rho^2)), \quad (25)$$

$$\widetilde{\sigma_1}^2 := \sum_{j=1}^{N} k_j\sigma_j^2,$$

$$\widetilde{\sigma_1}^2(1-\rho^2) = \frac{\sum_{j=1}^{N} k_j\sigma_j^2}{\sum_{j=1}^{N}(k_j+l_j)\sigma_j^2}\left(\sum_{j=1}^{N} l_j\sigma_j^2\right), \quad (26)$$

and

$$m(\lambda) = \widetilde{\mu_1} + \rho\frac{\widetilde{\sigma_1}}{\widetilde{\sigma_2}}(\lambda - \widetilde{\mu_2}),$$

$$m(\lambda) = \sum_{j=1}^{N} k_j\mu_j + \sqrt{\frac{\rho^2\widetilde{\sigma_1}^2}{\widetilde{\sigma_2}^2}}\left(\lambda - \sum_{j=1}^{N}(k_j+l_j)\mu_j\right). \quad (27)$$

Here

$$\frac{\rho^2 \widetilde{\sigma_1}^2}{\widetilde{\sigma_2}^2} = \left(\frac{\sum_{j=1}^N k_j \sigma_j^2}{\sum_{j=1}^N (k_j + l_j)\sigma_j^2} \right)^2 .$$

Thus

$$m(\lambda) = \sum_{j=1}^N k_j \mu_j + \frac{\sum_{j=1}^N k_j \sigma_j^2}{\sum_{j=1}^N (k_j + l_j)\sigma_j^2} \left(\lambda - \sum_{j=1}^N (k_j + l_j)\mu_j \right). \qquad (28)$$

As a result we have a conditional density $f_{Y_{(i)}|\Lambda}(y|\lambda)$, under the class $A_{k_1 \ldots k_N l_1 \ldots l_N}$, if a joint distribution of $(Y_{(i)}, \Lambda)$ equals (24).

Next,

$$E_{k_1 \ldots k_N l_1 \ldots l_N}(e^{-Y_{(i)}}|\Lambda) = E[e^{-m(\lambda) + \widetilde{\sigma_1}\sqrt{1-\rho^2}Z}|\Lambda],$$

where $Z \sim N(0,1)$, and Z is independent of Λ. Then

$$E_{k_1 \ldots k_N l_1 \ldots l_N}(e^{-Y_{(i)}}|\Lambda) = e^{-m(\lambda)} e^{\frac{\widetilde{\sigma_1}^2 (1-\rho^2)}{2}}.$$

Finally

$$S \geq_{cx} \sum_{i=1}^n \alpha_i \sum_{k_1 + \ldots + k_N = i} \; \sum_{l_1 + \ldots + l_N = n-i} q_{k_1 \ldots k_N l_1 \ldots l_N}(\Lambda) \times$$
$$\times \quad \exp\left\{ -m(\Lambda) + \frac{1}{2} \frac{\sum_{j=1}^N k_j \sigma_j^2}{\sum_{j=1}^N (k_j + l_j)\sigma_j^2} \left(\sum_{j=1}^N l_j \sigma_j^2 \right) \right\},$$

where $q_{k_1 \ldots k_N l_1 \ldots l_N}$ is given in (22), and $m(\Lambda)$ is given in (28), where we plug-in Λ instead of λ.

4.2. Mixture of N independent normals in Markovian way

In Yang (2006) a simple discrete-time model, consisting of one bank and one risky asset is considered. Trading of assets is allowed only at the beginning of each time period. The distribution of the return of risky asset depends on the market mode, which can switch among a finite number of states. Switching of the regime can represent a change in the economic environment. Regime is assumed to switch among a finite number of possible states in Markovian way.

We consider the problem similar to Section 4.1. We want to bound the sum

$$S := \sum_{i=1}^{n} \alpha_i e^{-(Y_1^{\xi_1}+\ldots+Y_i^{\xi_i})}, \tag{29}$$

where $Y_1^{\xi_1}, \ldots, Y_n^{\xi_n}$ are rv's, and $\{\xi_1, \ldots, \xi_n\}$ is a finite-state, time-homogeneous Markov chain, with phase space $S = \{1, \ldots s\}$. The transition probability matrix is denoted as

$$P = \left(\widetilde{p}_{ij}\right)_{i,j=1}^{s},$$

where $\sum_{j=1}^{s} \widetilde{p}_{ij} = 1$, $\forall i = \overline{1,s}$. Denote also $P(\xi_1 = k) = \widetilde{q}_k$, $\forall k = \overline{1,s}$, $\sum_{k=1}^{s} \widetilde{q}_k = 1$.

Let the conditional distribution of $Y_i^{\xi_i}$ given $\xi_i = k$ be

$$L(Y_i^{\xi_i}|\xi_i = k) = L(Y_i^k) = L(Y_1^k) = N(\mu_k, \sigma_k^2),\ k = \overline{1,s},\ \forall i = \overline{1,n}, \tag{30}$$

where $(\mu_i, \sigma_i^2) \neq (\mu_j, \sigma_j^2)$, $i \neq j$, $\sigma_i > 0$, $i = \overline{1,s}$, and the normals are independent.

Therefore $Y_1^k, \ldots, Y_n^k$ are i.i.d. rv's, for all $k = \overline{1,s}$.

We consider the conditioning r.v.

$$\Lambda := \sum_{i=1}^{n} Y_i^{\xi_i}.$$

We have by (10)

$$\sum_{i=1}^{n} \alpha_i E[e^{-(Y_1^{\xi_1}+\ldots+Y_i^{\xi_i})}|\Lambda] \leq_{cx} S. \tag{31}$$

Consider a joint distribution of $Y_{(i)} := Y_1^{\xi_1} + \ldots + Y_i^{\xi_i}$ and $\Lambda = Y_{(i)} + Y^{(i)}$, where $Y^{(i)} := Y_{i+1}^{\xi_{i+1}} + \ldots + Y_n^{\xi_n}$.

Let

$$\begin{aligned} A_{k_1\ldots k_s m} = \{(i_1, \ldots i_m) \in \{1, \ldots, s\}^m | \{i_1, \ldots i_m\} = \\ = \{\underbrace{1, \ldots, 1}_{k_1}, \ldots, \underbrace{s, \ldots, s}_{k_s}\}\}, \end{aligned} \tag{32}$$

$$k_1 + \ldots + k_s = m, \qquad 0 \leq k_i \leq m, \qquad i = \overline{1,s}, \qquad \forall m = \overline{0,n}.$$

Then introduce $a_{k_1\ldots k_s m} = a_{k_1\ldots k_s m}(\widetilde{q}_1, \ldots, \widetilde{q}_s, P)$ in the next way,

$$a_{k_1\ldots k_s m} = \sum_{(i_1,\ldots i_m)\in A_{k_1\ldots k_s m}} \widetilde{q}_{i_1}\widetilde{p}_{i_1 i_2}\widetilde{p}_{i_2 i_3} \cdot \ldots \cdot \widetilde{p}_{i_{m-1} i_m}. \tag{33}$$

We have

$$L(Y_{(i)}) = \sum_{k_1+\ldots+k_s=i} a_{k_1\ldots k_s i} N\left(\sum_{j=1}^{s} k_j\mu_j, \sum_{j=1}^{s} k_j\sigma_j^2\right), \tag{34}$$

$$L(Y^{(i)}) = L(Y_{(n-i)}) = \sum_{l_1+\ldots+l_s=n-i} a_{l_1\ldots l_s n-i} N\left(\sum_{j=1}^{s} l_j\mu_j, \sum_{j=1}^{s} l_j\sigma_j^2\right), \tag{35}$$

Similarly to Section 4.1 we obtain, using (34) and (35):

$$(Y_{(i)}, \Lambda) \sim \sum_{k_1+\ldots+k_s=i} \sum_{l_1+\ldots+l_s=n-i} a_{k_1\ldots k_s i} a_{l_1\ldots l_s n-i} \cdot N\Bigg(\sum_{j=1}^{s} k_j\mu_j, \\ \sum_{j=1}^{s}(k_j+l_j)\mu_j, \sum_{j=1}^{s} k_j\sigma_j^2, \sum_{j=1}^{s}(k_j+l_j)\sigma_j^2, \sqrt{\frac{\sum_{j=1}^{s} k_j\sigma_j^2}{\sum_{j=1}^{s}(k_j+l_j)\sigma_j^2}}\Bigg). \tag{36}$$

In particular,

$$\Lambda = Y_{(n)} \sim \sum_{k_1+\ldots+k_s=n} a_{k_1\ldots k_s n} N\left(\sum_{j=1}^{s} k_j\mu_j, \sum_{j=1}^{s} k_j\sigma_j^2\right),$$

but this also can be written from (36):

$$\Lambda \sim \sum_{k_1+\ldots+k_s=i} \sum_{l_1+\ldots+l_s=n-i} a_{k_1\ldots k_s i} a_{l_1\ldots l_s n-i} \times \\ \times \quad N\left(\sum_{j=1}^{s}(k_j+l_j)\mu_j, \sum_{j=1}^{s}(k_j+l_j)\sigma_j^2\right). \tag{37}$$

Now we use (11). In our case the prior probabilities for joint distribution of $(Y_{(i)}, \Lambda)$ are

$$p_{k_1\ldots k_s l_1\ldots l_s} = a_{k_1\ldots k_s i} a_{l_1\ldots l_s n-i}.$$

And the posterior probabilities given Λ are, see (13),

$$q_{k_1\ldots k_s l_1\ldots l_s}(\Lambda) = \frac{p_{k_1\ldots k_s l_1\ldots l_s} \cdot \rho^{\Lambda}_{k_1\ldots k_s l_1\ldots l_s}(\Lambda)}{\sum_{k_1+\ldots+k_s=i} \sum_{l_1+\ldots+l_s=n-i} p_{k_1\ldots k_s l_1\ldots l_s} \cdot \rho^{\Lambda}_{k_1\ldots k_s l_1\ldots l_s}(\Lambda)}, \tag{38}$$

where according to (37), $\rho^{\Lambda}_{k_1\ldots k_s l_1\ldots l_s}(\Lambda)$ is a density at point Λ of $N\left(\sum_{j=1}^{s}(k_j+l_j)\mu_j, \sum_{j=1}^{s}(k_j+l_j)\sigma_j^2\right)$.

Next we need

$$E_{k_1\dots k_s l_1\dots l_s}(e^{-Y_{(i)}}|\Lambda), \tag{39}$$

i.e., conditional expectation provided $(Y_{(i)}, \Lambda)$ has the following distribution, cf. (36):

$$N\left(\sum_{j=1}^{s} k_j\mu_j, \sum_{j=1}^{s}(k_j+l_j)\mu_j, \sum_{j=1}^{s} k_j\sigma_j^2, \sum_{j=1}^{s}(k_j+l_j)\sigma_j^2, \sqrt{\frac{\sum_{j=1}^{s} k_j\sigma_j^2}{\sum_{j=1}^{s}(k_j+l_j)\sigma_j^2}}\right) \tag{40}$$

Similarly to Section 4.1 we have that

$$S \geq_{cx} \sum_{i=1}^{n} \alpha_i \sum_{k_1+\dots+k_s=i} \sum_{l_1+\dots+l_s=n-i} q_{k_1\dots k_s l_1\dots l_s}(\Lambda) \times$$
$$\times \quad \exp\left\{-m(\Lambda) + \frac{1}{2}\frac{\sum_{j=1}^{s} k_j\sigma_j^2}{\sum_{j=1}^{s}(k_j+l_j)\sigma_j^2}\left(\sum_{j=1}^{s} l_j\sigma_j^2\right)\right\}.$$

Here $q_{k_1\dots k_N l_1\dots l_N}$ is given in (38), and $m(\Lambda)$ is given in (28), where we plug-in Λ instead of λ, and s instead of N.

5. Approximate evaluation of provisions under switching regime: upper bound

5.1. Mixture of N independent normals in linear way

We keep the notation from Section 4.1. From (1) we have

$$S \leq_{cx} \sum_{i=1}^{n} F^{-1}_{\alpha_i X_i|\Lambda}(U), \quad X_i := e^{-Y_{(i)}}, \quad i = \overline{1, n},$$

where U is uniform (0,1) and independent of Λ.

Now, we assume that $\alpha_i \neq 0$, for all $i = \overline{1, n}$. Then

$$F_{\alpha_i X_i|\Lambda=\lambda}(z) = P\{\alpha_i e^{-Y_{(i)}} \leq z|\Lambda = \lambda\}.$$

If $\alpha_i > 0$ then for $z > 0$

$$F_{\alpha_i X_i|\Lambda=\lambda}(z) = P\{Y_{(i)} \geq -\log\frac{z}{\alpha_i}|\Lambda = \lambda\} = \overline{F}_{Y_{(i)}|\Lambda=\lambda}(-\log\frac{z}{\alpha_i}).$$

Here we suppose that the conditional distribution is continuous, and $\overline{F} = 1 - F$ is a survival function.

Otherwise if $\alpha_i < 0$ then for $z > 0$

$$F_{\alpha_i X_i|\Lambda=\lambda}(z) = P\{Y_{(i)} \le -\log \frac{z}{\alpha_i}|\Lambda=\lambda\} = F_{Y_{(i)}|\Lambda=\lambda}(-\log \frac{z}{\alpha_i}).$$

In all the cases we need a conditional distribution of $Y_{(i)}$ provided $\Lambda = \lambda$. Such a distribution, for the class $A_{k_1\ldots k_N l_1 \ldots l_N}$, is given in (25). And the final conditional law of $Y_{(i)}$ provided $\Lambda = \lambda$, equals the mixture of independent normals:

$$\sum_{k_1+\ldots+k_N=i} \quad \sum_{l_1+\ldots+l_N=n-i} q_{k_1\ldots k_N l_1\ldots l_N}(\lambda) N\big(m_{k_1\ldots k_N l_1 \ldots l_N}(\lambda), \widetilde{\sigma_1}^2(k_1\ldots k_N)(1-\rho^2(k_1\ldots k_N l_1 \ldots l_N))\big).$$

Here $m_{k_1\ldots k_N l_1\ldots l_N}(\lambda)$ is given in (28), and $\widetilde{\sigma_1}^2(k_1\ldots k_N)(1-\rho^2(k_1\ldots k_N$ $l_1 \ldots l_N))$ is given in (26). Thus everything is ready to compute the upper bound numerically.

5.2. Mixture of N independent normals in Markovian way

The upper bound in this case can be taken from Section 5.1, where we have to plug-in $q_{k_1\ldots k_N l_1\ldots l_s}(\lambda)$ from (38) instead of $q_{k_1\ldots k_N l_1\ldots l_N}(\lambda)$, $m_{k_1\ldots k_N l_1 \ldots l_N}(\lambda)$ from (28), and $\widetilde{\sigma_1}^2(k_1\ldots k_N)(1-\rho^2(k_1\ldots k_N l_1\ldots l_N))$ from (26) with s instead of N.

6. Numerical illustrations

In this section, we illustrate numerically the bounds we derived for $S = \sum_{i=1}^5 \alpha_i e^{-(Y_1+Y_2+\ldots+Y_i)}$. We assume that the random variables Y_i are i.i.d. and have the distribution $\pi_1 N(\mu_1,\sigma_1^2)+\pi_2 N(\mu_2,\sigma_2^2)$. The conditional random variable Λ is defined as above:

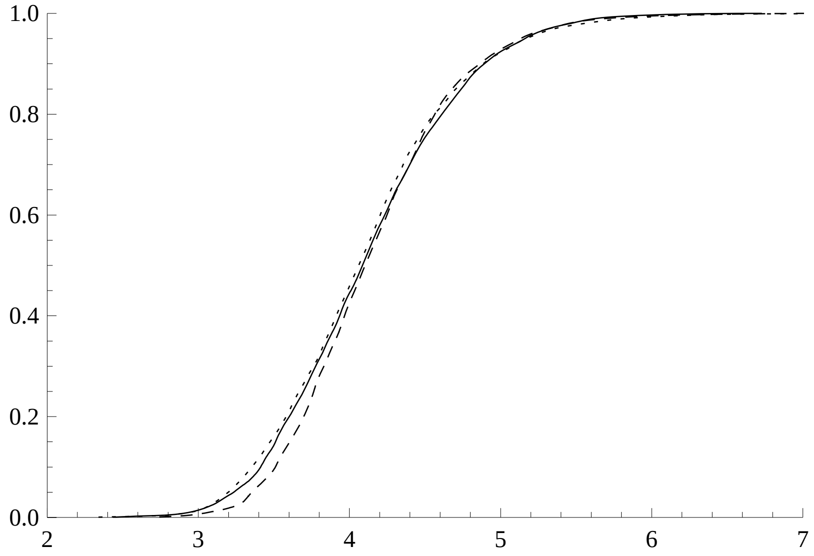
$$\Lambda = \sum_{i=1}^5 Y_i.$$

In our numerical illustration, we choose the parameters of the involved normal distributions as follows:

$\pi_1 = 0.25$, $\mu_1 = 0.04$, $\sigma_1^2 = 0.07$, $\pi_2 = 0.75$, $\mu_2 = 0.08$, $\sigma_2^2 = 0.01$.

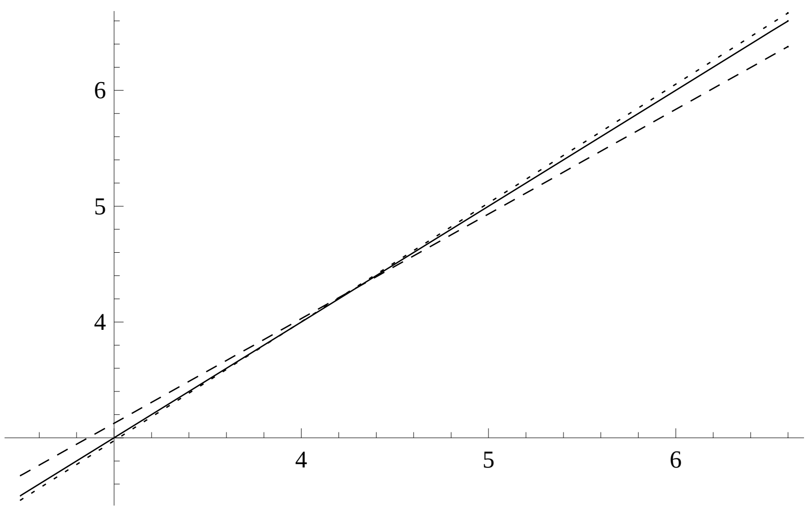
First we show the cdf's of S(solid black line), S^l-lower bound (dashed line), S^u-upper bound (dotted line) for the following payments:

$\alpha_k = 1, \qquad k = \overline{1,5}.$



In order to have a better view on the behaviour of the lower S^l and upper S^u bounds in the tails, we consider a QQ-plot where the quantiles of S^l and S^u are plotted against the quantiles of S obtained by simulation. The lower S^l and upper S^u bounds will be a good approximation for S if the plotted points $(F_S^{-1}(p), F_{S^l}^{-1}(p))$ and $(F_S^{-1}(p), F_{S^u}^{-1}(p))$ for all values of p in (0,1) do not deviate too much from the straight line $y = x$ respectively.

Hereafter, we present a QQ-plot illustrating the accurateness of the approximations.The dashed line represents the quantiles of the lower bound versus the 'exact' quantiles, whereas the dotted line represents quantiles of the upper bound versus the 'exact' quantiles. The solid black line represents the straight line $y = x$.

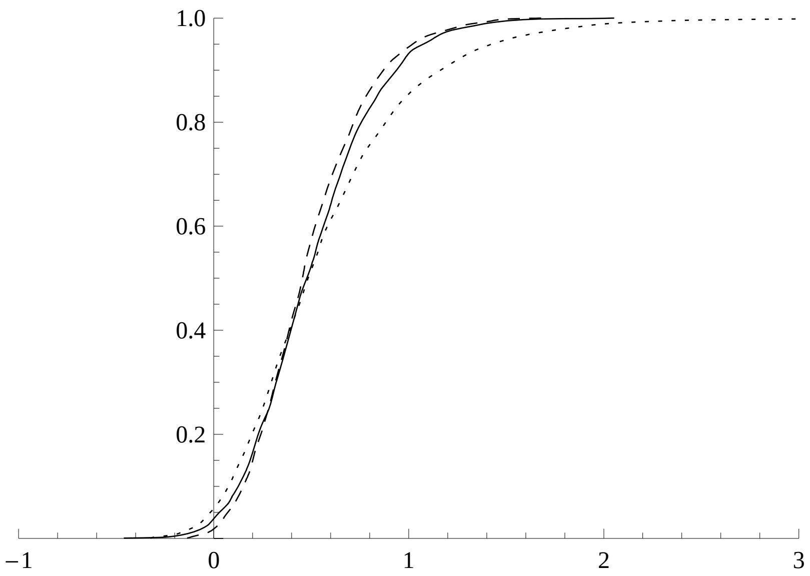


Now we will present some quantiles in the following table.

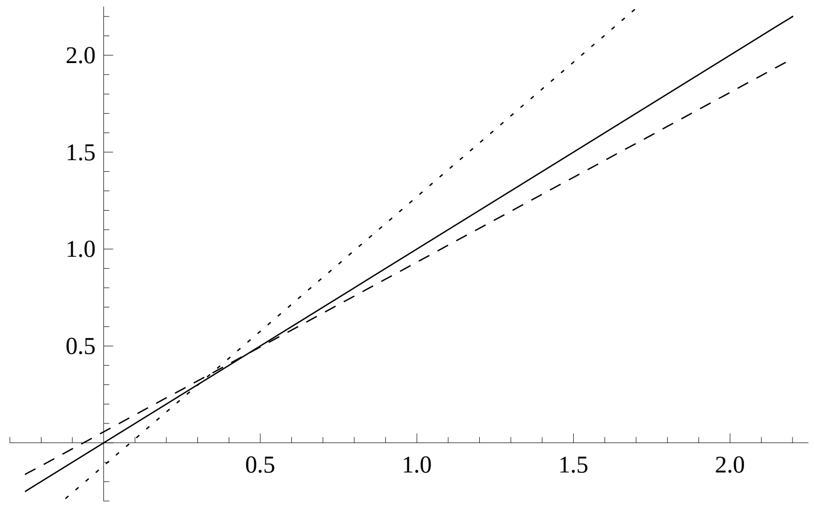
p	$F_{S^l}^{-1}(p)$	$F_S^{-1}(p)$	$F_{S^u}^{-1}(p)$
0.95	5.1384	5.2228	5.2146
0.975	5.4018	5.4581	5.4490
0.99	5.6312	5.8464	5.8273
0.995	5.8545	6.0262	6.0346
0.999	6.2806	6.2921	6.3544

Next, we consider a series of negative and positive payments:

$$a_k = \begin{cases} -1, & k = 1, 2 \\ 1, & k = 3, 4, 5. \end{cases}$$



And a QQ-plot and quantiles are as follows.



p	$F_{S^l}^{-1}(p)$	$F_S^{-1}(p)$	$F_{S^u}^{-1}(p)$
0.95	1.0212	1.0995	1.4312
0.975	1.2001	1.2432	1.7314
0.99	1.3094	1.3675	1.9813
0.995	1.4340	1.4234	2.3021
0.999	1.4873	1.8813	3.2011

The solid black line in the first and third pictures is the ”exact” cdf of S, which was obtained by generating 10,000 quasi-random paths.

7. Conclusions

In the papers Dhaene et al. (2002a) and (2002b) the approximations for sums of rv's were derived when the distributions of the components are lognormal and known, but the stochastic dependence structure is unknown or too cumbersome to work with. Any distribution can be approximated by a mixture of normals, in the sense of weak convergence. We considered the case of mixture of N normals. We got more complicated formulas compared with Dhaene et al. (2002b). Also we considered the case of switching among finite number of possible states in Markovian way. The result can be applied in finance and actuarial science, see Yang (2006).

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