APPORXIMATION OF RANDOM PROCESSES IN
THE SPACE $L_2(T)$

The estimation for distribution of the norms of strictly sub-Gaussian random processes in the space $L_2(T)$ is obtained. The approximation of some classes of strictly sub-Gaussian random processes with given accuracy and reliability is considered.

1. INTRODUCTION

In the paper [3] we constructed the approximations of strictly $\varphi$-sub-Gaussian random processes by broken lines such that this broken line approximates the process with given accuracy and reliability in the norm of $C[0,1]$.

In this paper we consider the approximation of strictly sub-Gaussian random processes by broken lines in the space $L_2(T)$. We obtain the inequality for the norm of strictly sub-Gaussian random process and use it to construct the approximation of the initial process.

We recall some basic facts about strictly sub-Gaussian random processes.

Let $(\Omega, B, P)$ be a standard probability space.

Definition 1. [1] A random variable $\xi$ is called sub-Gaussian ($\xi \in \text{Sub}(\Omega)$), if $E\xi = 0$ and $\exists a > 0$ such that $E \exp\{\lambda \xi\} \leq \exp\left\{\frac{\lambda^2 a^2}{2}\right\}$ for all $\lambda \in \mathbb{R}$.

Proposition 1. [1] The space $\text{Sub}(\Omega)$ is a Banach space with respect to the norm $\tau_{\varphi}(\xi) = \inf\{a \geq 0 : E \exp(\lambda \xi) \leq \exp(\varphi(a\lambda)), \lambda \in \mathbb{R}\}$.

Definition 2. [2] A random variable $\xi$ is called strictly sub-Gaussian if $E\xi = 0$ and $E\xi^2 = \tau^2(\xi)$.

Definition 3. [2] A family $\Delta$ of sub-Gaussian random variables is called strictly sub-Gaussian if for any finite or countable set $\Delta$ of random variables

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\{\xi_i, i \in I\} \text{ and for all } \lambda_i \in R: 
\tau^2 \left( \sum_{i \in I} \lambda_i \xi_i \right) = E \left( \sum_{i \in I} \lambda_i \xi_i \right)^2.

**Definition 4.** [2] A vector $\overrightarrow{\xi}^T = (\xi_1, ..., \xi_n)$, where $\xi_k$ are random variables from the family of strictly sub-Gaussian random variables, is called a strictly sub-Gaussian random vector.

**Definition 5.** [2] A random process $X = \{X(t), t \in T\}$ is called a strictly sub-Gaussian ($X(t) \in SSub(\Omega)$) if a family of random variables $\{X(t), t \in T\}$ is strictly sub-Gaussian.

Let $X = \{X(t), t \in T\}, T = [0, 1]$, be a strictly sub-Gaussian process.

Denote by $S := \{t_k\}_{k=0}^{k=N} = \{\frac{k}{N}, k = 0, N\}$ the uniform partition of the segment $[0, 1]$ into $N$ parts. We approximate the random process $\{X(t), t \in T\}$ by an interpolation broken line $X_N(t)$ for given values $\{X(t_k)\}, k = 0, N$, i.e.

$$X_N(t) = \alpha_1 X(t_k) + \alpha_2 X(t_{k+1}), t \in [t_k, t_{k+1}], k = 0, N - 1,$$

where $\alpha_1 = 1 - (t - t_k)N, \alpha_2 = (t - t_k)N$.

The problem is to restore the process $\{X(t), t \in T\}$ by the broken line $\{X_N(t), t \in T\}$ with given accuracy $\varepsilon$ and reliability $1 - \delta$ in the norm of $L_2(T)$ knowing the values of given process in corresponding points $\{k/N, k = 0, N\}$.

Denote by $Y_N(t) := X(t) - X_N(t), t \in T$, the deviation random process.

We assume that for given process $\{X(t), t \in T\}$ the next inequality is satisfied:

$$\sup_{t \in T} E|X(t + h) - X(t)|^2 \leq b^2(h), \quad (1)$$

where $b(h), h > 0$ is a known monotonically increasing continuous function and $b(h) \downarrow 0$ as $h \downarrow 0$.

As an example we consider power an logarithmic deviation functions $b(h)$.

**2. Accuracy of Approximation of Strictly Sub-Gaussian Processes in $L_2(T)$**

**Definition 6.** The broken line $X_N(t)$ approximates the process $X(t)$ with given accuracy $\varepsilon > 0$ and reliability $1 - \delta, 0 < \delta < 1$ in $L_2(T)$ if the next inequality is satisfied:

$$P \left\{ \left( \int_T |X(t) - X_N(t)|^2 dt \right)^{1/2} > \varepsilon \right\} \leq \delta.$$
Theorem. Let $X = \{X(t), t \in T\}$ be a strictly sub-Gaussian random process, $(T, L, \mu)$ be a measurable space. Assume $\int_T (EX^2(t))d\mu(t) < \infty$, then with probability one there exists $\int_T X^2(t)d\mu(t)$ and for any $\varepsilon > \int_T (EX^2(t))d\mu(t)$ the inequality holds

$$P \left\{ \int_T X^2(t)d\mu(t) > \varepsilon \right\} \leq$$

$$\leq e^{\frac{1}{2}} \left( \frac{\varepsilon}{\int_T (EX^2(t))d\mu(t)} \right)^{\frac{1}{2}} \exp \left\{ -\frac{\varepsilon}{2 \int_T (EX^2(t))d\mu(t)} \right\}.$$  \hspace{1cm} (2)

Proof. The existence of $\int_T X^2(t)d\mu(t)$ follows from the Fubini's theorem.

Assume $\xi^T = (\xi_1, ..., \xi_n)$ is a strictly sub-Gaussian random vector, $A$ - a symmetrical non-negatively defined matrix, $\eta = \xi^T A \xi$, then for $\varepsilon > Z_1$ the next inequality is satisfied (ex. 1.2.2, [2]):

$$P\{\eta > \varepsilon\} \leq e^{\frac{1}{2}} \left( \frac{\varepsilon}{Z_1} \right)^{\frac{1}{2}} \exp \left\{ -\frac{\varepsilon}{2Z_1} \right\},$$  \hspace{1cm} (3)

where $Z_1 = E \xi^T A \xi$. Let $\Lambda = \{t_i\}_{i=0}^{i=n} = \{0 = t_0 < ... < t_n = 1\}$ be a partition of the segment $T$. Let $\xi_i = X(t_i), i = 1, n$ and let

$$A = \begin{pmatrix}
\sqrt{\Delta t_1} & 0 & \ldots & 0 \\
0 & \sqrt{\Delta t_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \sqrt{\Delta t_n}
\end{pmatrix}.$$

Then the inequality (3) becomes

$$P \left\{ \sum_{i=1}^{n} X^2(t_i)\Delta t_i > \varepsilon \right\} \leq$$

$$\leq e^{\frac{1}{2}} \left( \frac{\varepsilon}{\sum_{i=1}^{n} X^2(t_i)\Delta t_i} \right)^{\frac{1}{2}} \exp \left\{ -\frac{\varepsilon}{2 \sum_{i=1}^{n} X^2(t_i)\Delta t_i} \right\},$$

where $\varepsilon > E \sum_{i=1}^{n} X^2(t_i)\Delta t_i$. 


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In the last inequality we proceed to the limit in the mean square when $\max_{1\leq i\leq n} \Delta t_i \to 0$. As $\int T X^2(t)dt = l.i.m. \sum_{i=1}^n X^2(t_i) \Delta t_i$, we obtain (2). □

3. SOME EXAMPLES OF APPROXIMATION IN $L^2(T)$

As the process $X = \{X(t), t \in T\}$ is a strictly sub-Gaussian, the processes $\{X_N(t), t \in T\}$ and $\{Y_N(t), t \in T\}$ are also strictly sub-Gaussian ([3]).

Let’s apply the theorem above to the deviation process $Y_N(t)$.

Assume the process $\{X(t), t \in T\}$ is a stationary. The right side of the expression in (2) increases on $\int T (EX^2(t))d\mu(t)$ (if $\int T (EX^2(t))d\mu(t) > \varepsilon$) so using the inequality $\sup_{t \in T} EY^2_N(t) \leq b^2(\frac{1}{N})$ ([3]), we obtain the next estimation:

$$P\{|Y_N(t)|_{L^2} > \varepsilon\} \leq \frac{\varepsilon^2}{b^2(\frac{1}{N})} \cdot \exp \left\{ \frac{-\varepsilon^2}{2b^2(\frac{1}{N})} \right\},$$

where $\varepsilon > b(\frac{1}{N})$.

So the desired rate of interpolation $N$ for approximation of stationary strictly sub-Gaussian random process by the broken line with given accuracy $\varepsilon > 0$ and reliability $1 - \delta, 0 < \delta < 1$ in $L^2([0,1])$ can be found from the inequalities

$$\left\{ \begin{array}{l}
\varepsilon > b(\frac{1}{N}), \\
\frac{-\varepsilon^2}{2b^2(\frac{1}{N})} \leq \delta,
\end{array} \right.$$

where $b(h)$ is a deviation function of the process $X(t)$.

**Example 1.** Power function $b(h)$.

Assume in (1) $b(h) = ch^\alpha, 0 \leq \alpha \leq 1, c$ is a positive constant.

Let $\varepsilon = 0.01, \delta = 0.01, c = 1, \alpha = 1$. Then the condition (4) is satisfied for $N \geq 358$.

**Example 2.** Logarithmic function $b(h)$.

Assume in (1) $b(h) = \frac{\varepsilon}{(\ln(1+\frac{1}{h}))^\mu}, \mu > \frac{1}{2}, c$ is a positive constant.

Let $\varepsilon = 0.01, \delta = 0.01, c = 1, \mu = 4$. Then we obtain that the condition (4) is satisfied for $N \geq 1204$.

**REFERENCES**


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