Theory of Stochastic Processes
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APPROXIMATION OF RANDOM PROCESSES IN THE SPACE $L_2(T)$

The estimation for distribution of the norms of strictly sub-Gaussian random processes in the space $L_2(T)$ is obtained. The approximation of some classes of strictly sub-Gaussian random processes with given accuracy and reliability is considered.

1. Introduction

In the paper [3] we constructed the approximations of strictly φ-sub-Gaussian random processes by broken lines such that this broken line approximates the process with given accuracy and reliability in the norm of $C[0,1]$.

In this paper we consider the approximation of strictly sub-Gaussian random processes by broken lines in the space $L_2(T)$. We obtain the inequality for the norm of strictly sub-Gaussian random process and use it to construct the approximation of the initial process.

We recall some basic facts about strictly sub-Gaussian random processes.

Let (Ω, B, P) be a standard probability space.

Definition 1. [1] A random variable ξ is called sub-Gaussian ($\xi \in Sub(\Omega)$), if $E\xi = 0$ and $\exists a > 0$ such that $E\exp\{\lambda\xi\} \le \exp\left\{\frac{\lambda^2 a^2}{2}\right\}$ for all $\lambda \in R^1$.

Proposition 1. [1] The space $Sub(\Omega)$ is a Banach space with respect to the norm $\tau_\varphi(\xi) = \inf\{a \ge 0 : E\exp(\lambda\xi) \le \exp(\varphi(a\lambda)), \lambda \in R\}$.

Definition 2. [2] A random variable ξ is called strictly sub-Gaussian if $E\xi = 0$ and $E\xi^2 = \tau^2(\xi)$.

Definition 3. [2] A family Δ of sub-Gaussian random variables is called strictly sub-Gaussian if for any finite or countable set Δ of random variables

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$\{\xi_i, i \in I\}$ and for all $\lambda_i \in R : \tau^2 \left(\sum\limits_{i\in I} \lambda_i \xi_i\right) = E \left(\sum\limits_{i\in I} \lambda_i \xi_i\right)^2$.

Definition 4. [2] A vector $\overrightarrow{\xi}^T = (\xi_1, ..., \xi_n)$, where ξ_k are random variables from the family of strictly sub-Gaussian random variables, is called a strictly sub-Gaussian random vector.

Definition 5. [2] A random process $X = \{X(t), t \in T\}$ is called a strictly sub-Gaussian ($X(t) \in SSub(\Omega)$) if a family of random variables $\{X(t), t \in T\}$ is strictly sub-Gaussian.

Let $X = \{X(t), t \in T\}, T = [0, 1]$, be a strictly sub-Gaussian process.

Denote by $S := \{t_k\}_{k=0}^{k=N} = \{\frac{k}{N}, k = \overline{0, N}\}$ the uniform partition of the segment $[0, 1]$ into N parts. We approximate the random process $\{X(t), t \in T\}$ by an interpolation broken line $X_N(t)$ for given values $\{X(t_k)\}, k = \overline{0, N}$, i.e.

$$X_N(t) = \alpha_1 X(t_k) + \alpha_2 X(t_{k+1}), t \in [t_k, t_{k+1}], k = \overline{0, N-1},$$

where $\alpha_1 = 1 - (t - t_k)N$, $\alpha_2 = (t - t_k)N$.

The problem is to restore the process $\{X(t), t \in T\}$ by the broken line $\{X_N(t), t \in T\}$ with given accuracy ε and reliability $1 - \delta$ in the norm of $L_2(T)$ knowing the values of given process in corresponding points $\{k/N, k = \overline{0, N}\}$.

Denote by $Y_N(t) := X(t) - X_N(t), t \in T$, the deviation random process.

We assume that for given process $\{X(t), t \in T\}$ the next inequality is satisfied:

$$\sup_{t\in T} E|X(t+h) - X(t)|^2 \le b^2(h), \tag{1}$$

where $b(h), h > 0$ is a known monotonically increasing continuous function and $b(h) \downarrow 0$ as $h \downarrow 0$.

As an example we consider power an logarithmic deviation functions $b(h)$.

2. Accuracy of approximation of strictly sub-Gaussian processes in $L_2(T)$

Definition 6. The broken line $X_N(t)$ approximates the process $X(t)$ with given accuracy $\varepsilon > 0$ and reliability $1 - \delta, 0 < \delta < 1$ in $L_2(T)$ if the next inequality is satisfied:

$$P\left\{\left(\int_T |X(t) - X_N(t)|^2 \, dt\right)^{1/2} > \varepsilon\right\} \le \delta.$$

Theorem. *Let $X = \{X(t), t \in T\}$ be a strictly sub-Gaussian random process, (T, L, μ) be a measurable space. Assume $\int_T (EX^2(t))d\mu(t) < \infty$, then with probability one there exists $\int_T X^2(t)d\mu(t)$ and for any $\varepsilon > \int_T (EX^2(t))d\mu(t)$ the inequality holds*

$$P\left\{\int_T X^2(t)d\mu(t) > \varepsilon\right\} \leq$$

$$\leq e^{\frac{1}{2}}\left(\frac{\varepsilon}{\int_T (EX^2(t))d\mu(t)}\right)^{\frac{1}{2}} \cdot \exp\left\{\frac{-\varepsilon}{2\int_T (EX^2(t))d\mu(t)}\right\}. \qquad (2)$$

Proof. The existence of $\int_T X^2(t)d\mu(t)$ follows from the Fubini's theorem.

Assume $\overrightarrow{\xi}^T = (\xi_1, ..., \xi_n)$ is a strictly sub-Gaussian random vector, A – a symmetrical non-negatively defined matrix, $\eta = \overrightarrow{\xi}^T A \overrightarrow{\xi}$, then for $\varepsilon > Z_1$ the next inequality is satisfied (ex. 1.2.2, [2]):

$$P\{\eta > \varepsilon\} \leq e^{\frac{1}{2}}\left(\frac{\varepsilon}{Z_1}\right)^{\frac{1}{2}} \exp\left\{-\frac{\varepsilon}{2Z_1}\right\}, \qquad (3)$$

where $Z_1 = E\overrightarrow{\xi}^T A \overrightarrow{\xi}$. Let $\Lambda = \{t_i\}_{i=0}^{i=n} = \{0 = t_0 < ... < t_n = 1\}$ be a partition of the segment T. Let $\xi_i = X(t_i), i = \overline{1,n}$ and let

$$A = \begin{pmatrix} \sqrt{\Delta t_1} & 0 & \dots & 0 \\ 0 & \sqrt{\Delta t_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{\Delta t_n} \end{pmatrix}.$$

Then the inequality (3) becomes

$$P\left\{\sum_{i=1}^n X^2(t_i)\Delta t_i > \varepsilon\right\} \leq$$

$$\leq e^{\frac{1}{2}}\left(\frac{\varepsilon}{E\sum_{i=1}^n X^2(t_i)\Delta t_i}\right)^{\frac{1}{2}} \exp\left\{-\frac{\varepsilon}{2E\sum_{i=1}^n X^2(t_i)\Delta t_i}\right\},$$

where $\varepsilon > E\sum_{i=1}^n X^2(t_i)\Delta t_i$.

In the last inequality we proceed to the limit in the mean square when $\max_{1\le i\le n}\Delta t_i \to 0$. As $\int\limits_T X^2(t)dt = l.i.m. \sum\limits_{i=1}^{n} X^2(t_i)\Delta t_i$, we obtain (2). $\square$

3. Some examples of approximation in $L_2(T)$

As the process $X = \{X(t), t \in T\}$ is a strictly sub-Gaussian, the processes $\{X_N(t), t \in T\}$ and $\{Y_N(t), t \in T\}$ are also strictly sub-Gaussian ([3]).

Let's apply the theorem above to the deviation process $Y_N(t)$.

Assume the process $\{X(t), t \in T\}$ is a stationary. The right side of the expression in (2) increases on $\int\limits_T (EX^2(t))d\mu(t)$ (if $\int\limits_T (EX^2(t))d\mu(t) > \varepsilon$) so using the inequality $\sup_{t\in T} EY_N^2(t) \le b^2(\frac{1}{N})$ ([3]), we obtain the next estimation:

$$P\left\{\|Y_N(t)\|_{L_2} > \varepsilon\right\} \le \frac{e^{\frac{1}{2}}\varepsilon}{b(\frac{1}{N})} \cdot \exp\left\{\frac{-\varepsilon^2}{2b^2(\frac{1}{N})}\right\},$$

where $\varepsilon > b(\frac{1}{N})$.

So the desired rate of interpolation N for approximation of stationary strictly sub-Gaussian random process by the broken line with given accuracy $\varepsilon > 0$ and reliability $1-\delta, 0 < \delta < 1$ in $L_2([0,1])$ can be found from the inequalities

$$\begin{cases} \frac{e^{\frac{1}{2}}\varepsilon}{b(\frac{1}{N})} \cdot \exp\left\{\frac{-\varepsilon^2}{2b^2(\frac{1}{N})}\right\} \le \delta, \\ \varepsilon > b(\frac{1}{N}), \end{cases} \tag{4}$$

where $b(h)$ is a deviation function of the process $X(t)$.

Example 1. Power function $b(h)$.

Assume in (1) $b(h) = ch^\alpha, 0 \le \alpha \le 1, c$ is a positive constant.

Let $\varepsilon = 0.01, \delta = 0.01, c = 1, \alpha = 1$. Then the condition (4) is satisfied for $N \ge 358$.

Example 2. Logarithmic function $b(h)$.

Assume in (1) $b(h) = \frac{c}{(\ln(1+\frac{1}{h}))^\mu}, \mu > \frac{1}{2}, c$ is a positive constant.

Let $\varepsilon = 0.01, \delta = 0.01, c = 1, \mu = 4$. Then we obtain that the condition (4) is satisfied for $N \ge 1204$.

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