Theory of Stochastic Processes
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LIMIT BEHAVIOR OF AUTONOMOUS RANDOM OSCILLATING SYSTEM OF THIRD ORDER

The asymptotic behavior of the general type third order autonomous oscillating system under the action of small non-linear random perturbations of "white" and "Poisson" types is investigated.

1. Introduction

The averaging method proposed by N.M.Krylov, N.N.Bogolyubov and Yu.A.Mytropolskij ([1], [2]) is one of the main tool in studying of the deterministic oscillating systems under the action of small non-linear perturbations. The case of small random "white noise" type disturbances in oscillating systems of second order is considered in papers of Yu.A.Mytropolskij, V.G.Kolomiets ([3]). The autonomous and non-autonomous oscillating systems of second order under the action of "white noise" and Poisson type noise perturbations are studied in the papers of O.V.Borysenko ([4], [5]). Particular case of the third order oscillating systems are investigated in articles of O.D.Borysenko, O.V.Borysenko ([6]), O.D.Borysenko, O.V.Borysenko and I.G.Malyshev ([7], [8]).

This paper deals with investigation of the behaviour, as $\varepsilon \to 0$, of the general type third order autonomous oscillating system described by stochastic differential equation

$$x'''(t) + ax''(t) + b^2x'(t) + ab^2x(t) = \\ = \varepsilon^{k_1} f_1(x(t), x'(t), x''(t)) + f_\varepsilon(x(t), x'(t), x''(t)) \tag{1}$$

with non-random initial conditions $x(0) = x_0, x'(0) = x'_0, x''(0) = x''_0$, where $\varepsilon > 0$ is a small parameter, $f_\varepsilon(x, x', x'')$ is a random function such that

$$\int_0^t f_\varepsilon(x(s), x'(s), x''(s))\, ds = \varepsilon^{k_2} \int_0^t f_2(x(s), x'(s), x''(s))\, dw(s)+$$

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$$+\varepsilon^{k_3}\int_0^t\int_{\mathrm{R}} f_3(x(s),x'(s),x''(s),z)\,\tilde{\nu}(ds,dz),$$

$k_i>0, i=1,2,3$; $f_i, i=1,2,3$ are non-random functions; $w(t)$ is a standard Wiener process; $\tilde{\nu}(dt,dy)=\nu(dt,dy)-\Pi(dy)dt$, $E\nu(dt,dy)=\Pi(dy)dt$, $\nu(dt,dy)$ is the Poisson measure independent on $w(t)$; $\Pi(A)$ is a finite measure on Borel sets $A\in \mathrm{R}$, $a>0, b>0$.

We will consider the equation (1) as the system of stochastic differential equations

$$\begin{aligned}
dx(t) &= x'(t)dt,\\
dx'(t) &= x''(t)dt,\\
dx''(t) &= [-ax''(t)-b^2x'(t)-ab^2x(t)+\\
&\quad +\varepsilon^{k_1}f_1(x(t),x'(t),x''(t))]dt+\\
&\quad +\varepsilon^{k_2}f_2(x(t),x'(t),x''(t))dw(t)+\\
&\quad +\varepsilon^{k_3}\int_{\mathrm{R}} f_3(x(t),x'(t),x''(t),z)\tilde{\nu}(dt,dz),\\
&\quad x(0)=x_0,\ x'(0)=x'_0,\ x''(0)=x''_0.
\end{aligned}\tag{2}$$

In what follows we will use the constant $K>0$ for the notation of different constants, which are not depend on ε.

2. Auxiliary result

From Borysenko O. and Malyshev I. [9], using the obvious modifications we obtain following results

Lemma. *Let for each $x\in\mathrm{R}^d$ there exists*

$$\lim_{T\to\infty}\frac{1}{T}\int_A^{T+A} f(t,x)\,dt=\bar{f}(x)$$

uniformly with respect to A, the function $\bar{f}(x)$ is bounded, continuous, function $f(t,x)$ is bounded and continuous in x uniformly with respect to (t,x) in any region $t\in[0,\infty), |x|\le K$, and stochastic processes $\xi(t)\in\mathrm{R}^d$, $\eta(t)\in\mathrm{R}$ are continuous, then

$$\lim_{\varepsilon\to 0}\int_0^t f\left(\frac{s}{\varepsilon}+\eta(s),\xi(s)\right)ds=\int_0^t \bar{f}(\xi(s))\,ds$$

almost surely for all arbitrary $t\in[0,T]$.

Remark. Let $f(t,x,z)$ is bounded and uniformly continuous in x with respect to $t\in[0,\infty)$ and $z\in \mathrm{R}$ in every compact set $|x|\le K, x\in \mathrm{R}^d$. Let $\Pi(\cdot)$ be a finite measure on the σ-algebra of Borel sets in R and let

$$\lim_{T\to\infty}\frac{1}{T}\int_A^{T+A} f(t,x,z)\,dt=\bar f(x,z),$$

uniformly with respect to A for each $x\in\mathrm{R}^d, z\in\mathrm{R}$, where $\bar f(x,z)$ is bounded, uniformly continuous in x with respect to $z\in\mathrm{R}$ in every compact set $|x|\le K$. Then for any continuous processes $\xi(t)\in\mathrm{R}^d$ and $\eta(t)\in\mathrm{R}$ we have

$$\lim_{\varepsilon\to0}\int_0^t\int_{\mathrm{R}} f\left(\frac{s}{\varepsilon}+\eta(s),\xi(s),z\right)\Pi(dz)ds=\int_0^t\int_{\mathrm{R}}\bar f(\xi(s),z)\,\Pi(dz)ds.$$

3. Main result

Let us consider the following representation of processes $x(t), x'(t), x''(t)$:

$$x(t)=C(t)\exp\{-at\}+A_1(t)\cos(bt)+A_2(t)\sin(bt),$$

$$x'(t)=-aC(t)\exp\{-at\}-bA_1(t)\sin(bt)+bA_2(t)\cos(bt),$$
$$x''(t)=a^2C(t)\exp\{-at\}-b^2A_1(t)\cos(bt)-b^2A_2(t)\sin(bt),$$
$$N(t)=C(t)\exp\{-at\}.$$

Then

$$N(t)=\frac{b^2x(t)+x''(t)}{a^2+b^2},$$

$$A_1(t)=\cos\alpha\cos(bt+\alpha)x(t)-\frac{\sin bt}{b}x'(t)-\frac{\sin\alpha\sin(bt+\alpha)}{b^2}x''(t),$$

$$A_2(t)=\cos\alpha\sin(bt+\alpha)x(t)+\frac{\cos bt}{b}x'(t)+\frac{\sin\alpha\cos(bt+\alpha)}{b^2}x''(t),$$

where $\alpha=\operatorname{arctg}(b/a)$. We can apply Ito formula [10] to stochastic process $\xi(t)=(N(t),A_1(t),A_2(t))$ and obtain for the process $\xi(t)$ the system of stochastic differential equations

$$dN(t)=\left[-aN(t)+\frac{\varepsilon^{k_1}}{a^2+b^2}\tilde f_1(t,N(t),A_1(t),A_2(t))\right]dt+$$

$$+\frac{\varepsilon^{k_2}}{a^2+b^2}\tilde f_2(t,N(t),A_1(t),A_2(t))dw(t)+$$

$$+\frac{\varepsilon^{k_3}}{a^2+b^2}\int_{\mathrm{R}}\tilde f_3(t,N(t),A_1(t),A_2(t),z)\tilde\nu(dt,dz),$$

$$dA_1(t) = -\frac{\sin\alpha\sin(bt+\alpha)}{b^2}[\varepsilon^{k_1}\tilde f_1(t,N(t),A_1(t),A_2(t))dt+ \tag{3}$$
$$+\varepsilon^{k_2}\tilde f_2(t,N(t),A_1(t),A_2(t))dw(t)+$$
$$+\varepsilon^{k_3}\int_{\mathrm{R}}\tilde f_3(t,N(t),A_1(t),A_2(t),z)\tilde\nu(dt,dz)],$$
$$dA_2(t) = \frac{\sin\alpha\cos(bt+\alpha)}{b^2}[\varepsilon^{k_1}\tilde f_1(t,N(t),A_1(t),A_2(t))dt+$$
$$+\varepsilon^{k_2}\tilde f_2(t,N(t),A_1(t),A_2(t))dw(t)+$$
$$+\varepsilon^{k_3}\int_{\mathrm{R}}\tilde f_3(t,N(t),A_1(t),A_2(t),z)\tilde\nu(dt,dz)],$$
$$N(0)=\frac{b^2x_0+x_0''}{a^2+b^2}, A_1(0)=\frac{a^2x_0-x_0''}{a^2+b^2}, A_2(0)=\frac{ax_0''+(a^2+b^2)x_0'+ab^2x_0}{b(a^2+b^2)},$$

where $\tilde f_i(t,N,A_1,A_2) = f_i(N+A_1\cos bt+A_2\sin bt, -aN-bA_1\sin bt+bA_2\cos bt, a^2N-b^2A_1\cos bt-b^2A_2\sin bt)$, $i=1,2$, $\tilde f_3(t,N,A_1,A_2,z) = f_3(N+A_1\cos bt+A_2\sin bt, -aN-bA_1\sin bt+bA_2\cos bt, a^2N-b^2A_1\cos bt-b^2A_2\sin bt, z)$.

Theorem. *Let $\Pi(\mathrm{R})<\infty$, $t\in[0,t_0]$, $k=\min(k_1,2k_2,2k_3)$. Let us suppose, that functions $f_i, i=\overline{1,3}$ bounded and satisfy Lipschitz condition on x,x',x''. If given below matrix $\bar\sigma^2(A_1,A_2)$ is non-negative definite, then*

1. For $k_1=2k_2=2k_3$ the stochastic process $\xi_\varepsilon(t)=\xi(t/\varepsilon^k)$ weakly converges, as $\varepsilon\to0$, to the stochastic process $\bar\xi(t)=(0,\bar A_1(t),\bar A_2(t))$, where $\bar A(t)=(\bar A_1(t),\bar A_2(t))$ is the solution to the system of stochastic differential equations

$$d\bar A(t)=\bar\alpha(\bar A(t))dt+\bar\sigma(\bar A(t))d\bar w(t),\ \bar A(0)=(A_1(0),A_2(0)), \tag{4}$$

where $\bar\alpha(\bar A)=(\bar\alpha^{(1)}(A_1,A_2),\bar\alpha^{(2)}(A_1,A_2))$,

$$\bar\alpha^{(1)}(A_1,A_2)=-\frac{1}{2\pi b(a^2+b^2)}\int_0^{2\pi}\hat f_1(\psi,A_1,A_2)(a\sin\psi+b\cos\psi)\,d\psi,$$

$$\bar\alpha^{(2)}(A_1,A_2)=\frac{1}{2\pi b(a^2+b^2)}\int_0^{2\pi}\hat f_1(\psi,A_1,A_2)(a\cos\psi-b\sin\psi)\,d\psi,$$

$$\bar\sigma(A_1,A_2)=\left\{\bar B(A_1,A_2)\right\}^{\frac12}=\left\{\frac{1}{2\pi b^2(a^2+b^2)^2}\int_0^{2\pi}\hat f(\psi,A_1,A_2)B(\psi)\,d\psi\right\}^{\frac12},$$

$$B(\psi)=(B_{ij}(\psi),i,j=1,2),\quad B_{11}(\psi)=(a\sin\psi+b\cos\psi)^2,$$

$$B_{12}(\psi) = B_{21}(\psi) = -(a\sin\psi + b\cos\psi)(a\cos\psi - b\sin\psi),$$
$$B_{22}(\psi) = (a\cos\psi - b\sin\psi)^2,\ \hat{f}_i(\psi, A_1, A_2) = \tilde{f}_i(\psi, 0, A_1, A_2),\ i = 1, 2,$$
$$\hat{f}_3(\psi, A_1, A_2, z) = \tilde{f}_3(\psi, 0, A_1, A_2, z),$$
$$\hat{f}(\psi, A_1, A_2) = \hat{f}_2^2(\psi, A_1, A_2) + \int_R \hat{f}_3^2(\psi, A_1, A_2, z)\,\Pi(dz),$$

$\bar{w}(t) = (w_i(t), i = 1, 2)$, $w_i(t), i = 1, 2$ – *independent one-dimensional Wiener processes.*

2. If $k < k_1$ *then in the averaging equation (4) we must put* $\tilde{f}_1 \equiv 0$*; if* $k < 2k_2$ *then in the averaging equation (4) we must put* $\tilde{f}_2 \equiv 0$*; if* $k < 2k_3$ *then in the averaging equation (4) we must put* $\tilde{f}_3 \equiv 0$*.*

Proof. Let us make a change of variable $t \to t/\varepsilon^k$ in equation (3) and obtain for the process $\xi_\varepsilon(t) = (N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t)) = (N(t/\varepsilon^k), A_1(t/\varepsilon^k), A_2(t/\varepsilon^k))$ the system of stochastic differential equations

$$dN_\varepsilon(t) = \left[-\frac{a}{\varepsilon^k}N_\varepsilon(t) + \frac{\varepsilon^{k_1-k}}{a^2+b^2}\tilde{f}_1(t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t))\right]dt+$$
$$+\frac{\varepsilon^{k_2-k/2}}{a^2+b^2}\tilde{f}_2(t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t))dw_\varepsilon(t)+$$
$$+\frac{\varepsilon^{k_3}}{a^2+b^2}\int_{\mathrm{R}}\tilde{f}_3(t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t), z)\tilde{\nu}_\varepsilon(dt, dz),$$
$$dA_1^\varepsilon(t) = -\frac{\sin\alpha\sin(bt/\varepsilon^k+\alpha)}{b^2}[\varepsilon^{k_1-k}\tilde{f}_1(t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t))dt+ \quad (5)$$
$$+\varepsilon^{k_2-k/2}\tilde{f}_2(t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t))dw_\varepsilon(t)+$$
$$+\varepsilon^{k_3}\int_{\mathrm{R}}\tilde{f}_3(t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t), z)\tilde{\nu}_\varepsilon(dt, dz)],$$
$$dA_2^\varepsilon(t) = \frac{\sin\alpha\cos(bt/\varepsilon^k+\alpha)}{b^2}[\varepsilon^{k_1-k}\tilde{f}_1(t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t))+$$
$$+\varepsilon^{k_2-k/2}\tilde{f}_2(t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t))dw_\varepsilon(t)+$$
$$+\varepsilon^{k_3}\int_{\mathrm{R}}\tilde{f}_3(t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t), z)\tilde{\nu}_\varepsilon(dt, dz)],$$

where $w_\varepsilon(t) = \varepsilon^{k/2}w(t/\varepsilon^k)$, $\tilde{\nu}_\varepsilon(t, A) = \nu(t/\varepsilon^k, A) - \Pi(A)t/\varepsilon^k$, here A is Borel set in R. For any $\varepsilon > 0$ the process $w_\varepsilon(t)$ is the Wiener process and $\tilde{\nu}_\varepsilon(t, A)$ is the centered Poisson measure independent on $w_\varepsilon(t)$.

Since we have relationship $N_\varepsilon(t) = \exp\{-at/\varepsilon^k\}C(t/\varepsilon^k)$ and process $C_\varepsilon(t) = C(t/\varepsilon^k)$ satisfies the stochastic equation

$$C_\varepsilon(t) = C(0) + \varepsilon^{k_1-k}\int_0^t \frac{\exp\{as/\varepsilon^k\}}{a^2+b^2}\tilde{f}_1(s/\varepsilon^k, N_\varepsilon(s), A_1^\varepsilon(s), A_2^\varepsilon(s))\,ds+$$

$$+\varepsilon^{k_2-k/2}\int_0^t \frac{\exp\{as/\varepsilon^k\}}{a^2+b^2}\tilde{f}_2(s/\varepsilon^k, N_\varepsilon(s), A_1^\varepsilon(s), A_2^\varepsilon(s))\, dw_\varepsilon(s)+$$

$$+\varepsilon^{k_3}\int_0^t\int_{\mathrm{R}} \frac{\exp\{as/\varepsilon^k\}}{a^2+b^2}\tilde{f}_3(s/\varepsilon^k, N_\varepsilon(s), A_1^\varepsilon(s), A_2^\varepsilon(s), z)\, \tilde{\nu}_\varepsilon(dt, dz),$$

where $C(0)=\frac{b^2x_0+x_0''}{a^2+b^2}$, we can obtain estimate

$$\mathrm{E}|N_\varepsilon(t)|^2 \le K[e^{-2at/\varepsilon^k}+\varepsilon^k(1-e^{-2at/\varepsilon^k})(t\varepsilon^{2(k_1-k)}+\varepsilon^{2k_2-k}+\varepsilon^{2k_3-k})].$$

Therefore $\lim_{\varepsilon\to 0}\mathrm{E}|N_\varepsilon(t)|^2=0$ and it is sufficient to study the behaviour, as $\varepsilon\to 0$, of solution to the system of stochastic differential equations

$$dA_1^\varepsilon(t) = -\frac{\sin\alpha\sin(bt/\varepsilon^k+\alpha)}{b^2}[\varepsilon^{k_1-k}\hat{f}_1(t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t))dt+$$

$$+\varepsilon^{k_2-k/2}\hat{f}_2(t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t))dw_\varepsilon(t)+$$

$$+\varepsilon^{k_3}\int_{\mathrm{R}}\hat{f}_3(t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t), z)\tilde{\nu}_\varepsilon(dt, dz)], \qquad (6)$$

$$dA_2^\varepsilon(t) = \frac{\sin\alpha\cos(bt/\varepsilon^k+\alpha)}{b^2}[\varepsilon^{k_1-k}\hat{f}_1(t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t))+$$

$$+\varepsilon^{k_2-k/2}\hat{f}_2(t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t))dw_\varepsilon(t)+$$

$$+\varepsilon^{k_3}\int_{\mathrm{R}}\hat{f}_3(t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t), z)\tilde{\nu}_\varepsilon(dt, dz)],$$

with initial conditions $A_1^\varepsilon(0)=A_1(0)$, $A_2^\varepsilon(0)=A_2(0)$.

Let us denote $A_\varepsilon(t)=(A_1^\varepsilon(t), A_2^\varepsilon(t))$. Using conditions on coefficients of equation (6) and properties of stochastic integrals we obtain estimates

$$\mathrm{E}||A_\varepsilon(t)||^2 \le K[1+t^2\varepsilon^{2(k_1-k)}+t(\varepsilon^{2k_2-k}+\varepsilon^{2k_3-k})],$$

$$\mathrm{E}||A_\varepsilon(t)-A_\varepsilon(s)||^2 \le K[|t-s|^2\varepsilon^{2(k_1-k)}+|t-s|(\varepsilon^{2k_2-k}+\varepsilon^{2k_3-k})].$$

Similarly for the process $\zeta_\varepsilon(t)=(\zeta_1^\varepsilon(t), \zeta_2^\varepsilon(t))$, where

$$\zeta_1^\varepsilon(t) = -\varepsilon^{k_2-k/2}\int_0^t\frac{\sin\alpha\sin(bs/\varepsilon^k+\alpha)}{b^2}\hat{f}_2(s/\varepsilon^k, A_1^\varepsilon(s), A_2^\varepsilon(s))dw_\varepsilon(s)-$$

$$-\varepsilon^{k_3}\int_0^t\int_{\mathrm{R}}\frac{\sin\alpha\sin(bs/\varepsilon^k+\alpha)}{b^2}\hat{f}_3(s/\varepsilon^k, A_1^\varepsilon(s), A_2^\varepsilon(s), z)\tilde{\nu}_\varepsilon(ds, dz)],$$

$$\zeta_2^\varepsilon(t) = \varepsilon^{k_2-k/2}\int_0^t\frac{\sin\alpha\cos(bs/\varepsilon^k+\alpha)}{b^2}\hat{f}_2(s/\varepsilon^k, A_1^\varepsilon(s), A_2^\varepsilon(s))dw_\varepsilon(s)+$$

$$+\varepsilon^{k_3}\int_0^t\int_{\mathrm{R}}\frac{\sin\alpha\cos(bs/\varepsilon^k+\alpha)}{b^2}\hat{f}_3(s/\varepsilon^k, A_1^\varepsilon(s), A_2^\varepsilon(s), z)\tilde{\nu}_\varepsilon(ds, dz)]$$

we derive estimates

$$\mathrm{E}||\zeta_\varepsilon(t)||^2 \le Kt(\varepsilon^{2k_2-k}+\varepsilon^{2k_3-k}),\ \mathrm{E}||\zeta_\varepsilon(t)-\zeta_\varepsilon(s)||^2 \le K|t-s|(\varepsilon^{2k_2-k}+\varepsilon^{2k_3-k}).$$

Therefore for stochastic process $\eta_\varepsilon(t) = (A_\varepsilon(t), \zeta_\varepsilon(t))$ conditions of weak compactness [11] are fulfilled

$$\lim_{h\downarrow 0} \overline{\lim_{\varepsilon\to 0}} \sup_{|t-s|<h} \mathrm{P}\{|\eta_\varepsilon(t) - \eta_\varepsilon(s)| > \delta\} = 0 \text{ for any } \delta > 0,\ t, s \in [0, T],$$

$$\lim_{N\to\infty} \overline{\lim_{\varepsilon\to 0}} \sup_{t\in[0,T]} \mathrm{P}\{|\eta_\varepsilon(t)| > N\} = 0,$$

and for any sequence $\varepsilon_n \to 0, n = 1, 2, \ldots$ there exists a subsequence $\varepsilon_m = \varepsilon_{n(m)} \to 0, m = 1, 2, \ldots$, probability space, stochastic processes $\bar{A}_{\varepsilon_m}(t) = (\bar{A}_1^{\varepsilon_m}(t), \bar{A}_2^{\varepsilon_m}(t))$, $\bar{\zeta}_{\varepsilon_m}(t)$, $\bar{A}(t) = (\bar{A}_1(t), \bar{A}_2(t))$, $\bar{\zeta}(t)$ defined on this space, such that $\bar{A}_{\varepsilon_m}(t) \to \bar{A}(t), \bar{\zeta}_{\varepsilon_m}(t) \to \bar{\zeta}(t)$ in probability, as $\varepsilon_m \to 0$, and finite-dimensional distributions of $\bar{A}_{\varepsilon_m}(t), \bar{\zeta}_{\varepsilon_m}(t)$ are coincide with finite-dimensional distributions of $A_{\varepsilon_m}(t), \zeta_{\varepsilon_m}(t)$. Since we interesting in limit behaviour of distributions, we can consider processes $A_{\varepsilon_m}(t)$, and $\zeta_{\varepsilon_m}(t)$ instead of $\bar{A}_{\varepsilon_m}(t), \bar{\zeta}_{\varepsilon_m}(t)$. From (6) we obtain equation

$$A_{\varepsilon_m}(t) = A(0) + \int_0^t \alpha_{\varepsilon_m}(s, A_{\varepsilon_m}(s))\, ds + \zeta_{\varepsilon_m}(t), \quad A_0 = (A_1(0), A_2(0)), \quad (7)$$

where $\alpha_\varepsilon(t, A) = (\alpha_\varepsilon^{(1)}(t, A_1, A_2), \alpha_\varepsilon^{(2)}(t, A_1, A_2))$,

$$\alpha_\varepsilon^{(1)}(t, A_1, A_2) = -\varepsilon^{k_1-k} \frac{\sin\alpha \sin(bt/\varepsilon^k + \alpha)}{b^2} \hat{f}_1(t/\varepsilon^k, A_1, A_2),$$

$$\alpha_\varepsilon^{(2)}(t, A_1, A_2) = \varepsilon^{k_1-k} \frac{\sin\alpha \cos(bt/\varepsilon^k + \alpha)}{b^2} \hat{f}_1(t/\varepsilon^k, A_1, A_2).$$

It should be noted that process $\zeta_\varepsilon(t)$ is the vector-valued square integrable martingale with matrix characteristic

$$\langle\zeta_\varepsilon^{(i)}, \zeta_\varepsilon^{(j)}\rangle(t) = \int_0^t \sigma_\varepsilon^{(i)}(s, A_1^\varepsilon(s), A_2^\varepsilon(s))\sigma_\varepsilon^{(j)}(s, A_1^\varepsilon(s), A_2^\varepsilon(s))\, ds +$$

$$+\frac{1}{\varepsilon^k} \int_0^t \int_{\mathrm{R}} \gamma_\varepsilon^{(i)}(s, A_1^\varepsilon(s), A_2^\varepsilon(s), z)\gamma_\varepsilon^{(j)}(s, A_1^\varepsilon(s), A_2^\varepsilon(s), z)\, \Pi(dz)ds,\ i, j = 1, 2,$$

where

$$\sigma_\varepsilon^{(1)}(s, A_1, A_2) = -\varepsilon^{k_2-k/2} \frac{\sin\alpha \sin(bs/\varepsilon^k + \alpha)}{b^2} \hat{f}_2(s/\varepsilon^k, A_1, A_2),$$

$$\sigma_\varepsilon^{(2)}(s,A_1,A_2)=\varepsilon^{k_2-k/2}\frac{\sin\alpha\cos(bs/\varepsilon^k+\alpha)}{b^2}\hat f_2(s/\varepsilon^k,A_1,A_2),$$

$$\gamma_\varepsilon^{(1)}(s,A_1,A_2,z)=-\varepsilon^{k_3}\frac{\sin\alpha\sin(bs/\varepsilon^k+\alpha)}{b^2}\hat f_3(s/\varepsilon^k,A_1,A_2,z),$$

$$\gamma_\varepsilon^{(2)}(s,A_1,A_2,z)=\varepsilon^{k_3}\frac{\sin\alpha\cos(bs/\varepsilon^k+\alpha)}{b^2}\hat f_3(s/\varepsilon^k,A_1,A_2,z).$$

For processes $A_\varepsilon(t)$ and $\zeta_\varepsilon(t)$ following estimates hold

$$\mathrm{E}||A_\varepsilon(t)-A_\varepsilon(s)||^4\le K\left[\varepsilon^{4(k_1-k)}|t-s|^4+\mathrm{E}||\zeta_\varepsilon(t)-\zeta_\varepsilon(s)||^4\right], \qquad (8)$$

$$\mathrm{E}||\zeta_\varepsilon(t)-\zeta_\varepsilon(s)||^4\le K\left[(\varepsilon^{4k_2-2k}+\varepsilon^{4k_3-2k})|t-s|^2+\right.$$
$$\left.+\varepsilon^{4k_3-3k/2}|t-s|^{3/2}+\varepsilon^{4k_3-k}|t-s|\right], \qquad (9)$$

$$\mathrm{E}||A_\varepsilon(t)-A_\varepsilon(s)||^8\le K,\quad \mathrm{E}||\zeta_\varepsilon(t)-\zeta_\varepsilon(s)||^8\le K. \qquad (10)$$

Since $A_{\varepsilon_m}(t)\to\bar A(t),\zeta_{\varepsilon_m}(t)\to\bar\zeta(t)$ in probability, as $\varepsilon_m\to0$, then, using (10), from (8) and (9) we obtain estimates

$$\mathrm{E}||\bar A(t)-\bar A(s)||^4\le K(|t-s|^4+|t-s|^2),\quad \mathrm{E}||\bar\zeta(t)-\bar\zeta(s)||^4\le C|t-s|^2.$$

Therefore processes $\bar A(t)$ and $\bar\zeta(t)$ satisfy the Kolmogorov's continuity condition [12].

Let us consider the case $k_1=2k_2=2k_3$. Under these conditions we have for $i,j=1,2$

$$\lim_{\varepsilon\to0}\frac1t\int_0^t\alpha_\varepsilon^{(i)}(s,A_1,A_2)ds=\bar\alpha^{(i)}(A_1,A_2),$$

$$\lim_{\varepsilon\to0}\frac1t\int_0^t\left[\sigma_\varepsilon^{(i)}(s,A_1,A_2)\sigma_\varepsilon^{(j)}(s,A_1,A_2)+\right. \qquad (11)$$

$$\left.+\frac{1}{\varepsilon^k}\int_R\gamma_\varepsilon^{(i)}(s,A_1,A_2,z)\gamma_\varepsilon^{(j)}(s,A_1,A_2,z)\Pi(dz)\right]ds=\bar B_{ij}(A_1,A_2),$$

where functions $\bar\alpha^{(i)}(A_1,A_2)$ and $\bar B(A_1,A_2)=\{\bar B_{ij}(A_1,A_2),\ i,j=1,2\}$ are defined in the condition of theorem. Since processes $\bar A(t),\bar\zeta(t)$ are continuous, then from Lemma and relationships (7), (11) it follows

$$\bar A(t)=A(0)+\int_0^t\bar\alpha(\bar A_1(s),\bar A_2(s))ds+\bar\zeta(t),\quad A(0)=(A_1(0),A_2(0)), \quad (12)$$

where $\bar{\zeta}(t)$ is continuous vector-valued martingale with matrix characteristic

$$\langle \bar{\zeta}^{(i)}, \bar{\zeta}^{(j)} \rangle (t) = \int_0^t \bar{B}_{ij}(\bar{A}_1(s), \bar{A}_2(s))ds, \quad i,j = 1,2.$$

Hence [13] there exists Wiener process $\bar{w}(t) = (w_i(t), i = 1,2)$, such that

$$\bar{\zeta}(t) = \int_0^t \bar{\sigma}(\bar{A}_1(s), \bar{A}_2(s))\, d\bar{w}(s),\ \ \bar{\sigma}(A_1, A_2) = \left\{\bar{B}(A_1, A_2)\right\}^{1/2}. \qquad (13)$$

Relationships (12), (13) mean that process $\bar{A}(t)$ satisfies equation (4). Under conditions of theorem the equation (4) has unique solution. Therefore process $\bar{A}(t)$ does not depend on choosing of sub-sequence $\varepsilon_m \to 0$, and finite-dimensional distributions of process $A_{\varepsilon_m}(t)$ converge to finite-dimensional distributions of process $\bar{A}(t)$. Since processes $A_{\varepsilon_m}(t)$ and $\bar{A}(t)$ are Markov processes then using the conditions for weak convergence of Markov processes [12] we finish the proof of statement 1 of theorem.

Let us consider the case $k < k_1$. Then coefficients $\alpha_\varepsilon^{(i)}(t, A_1, A_2)$, $i = 1,2$ of equation (7) tend to zero, as $\varepsilon \to 0$. Repeating with obvious modifications the proof of statement 1) of theorem we obtain proof of statement 2).

In the case $k < 2k_2$ in (11) we have $\sigma_\varepsilon^{(i)}(t, A_1, A_2)\sigma_\varepsilon^{(j)}(t, A_1, A_2) = O(\varepsilon^{2k_2 - k})$, $i,j = 1,2$. Then we finish the proof in this case as above. In the same way we consider the case $k < 2k_3$.

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