Theory of Stochastic Processes
Vol. 13 (29), no. 3, 2007, pp. 29–37

UDC 519.21

ANDREY A. DOROGOVTSEV

CONDITIONING OF GAUSSIAN FUNCTIONALS AND ORTHOGONAL EXPANSION

In the article, we consider terms of the Gaussian chaotic expansion under conditioning with respect to some sigma-field and discuss the possibility to organize the orthogonal expansion from them.

1. Introduction

Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. On a probability space $(\Omega,\mathcal{F},P)$, define a Gaussian random element ξ in H with zero mean and identity covariation. If $\dim H=\infty$, then ξ is not a random element in a usual sense. In this case, ξ is a family of jointly Gaussian random variables $\{(\varphi,\xi),\varphi\in H\}$ which linearly depend on φ and have the properties

1) $E(\varphi,\xi)=0$,
2) $E(\varphi,\xi)(\psi,\xi)=(\varphi,\psi)$.

Such ξ is often called in the literature as a generalized Gaussian random element or white noise in H [1, 2]. Suppose that the σ-field $\mathcal{F}$ is generated by ξ. Under this condition, the space $L_2(\Omega,\mathcal{F},P)$ has a chaotic expansion [1]. The members of this expansion are multidimensional Hermite polynomials (infinite-dimensional if $\dim H=\infty$) which can be described as follows. Denote, by $\mathcal{P}_n$, the set of all polynomials of degree not greater than n from $\{(\varphi,\xi);\varphi\in H\}$. Let $\widetilde{\mathcal{P}}_n$ be the closure of $\mathcal{P}_n$ in $L_2(\Omega,\mathcal{F},P)$. Define, for $n\geq 0$,

$$\mathcal{K}_{n+1}=\widetilde{\mathcal{P}}_{n+1}\ominus\widetilde{\mathcal{P}}_n$$

and $\mathcal{K}_0=\mathcal{P}_0$. It is well known that

$$L_2(\Omega,\mathcal{F},P)=\bigoplus_{n=0}^{\infty}\mathcal{K}_n.$$

The elements of $\mathcal{K}_n$ can be described using the one-dimensional Hermite polynomials [3]. Let

$$H_n(x)=(-1)^n e^{\frac{x^2}{2}}\left(\frac{d}{dx}\right)^n e^{-\frac{x^2}{2}},\ n\geq 0,\ x\in\mathbb{R}$$

be one-dimensional Hermite polynomials.

Consider the orthonormal basis $\{e_k;k\geq 1\}$ in H (here we suppose that $\dim H=\infty$; in the opposite case, the explanation will be the same with the trivial corrections). Denote $\xi_k=(e_k,\xi)$ for $k\geq 1$ and, by $\xi_{r_1}*\ldots*\xi_{r_n}$ for $r_1,\ldots,r_n\geq 1$, the product $\xi_{r_1}\cdot\ldots\cdot\xi_{r_n}$, in which the powers are substituted by the Hermite polynomials of the same degree. For example,

$$\xi_1*\xi_2*\xi_3*\xi_2=\xi_1H_2(\xi_2)\xi_3.$$

2000 *AMS Mathematics Subject Classification.* Primary 60H05, 60H40, 60B11.

Key words and phrases. Gaussian white noise, Ito–Wiener expansion, conditional expectation, polynomially nondegenerate measure.

This research has been partially supported by the Ministry of Education and Science of Ukraine, project N GP/F13/0195

Let A_n be a symmetric n-linear Hilbert–Schmidt form on H. The form A_n can be described in terms of the basis $\{e_k; k \geq 1\}$ as follows

$$A_n = \sum_{r_1 \ldots r_n = 1}^{\infty} a_{r_1 \ldots r_n} e_{r_1} \otimes \ldots \otimes e_{r_n},$$

where $e_{r_1} \otimes \ldots \otimes e_{r_n}$ is a tensor product and $a_{r_1 \ldots r_n}$ are symmetric with respect to all permutations of indices. Define

$$A_n(\xi, \ldots, \xi) = \sum_{r_1 \ldots r_n = 1}^{\infty} a_{r_1 \ldots r_n} \xi_{r_1} * \ldots * \xi_{r_n}. \tag{1.1}$$

It can be proved [3] that (1.1) converges in the square mean and that every element of $\mathcal{K}_n$ has a unique representation (1.1). Moreover,

$$EA_n(\xi, \ldots, \xi) B_n(\xi, \ldots, \xi) = n!(A_n, B_n)_n.$$

Here, $(\cdot, \cdot)_n$ is the scalar product in the tensor power $H^{\otimes n}$. Note that the elements of $\mathcal{K}_n$ can be described as a multiple Wiener integrals if ξ is built with the help of a random Gaussian measure [4]. The most important case for us is that, when the white noise ξ is generated by the standard Wiener process w. Namely, put $H = L_2([0; 1])$ with the usual inner product and define

$$(\varphi, \xi) = \int_0^1 \varphi(r) dw(r)$$

for $\varphi \in L_2([0; 1])$. It can be easily verified that ξ is a Gaussian white noise. Now the elements of $\mathcal{K}_n$ can be described as follows. Every A_n from the symmetric part of $H^{\otimes n}$ has the representation as a multiple integral, i.e., for $\varphi_1, \ldots, \varphi_n \in H$,

$$A_n[\varphi_1, \ldots, \varphi_n] = \int_0^1 \overset{n}{\cdots} \int a_n(r_1, \ldots, r_n) \varphi(r_1) \ldots \varphi(r_n) dr_1 \ldots dr_n.$$

Here, the symmetric kernel a_n belongs to the space $L_2([0; 1]^n)$ and

$$A_n(\xi, \ldots, \xi) = \int_0^1 \overset{n}{\cdots} \int a_n(r_1, \ldots, r_n) dw(r_1) \ldots dw(r_n).$$

The last integral is a useful notation for

$$n! \int_{0 \leq r_1 \leq \ldots \leq r_n \leq 1} a_n(r_1, \ldots, r_n) dw(r_1) \ldots dw(r_n).$$

As follows from the stated above, every square-integrable functional α of ξ has a representation of the form

$$\alpha = \sum_{n=0}^{\infty} A_n(\xi, \ldots, \xi), \tag{1.2}$$

where $A_0 = E\alpha$ and

$$E\alpha^2 = \sum_{n=0}^{\infty} n! \|A_n\|_n^2.$$

In a particular case where ξ is generated by w, representation (1.2) has the form

$$\alpha = \sum_{n=0}^{\infty} \int_0^1 \overset{n}{\cdots} \int a_n(r_1, \ldots, r_n) dw(r_1) \ldots dw(r_n) \tag{1.3}$$

and

$$E\alpha^2 = \sum_{n=0}^{\infty} n! \int \overset{1}{\underset{0}{\overset{n}{\cdots}}} \int a_n(r_1, \dots, r_n)^2 dr_1 \dots dr_n.$$

Series (1.2) or (1.3) is often referred to as the Itô-Wiener expansion. We will also use this term. The Itô-Wiener expansion is a powerful tool in the studying of Gaussian functionals due to its various and deep relations with the infinite-dimensional measure theory, stochastic calculus and martingale theory. In the modern time increases the interest in non-Gaussian objects that are built from the initial Gaussian noise. Examples of such objects are the stopped Wiener process, skew Brownian motion, and Arratia flow of coalescing Brownian particles [5–7]. The investigation of functionals of such processes leads to the natural question about the modification of the Itô–Wiener expansion for these cases. In this article, we consider how the Itô–Wiener expansion changes under the conditioning. More precisely the problem can be formulated as follows. Let $\mathcal{F}'$ be a sub-σ-field of $\mathcal{F}$. From the previous considerations, one can conclude that the set of random variables

$$\{E(A_n(\xi, \dots, \xi)/\mathcal{F}'),\ A_n \in H^{\otimes n},\ n \geq 0\} \tag{1.4}$$

is dense in $L_2(\Omega, \mathcal{F}', P)$. But in general under conditioning with respect to $\mathcal{F}'$ we loose the orthogonality. The following example shows this and will be useful in the future.

Example 1.1. Suppose that $H = L_2([0;1])$ and ξ is generated by the Wiener process w as it was described above. Let τ be the stopping time with respect to the flow of σ-fields generated by w. Consider the σ-field $\mathcal{F}_\tau$ of random events associated with the random time τ. Note that $\mathcal{F}_\tau$ coincides with the σ-field generated by the stopped process $\{w(s \wedge \tau); s \in [0;1]\}$. For the symmetric kernel $a_n \in L_2([0;1]^n)$ the conditional expectation is given by

$$E\left(\int_{0\leq t_1\leq \dots \leq t_n \leq 1} a_n(t_1, \dots, t_n) dw(t_1) \dots dw(t_n)/\mathcal{F}_\tau\right) =$$
$$= \int_{0\leq t_1\leq \dots \leq t_n \leq \tau} a_n(t_1, \dots, t_n) dw(t_1) \dots dw(t_n).$$

Two such random variables for different n can be not orthogonal. Really

$$E\int_0^\tau w(t)dw(t)w(\tau) = E\int_0^\tau w(t)dt = \int_0^1 Ew(t)1\!\!1_{\{\tau\geq t\}}dt.$$

The last value is not equal to zero for example when

$$\tau = \inf\{1, t:\ w(t) = 1\}.$$

Thus the aim of the article is to describe the orthogonal expansion of $L_2(\Omega, \mathcal{F}', P)$ which is built from the set (1.4).

2. Projections of $\mathcal{K}_n$ and polynomially nondegenerated measures

This section is devoted to properties of the set

$$\mathcal{H}_n = \{E(A_n(\xi, \dots, \xi)/\mathcal{F}') :\ A_n \in H^{\otimes n}\}. \tag{2.1}$$

It is well known that the conditional expectation with respect to a sub-σ-field $\mathcal{F}'$ is an orthogonal projector in $L_2(\Omega, \mathcal{F}, P)$. Denote it by R. Then

$$\mathcal{H}_n = R(\mathcal{K}_n).$$

This section contains the conditions under which $R(\mathcal{K}_n)$ is closed in $L_2(\Omega, \mathcal{F}, P)$. Note that this property does not hold without additional assumptions. Consider the following example.

Example 2.1. Let $\{e_k; k \geq 1\}$ be an orthonormal basis in H. Denote $\xi_k = (e_k, \xi)$. Then $\{\xi_k; k \geq 1\}$ are the independent standard normal variables. For every $k \geq 1$, consider the random variable

$$\eta_k = 1\!\mathrm{I}_{[-1+a_k;1+a_k]}(\xi_k).$$

Here, the positive constants $\{a_k; k \geq 1\}$ are chosen in such a way that the series

$$\sum_{k=1}^{\infty} \left| \left[\int_{-\infty}^{-1+a_k} x p_1(x)dx + \int_{1+a_k}^{+\infty} x p_1(x)dx \right] \left[\int_{-\infty}^{-1+a_k} p_1(x)dx + \int_{1+a_k}^{+\infty} p_1(x)dx \right]^{-1} \right| +$$
$$+ \int_{-1+a_k}^{1+a_k} x p_1(x)dx \Big/ \int_{-1+a_k}^{1+a_k} p_1(x)dx$$

converges and consists of positive summands. Define, for every $k \geq 1$,

$$\zeta_k = E(\xi_k / \eta_k).$$

Let us take the σ-field $\mathcal{F}'$ as

$$\mathcal{F}' = \sigma(\eta_k; k \geq 1).$$

Due to the independence of $\{\eta_k; k \geq 1\}$, the operations of conditional expectation with respect to $\mathcal{F}'$ and a single η_k commute. Now consider the sequence from $\mathcal{K}_1$

$$S_n = \sum_{k=1}^{n} \xi_k, \ n \geq 1.$$

For every $n \geq 1$,

$$E(S_n / \mathcal{F}') = \sum_{k=1}^{n} \zeta_k.$$

Note that, due to the condition on $\{a_k; k \geq 1\}$, the series

$$\sum_{k=1}^{\infty} \zeta_k$$

converges in $L_2(\Omega, \mathcal{F}, P)$. But there is no random variable α in $\mathcal{K}_1$ such that

$$E(\alpha / \mathcal{F}') = \sum_{k=1}^{\infty} \zeta_k. \tag{2.2}$$

Really, every $\alpha \in \mathcal{K}_1$ has a unique representation

$$\alpha = \sum_{k=1}^{\infty} b_k \xi_k$$

with

$$\sum_{k=1}^{\infty} b_k^2 < +\infty. \tag{2.3}$$

Then

$$E(\alpha / \mathcal{F}') = \sum_{k=1}^{\infty} b_k \zeta_k. \tag{2.4}$$

Comparing (2.2) and (2.4) and taking the conditional expectation with respect to η_k , one can get

$$b_k = 1, \quad k \geq 1.$$

This proves our statement.

According to the previous example, additional conditions on $\mathcal{F}'$ must be imposed to assure us that $R(\mathcal{K}_n)$ is closed. The next theorem presents such a condition.

Theorem 2.1. *Suppose that there exists such $\Delta \in \mathcal{F}$ of positive probability which has the property*
$\forall\ n \geq 1\ \ \forall\ A_n \in \mathcal{K}_n$:

$$A_n/\Delta = E(A_n/\mathcal{F}')/\Delta. \tag{2.5}$$

Here, for a random variable η, the symbol η/Δ means the restriction of η on the set Δ. Then, for every $n \geq 1$, $R(\mathcal{K}_n)$ is a closed subspace of $L_2(\Omega, \mathcal{F}, P)$.

Before proving the theorem, let us consider an example when condition (2.5) holds.

Example 2.2. Suppose that H and ξ are the same as in Example 1.1. Consider a stopping time τ such that

$$P\{\tau = 1\} > 0.$$

As was mentioned above, for every multiple integral

$$I_k(a_k) = \int_0^1 \ldots \int_0^{t_k} a_k(t_1, \ldots, t_k) dw(t_1) \ldots dw(t_k),$$

we have

$$E(I_k(a_k)/\mathcal{F}_\tau) = \int_0^\tau \ldots \int_0^{t_k} a_k(t_1, \ldots, t_k) dw(t_1) \ldots dw(t_k).$$

Hence

$$I_k(a_k)\big/_{\{\tau=1\}} = E(I_k(a_k)/\mathcal{F}_\tau)\big/_{\{\tau=1\}}.$$

Consequently, $\mathcal{F}_\tau$ satisfies condition (2.5).

Consider the proof of the theorem.

Proof. The proof is based on the notion of polynomially nondegenerated measure which was introduced in [8]. Consider a probability measure μ on the separable Banach space B. Suppose that μ has the weak moments of any order and the polynomials are dense in $L_2(B, \mu)$. Also let a closure of the linear span of $\operatorname{supp} \mu$ coincide with B. Define H' as a closure of B^*in $L_2(B, \mu)$. For every $\varphi \in H'$, define

$$j(\varphi) = \int_B \langle \varphi, u \rangle u \mu(du),$$

where, on the right-hand side, we consider the Pettise integral. Denote, by H_1, the image $j(H')$. Since j is the one-to-one correspondence one can define the inner product in H_1 by the formula

$$(h, n)_1 := \int_B \langle j^{-1}(h), u \rangle \langle j^{-1}(n), u \rangle \mu(du).$$

It can be proved that, with this product, H_1 is a Hilbert space densely and compactly embedded in B [8]. So, every finite-dimensional symmetric n-linear form A_n on B can be considered as an element of the tensor power $H_1^{\otimes n}$. Denote, by $J_n(A_n)$, the orthogonal polynomial related to A_n. Note that, in general, the degree of A_n can be less than n (see [8] for the corresponding example).

The measure μ is referred to as a polynomially nondegenerated if, for every $n \geq 1$, there exist such constants $c'_n, c''_n > 0$ that, for an arbitrary finite-dimensional n-linear form A_n, the following inequality holds:

$$c'_n |A_n|_n \leq \int_B J_n(A_n)^2(u)\mu(du) \leq c''_n |A_n|_n.$$

It was proved in [8] that the restriction of the Gaussian measure on the ball is polynomially nondegenerated. Then it was proved in [9] that the restriction of the Gaussian measure on an arbitrary set of positive measure is nondegenerated. We will use the last fact in the following way. Let us consider a Hilbert space $\widetilde{H}$ such that H is densely embedded in $\widetilde{H}$ and the embedding i is a Hilbert–Schmidt operator. Then $\widetilde{\xi} = i(\xi)$ can be correctly defined as a usual random element in $\widetilde{H}$. Define the Gaussian measure μ as a distribution of $\widetilde{\xi}$ in $\widetilde{H}$. Since i is an embedding, there exists the one-to-one correspondence between the σ-field $\mathcal{F}$ and the Borel σ-field in $\widetilde{H}$ under which the probability P turns into the measure μ. Note also that the polynomials of $\widetilde{\xi}$ are the polynomials of ξ. The space H plays the same role for the measure μ as a space H_1 in the above construction. The elements of H are often called by the measurable linear functionals related to the measure μ. The polynomials of $\widetilde{\xi}$ with the coefficients, which are the Hilbert–Schmidt forms on H, are called by the measurable polynomials. The set of measurable polynomials contains the usual polynomials (see [8] for details). Also, every measurable polynomial of $\widetilde{\xi}$ is a polynomial of ξ, and every polynomial of ξ is a measurable polynomial of $\widetilde{\xi}$. Denote, by $\widetilde{\Delta}$, the subset of $\widetilde{H}$ which corresponds to Δ. Since $\widetilde{\Delta}$ has positive measure μ (as well as Δ by the condition of the theorem), the restriction of μ on $\widetilde{\Delta}$ is polynomially nondegenerated [9]. Consider the sequence $\{q_l; l \geq 1\}$ from $\mathcal{K}_n$ such that

$$R(q_l) \to \eta, \quad l \to \infty$$

in the square mean. Then the restrictions $R(q_l)/_{\widetilde{\Delta}}$ tend to the restriction $\eta/_{\widetilde{\Delta}}$. Due to the condition of the theorem, for every $l \geq 1$,

$$R(q_l)/_{\widetilde{\Delta}} = q_l/_{\widetilde{\Delta}}.$$

It follows from the above reasonings that the sequence $\{q_l; l \geq 1\}$ converges in $L_2(\widetilde{\Delta}, \mu)$. Then $\{\mathcal{J}_n(q_l);\ l \geq 1\}$ is a fundamental sequence in $L_2(\widetilde{\Delta}, \mu)$. Since μ is polynomially nondegenerated on $\widetilde{\Delta}$, $\{q_l;\ l \geq 1\}$ is fundamental in $\mathcal{K}_n$. Then there exists $q \in \mathcal{K}_n$ such that

$$q_l \to q, \quad l \to \infty$$

in the square-mean. The continuity of R implies now that $\eta = R(q)$. Hence, $R(\mathcal{K}_n)$ is closed. Theorem 2.1 is proved.

Remark. Note that condition (2.5) of the theorem can be reformulated as follows. For every $Q \in \mathcal{P}$,

$$R(Q)/_{\Delta} = Q/_{\Delta}.$$

But, in general, it is much more difficult to check it in this form.

3. The orthogonal expansion for $L_2(\Omega, \mathcal{F}', P)$

In this section, we suppose that $R(\mathcal{K}_n)$ is closed for every $n \geq 1$. The sufficient condition for this was given in Theorem 2.1. Let us recall that $\mathcal{K}_n$ is in one-to-one correspondence with the symmetric part of $H^{\otimes n}$. To use this correspondence, we also suppose that R is one-to-one on every $\mathcal{K}_n$. Note that this condition is fulfilled in Example 1.2. Under these assumptions, R has a continuous inverse as a linear operator from $\mathcal{K}_n$ to $R(\mathcal{K}_n)$.

Lemma 3.1. *For every $n \geq 1$, there exists a unique continuous linear operator Q_n : $L_2(\Omega, \mathcal{F}, P) \to \mathcal{K}_n$ such that*
$\forall\ \eta \in L_2(\Omega, \mathcal{F}, P),\quad B_n(\xi, \dots, \xi) \in \mathcal{H}_n$:

$$E\eta R(B_n(\xi, \dots, \xi)) = EQ_n(\eta)B_n(\xi, \dots, \xi).$$

Proof. Denote, by Π_n, the projector on $R(\mathcal{K}_n)$ (by our assumption, $R(\mathcal{K}_n)$ is a closed subspace). Then, due to the assumption on R, Q_n can be defined as $R^{-1}(\Pi_n)$. The uniqueness of Q_n is evident.

To describe the operators Q_n, it is enough to describe the expectations

$$EA_m(\xi, \dots, \xi)R(B_n(\xi, \dots, \xi)).$$

We will do this for $\mathcal{F}'$ from Example 1.2.

Example 3.1. Suppose that $H = L_2([0; 1])$ and ξ is generated by the Wiener process w. Take $\mathcal{F}' = \mathcal{F}_\tau$ for the stopping time τ which has the property

$$P\{\tau = 1\} > 0.$$

Then the assumptions on the operator R hold. Now, to describe Q_n, we have to find, for $m \leq n$,

$$\begin{aligned} &E\int_0^1 \dots \int_0^{t_{m-1}} a_m(t_1, \dots, t_m)dw(t_m)\dots dw(t_1)\\ &\int_0^{\tau} \dots \int_0^{t_{n-1}} a_n(t_1, \dots, t_n)dw(t_n)\dots dw(t_1) =\\ &= E\int_0^1 1\!\!\mathrm{I}_{\{\tau\geq t_1\}} \int_0^{t_1} \dots \int_0^{t_{m-1}} a_m(t_1, \dots, t_m)dw(t_m)\dots dw(t_2)\\ &\int_0^{t_1} \dots \int_0^{t_{n-1}} a_n(t_1, \dots, t_n)dw(t_n)\dots dw(t_2)dt_1. \end{aligned}$$

Let the random variable $1\!\!\mathrm{I}_{\{\tau\geq t\}}$ have the Itô–Wiener expansion

$$1\!\!\mathrm{I}_{\{\tau\geq t\}} = \sum_{k=0}^{\infty} \int_0^t \dots \int_0^{t_k} \alpha_k(t_1, \dots, t_k)dw(t_k)\dots dw(t_1).$$

Then [3, 4] one can find the Itô–Wiener expansion for the product of two multiple integrals as

$$\begin{aligned} &\int_0^{t_1} \dots \int_0^{t_m} a_m(t_1, \dots, t_m)dw(t_m)\dots dw(t_2)\\ &\int_0^{t_1} \dots \int_0^{t_n} a_n(t_1, \dots, t_m)dw(t_n)\dots dw(t_2) =\\ &= \frac{1}{(m-1)!(n-1)!} \int_0^{t_1} \dots \int a_m(t_1, t_2, \dots, t_m)dw(t_2)\dots dw(t_m)\\ &\int_0^{t_1} \dots \int a_n(t_1, t_2, \dots, t_n)dw(t_2)\dots dw(t_n) =\\ &= \frac{1}{(m-1)!(n-1)!} \sum_{i=0}^{m-1} \frac{1}{i!} \frac{(m-1)!(n-1)!}{(m-1-i)!(n-1-i)!} \cdot\\ &\cdot \int_0^{t_1} \overset{m+n-2-2i}{\dots} \int \wedge \int_0^{t_1} \overset{i}{\dots} \int a_m(t_1, r_1, \dots, r_{m-i-1}, s_1, \dots, s_i)\cdot \end{aligned}$$

$$\cdot a_n(t_1, r_{m-i}, \ldots, r_{m+n-2-2i}, s_1, \ldots, s_i)ds_1 \ldots ds_i dw(r_1) \ldots dw(r_{m+n-2-2i}) = \sum_{i=0}^{m-1} \frac{1}{i!}(a_m(t_1, \ldots), a_n(t_1, \ldots))_{\sigma_i}.$$

Here, Λ means the symmetrization with respect to variables $r_1, \ldots, r_{m+n-2-2i}$. Using the obtained relation, one can conclude that

$$E \int_0^1 \cdots \int_0^{t_{m-1}} a_m(t_1, \ldots, t_m)dw(t_m) \ldots dw(t_1) \cdot$$

$$\int_0^\tau \cdots \int_0^{t_{n-1}} a_n(t_1, \ldots, t_m)dw(t_m) \ldots dw(t_1) =$$

$$= \int_0^1 \sum_{i=0}^{m-1} \frac{1}{i!} \int \overset{t_1}{\underset{0}{\overset{i}{\cdots}}} \int \alpha_{m_n-2-2i}(r_1, \ldots, r_{m_n-2-2i}) \cdot$$

$$(a_m(t_1, \ldots, a_n(t_1, \ldots))_{\sigma_i}(r_1, \ldots, r_{m+n-2-2i})dr_1 \ldots dr_{m+n-2i}.$$

The last formula gives us the possibility to obtain the expression for $Q_n(a_m)$.

As was mentioned in Section 1, the union of $R(\mathcal{K}_n)$, $n \geq 0$ is dense in $L_2(\Omega, \mathcal{F}', P)$ and our aim is to built an orthogonal expansion using the elements of $R(\mathcal{K}_n)$, $n \geq 0$. For every symmetric $A_n \in H^{\otimes n}$, we are looking for such $X_k \in H^{\otimes n}$, $k = 0, \ldots, n-1$, that $\forall\ j = 0, \ldots, n-1$

$$\sum_{k=0}^{n-1} Q_j(X_k) = Q_j(A_n). \tag{3.1}$$

We will prove that the solution to (3.1) exists and is unique by induction. Note that, for $n = 1, 2$, it is true. Suppose that (3.1) has a solution for every $n \leq N$. Consider (3.1) for A_{N+1}. By $Y(A_k)$, we denote for $A_k,\ k \leq N$, the sum

$$Y(A_k) = A_k = \sum_{j=0}^{k-1} X_j,$$

where $X_0, \ldots, X_{k-1}$ is the solution to (3.1) for A_k. The next lemma will be useful.

Lemma 3.1. *For arbitrary B_k, there exists a unique A_k such that*

$$Q_k(Y(A_k)) = B_k. \tag{3.2}$$

Proof. As was mentioned above, under our conditions on $\mathcal{F}'$, equality (3.2) is equivalent to

$$R(Q_k(Y(A_k))) = R(B_k). \tag{3.3}$$

Let us denote, by G'_{k-1}, the orthogonal projector on $R(\widetilde{\mathcal{P}}_{k-1})$ and, by G''_k, the orthogonal projector on $R(\mathcal{K}_k)$. Then (3.3) can be written as

$$G''_k G'_{k-1}(R(A_k)) + R(A_k) = R(B_k). \tag{3.4}$$

To prove the statement of Lemma 3.1, it is enough to check that the operator norm of $G''_k G'_{k-1}$ is less than unity. The opposite suggestion ($\|G''_k G'_{k-1}\| = 1$) leads to the existence of the sequence $\{C^n_k;\ n \geq 1\}$ with the properties

1) $ER(C^n_k)^2 = 1,\ n \geq 1,$
2) $E(R(C^n_k) - G'_{k-1}R(C^n_k))^2 \to 0,\ n \to \infty.$

This contradicts the condition that

$$R(C_k^n)\big|_\Delta = C_k^n(\xi,\dots,\xi)\big|_\Delta,\ n \geq 1$$

and the above-mentioned fact [9] that the restriction of a Gaussian measure on the set of positive measure is polynomially nondegenerated. So the hypothesis $\|G_k''G_{k-1}'\| = 1$ is wrong. Consequently, the solution to (3.4) exists, is unique, and can be obtained by the iteration method. Lemma 3.1 is proved.

Remark. Note that, due to the properties of R, the iteration method can be applied directly to (3.2).

Now let us rewrite (3.1) in terms of Y. Let us look for $C_0,\dots,C_N$ such that $\forall\ j = 0,\dots,N$:

$$Q_j\left(\sum_{k=0}^{N} Y(C_k)\right) = Q_j(A_{N+1}). \tag{3.5}$$

Note that, due to the properties of Y, (3.5) actually is a triangular system: $\forall\ j = 0,\dots,N$

$$Q_j\left(\sum_{k=0}^{j} Y(C_k)\right) = Q_j(A_{N+1}). \tag{3.6}$$

Accordingly to Lemma 3.1, (3.6) has a unique solution.

Note that the orthogonal "polynomials" $Y(A_k)$ can be obtained by the iteration method. It is clearly seen in the case $\mathcal{F}' = \mathcal{F}_\tau$ because, in this situation, the operators Q (and G, respectively) become integral operators of the first kind.

Bibliography

1. D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, New York, 1995, pp. 266.
2. M. Sanz–Solé, *Malliavin Calculus. With Applications to Stochastic Partial Differential Equations*, EPEL Press, Lausanne, 2005, pp. 162.
3. A.A. Dorogovtsev, *Stochastic Analysis and Random Maps in Hilbert Space*, VSP, Utrecht, Tokyo, 1994, pp. 110.
4. T. Sekiguchi, Y. Shyota, *L_2-theory of non-causal stochastic integrals*, Math. Reports Toyama Univ. **85** (1985), no. 8, 119–195.
5. A.A. Dorogovtsev, *Stochastic integral with respect to Arratia flow*, Dokl. Akad. Nauk **410** (2006), no. 2, 156–157.
6. N.I. Portenko, *On multidimensional skew Brownian motion and the Feynman–Kac formula*, Stochastic Processes **4** (1998), no. 1–2, 60–70.
7. Le J. Yves, R. Olivier, *Flows, coalescense and noise*, Ann. Probab. **32** (2004), no. 2, 1247–1315.
8. Dorogovtsev A.A., *Measurable functionals and finitely absolutely continuous measures on Banach space*, Ukrainian. Math. Journ. **52** (2000), no. 9, 1194–1204.
9. V.E. Berezhnoy, *On the equivalence of norms on the space of γ-measurable polynomials*, Moscow Univ. Bull., Ser. Math. (2004), no. 4, 54–56.

Institute of Mathematics,National Academy of Science of Ukraine, 3, Tereshchenkivs'ka Str., Kyiv 10601, Ukraine

E-mail: adoro@imath.kiev.ua