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ASYMPTOTIC EXPANSIONS FOR QUASI-STATIONARY DISTRIBUTIONS OF NONLINEARLY PERTURBED SEMI-MARKOV PROCESSES

Asymptotic expansions are given for quasi-stationary distributions of nonlinearly perturbed semi-Markov processes.

1. Introduction

Quasi-stationary phenomena in stochastic systems are a subject of intensive studies that started in 60s of the 20th century. These phenomena describe the behaviour of stochastic systems with random lifetimes. The core of the quasi-stationary phenomenon is that one can observe something that resembles a stationary behaviour of the system before the lifetime goes to the end.

Examples of stochastic systems, in which quasi-stationary phenomena can be observed, are various queuing systems and reliability models, in which the lifetime is usually considered to be the time in which some kind of a fatal failure occurs in the system. Another class of examples of such stochastic systems is supplied by population dynamics or epidemic models. In population dynamics models, the lifetimes are usually the extinction times for the corresponding populations. In epidemic models, the role of the lifetime is played by the time of extinction of the epidemic in the population.

Usually the behaviour of a stochastic system can be described in terms of some Markov type stochastic process $\eta^{(\varepsilon)}(t)$ and its lifetime defined to be the time $\mu^{(\varepsilon)}$ at which the process $\eta^{(\varepsilon)}(t)$ hits a special absorption subset of the phase space of this process for the first time. A typical situation is when the process $\eta^{(\varepsilon)}(t)$ and the absorption time $\mu^{(\varepsilon)}$ depend on a small parameter $\varepsilon \geq 0$ in the sense that some of their local "transition" characteristics depend

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on the parameter ε. The parameter ε is involved in the model in such a way that the corresponding local characteristics are continuous at the point $\varepsilon = 0$, if regarded as functions of ε. These continuity conditions permit to consider the process $\eta^{(\varepsilon)}(t)$, for $\varepsilon > 0$ as a perturbed version of the process $\eta^{(0)}(t)$. In many cases, under some natural communication and aperiodicity conditions imposed on the local transition characteristics of the process $\eta^{(\varepsilon)}(t)$, there exists a so-called quasi-stationary distribution for the process $\eta^{(\varepsilon)}(t)$, which is given by the formula,

$$\pi^{(\varepsilon)}(A) = \lim_{t\to\infty} \mathsf{P}\{\eta^{(\varepsilon)}(t) \in A \;/\; \mu^{(\varepsilon)} > t\}. \tag{1}$$

In this paper, we give asymptotic expansions for quasi-stationary distributions of nonlinearly perturbed semi-Markov processes. The principal novelty of the result is that we consider the model with nonlinear perturbations. By a nonlinear perturbation we mean that local transition characteristics (that are some moment functionals of transition probabilities for the corresponding semi-Markov processes) are nonlinear functions of the perturbation parameter ε and that the assumptions made imply that the characteristics can be expanded in an asymptotic power series with respect to ε up to and including some order k.

2. Main result

Let, for every $\varepsilon \geq 0$, $(\eta_n^{(\varepsilon)}, \kappa_n^{(\varepsilon)}), n = 0, 1, \ldots$, be a Markov renewal process with the phase space $X = \{0, 1, \ldots, N\}$ and transition probabilities $Q_{ij}^{(\varepsilon)}(u)$. Let $\nu^{(\varepsilon)}(t) = \max(n : \tau^{(\varepsilon)}(n) \leq t),\ t \geq 0$, where $0 = \tau^{(\varepsilon)}(0), \tau^{(\varepsilon)}(n) = \kappa_1^{(\varepsilon)} + \ldots \kappa_n^{(\varepsilon)}, n \geq 1$, and $\eta^{(\varepsilon)}(t) = \eta^{(\varepsilon)}_{\nu^{(\varepsilon)}(t)}, t \geq 0$, be the corresponding semi-Markov process associated with the Markov renewal process $(\eta_n^{(\varepsilon)}, \kappa_n^{(\varepsilon)})$.

We write the transition probabilities as $Q_{ij}^{(\varepsilon)}(t) = p_{ij}^{(\varepsilon)} F_{ij}^{(\varepsilon)}(t)$, where $p_{ij}^{(\varepsilon)} = Q_{ij}^{(\varepsilon)}(\infty)$ are the transition probabilities of the corresponding imbedded Markov chain and $F_{ij}^{(\varepsilon)}(t)$ are the distribution functions of transition times. For simplicity we assume that the following condition preventing instant jumps holds:

I: $F_{ij}^{(\varepsilon)}(0) = 0,\ i, j \in X$ for all $\varepsilon \geq 0$.

We consider a model in which 0 is an absorbing state, that is, the following condition holds:

A: $p_{0j}^{(\varepsilon)} = 0, j \neq 0$, for all $\varepsilon \geq 0$.

Let also $\nu_j^{(\varepsilon)} = \min\{n \geq 1 : \eta_n^{(\varepsilon)} = j\}$ and $\mu_j^{(\varepsilon)} = \tau^{(\varepsilon)}(\nu_j^{(\varepsilon)})$ be the first hitting time at which the imbedded Markov chain $\eta_n^{(\varepsilon)}$ and the semi-Markov process $\eta^{(\varepsilon)}(t)$ hit the state j, respectively. The first hitting time $\mu_0^{(\varepsilon)}$ is the absorption time for the semi-Markov process $\eta^{(\varepsilon)}(t)$.

We also assume the following condition imposed on the transition probabilities of the semi-Markov process $\eta^{(\varepsilon)}(t)$, which allows to consider these processes for $\varepsilon > 0$ as perturbed versions of the semi-Markov process $\eta^{(0)}(t)$:

T: **(a)** $p_{ij}^{(\varepsilon)} \to p_{ij}^{(0)}$ as $\varepsilon \to 0,\ i \neq 0, j \in X$;

(b) $F_{ij}^{(\varepsilon)}(\cdot) \Rightarrow F_{ij}^{(0)}(\cdot)$ as $\varepsilon \to 0,\ i \neq 0, j \in X$.

Here and henceforth, the symbol $\Rightarrow$ is used to denote weak convergence for distribution functions.

Let us introduce the hitting-without-absorption probabilities ${}_0f_{ij}^{(\varepsilon)} = \mathsf{P}_i\{\nu_j^{(\varepsilon)} < \nu_0^{(\varepsilon)}\},\ i, j \neq 0$.

We assume that the set $X_0 = \{j \neq 0\}$ is a class of recurrent-without-absorption states for the limiting semi-Markov process, i.e., the following condition holds:

E: ${}_0f_{ij}^{(0)} > 0$ for all $i, j \neq 0$.

Let us introduce distribution functions

$$
{}_0G_{ij}^{(\varepsilon)}(t) = \mathsf{P}_i\{\mu_0^{(\varepsilon)} \leq t, \nu_j^{(\varepsilon)} < \nu_0^{(\varepsilon)}\},\ t \geq 0,
$$

and the following mixed power-exponential moment generating functions, for $i, j \neq 0, n = 0, 1, \ldots$,

$$
\psi_{ij}^{(\varepsilon)}[\rho, n] = \int_0^\infty t^n e^{\rho t} F_{ij}^{(\varepsilon)}(dt),\ \rho \geq 0,
$$

and

$$
{}_0\phi_{ij}^{(\varepsilon)}[\rho, n] = \int_0^\infty t^n e^{\rho t}\, {}_0G_{ij}^{(\varepsilon)}(dt) = \mathsf{E}_i(\mu_0^{(\varepsilon)})^n e^{\rho \mu_0^{(\varepsilon)}} \chi(\nu_j^{(\varepsilon)} < \nu_0^{(\varepsilon)}),\ \rho \geq 0.
$$

We assume also the following conditions:

$\mathbf{C_1}$: There exists $\delta > 0$ such that $\overline{\lim}_{0 \leq \varepsilon \to 0} \psi_{ij}^{(\varepsilon)}[\delta, 0] < \infty,\ i \neq 0, j \in X$,

and

$\mathbf{C_2}$: There exist $i \neq 0$ and $\beta_i \in (0, \delta]$, for δ defined in condition $\mathbf{C_1}$, such that ${}_0\phi_{ii}^{(0)}[\beta_i, 0] \in (1, \infty)$.

Let us consider the following characteristic equations, for $i \neq 0$,

$$
{}_0\phi_{ii}^{(0)}[\rho, 0] = 1. \tag{2}
$$

It can be shown that conditions **I**, **A**, **T**, **E**, and $\mathbf{C_1}$, $\mathbf{C_2}$ imply that there exists $\varepsilon_1 > 0$ such that for every $\varepsilon \leq \varepsilon_1$: **(a)** equation (2) have a unique root $\rho^{(\varepsilon)} \geq 0$; **(b)** the root $\rho^{(\varepsilon)}$ does not depend of the choice of the state $i \neq 0$; **(c)** there exists $\beta < \delta$ such that $\rho^{(\varepsilon)} < \beta$; **(d)** $\rho^{(\varepsilon)} \to \rho^{(0)}$ as $\varepsilon \to 0$; **(e)** $\rho^{(0)} > 0$ if and only if $\sum_{i\neq 0} p_{i0}^{(0)} > 0$.

Let also assume the following non-arithmetic condition:

N: There exists $i \neq 0$ such that the distribution function ${}_0G_{ii}^{(\varepsilon)}(t)$ is non-arithmetic.

It can be shown that conditions **I**, **A**, **T**, and **E** imply that ${}_0G_{ii}^{(\varepsilon)}(t)$ is arithmetic or non-arithmetic distribution function for all $i \neq 0$ simultaneously.

It can be shown that conditions **I**, **A**, **T**, **E**, **N**, and $\mathbf{C_1}$, $\mathbf{C_2}$ imply that there exist $\varepsilon_1 \geq \varepsilon_2 > 0$ such that for every $\varepsilon \leq \varepsilon_2$ the following limits exist for $i \neq 0$,

$$\lim_{t\to\infty} \mathsf{P}_i\{\eta^{(\varepsilon)}(t) = l/\mu_0^{(\varepsilon)} > t\} = \pi_l^{(\varepsilon)}(\rho^{(\varepsilon)}),\ l \neq 0. \tag{3}$$

The limits satisfy the following conditions: **(f)** they do not depend on $i \neq 0$; **(g)** $\pi_l^{(\varepsilon)}(\rho^{(\varepsilon)}) > 0, l \neq 0$; **(h)** $\sum_{l\neq 0} \pi_l^{(\varepsilon)}(\rho^{(\varepsilon)}) = 1$. Relations **(f)** – **(h)** clarify why it is natural to call the distribution $\pi_l^{(\varepsilon)}(\rho^{(\varepsilon)}), l \neq 0$ a quasi-stationary distribution for the semi-Markov process $\eta^{(\varepsilon)}(t)$ with absorption times $\mu_0^{(\varepsilon)}$.

Also let us consider the following mixed power-exponential moment generating functions, for $i, j, l \neq 0$ and $n = 0, 1, \ldots$.

$$\omega_{ijl}^{(\varepsilon)}[\rho, n] = \int_0^\infty s^n e^{\rho s} \mathsf{P}_i\{\eta^{(\varepsilon)}(s) = l, \mu_j^{(\varepsilon)} \wedge \mu_0^{(\varepsilon)} > s\} ds,\ \rho \geq 0,$$

and

$$\omega_{ij}^{(\varepsilon)}[\rho, n] = \int_0^\infty s^n e^{\rho s} \mathsf{P}_i\{\mu_j^{(\varepsilon)} \wedge \mu_0^{(\varepsilon)} > s\} ds = \sum_{l\neq 0} \omega_{ijl}^{(\varepsilon)}[\rho, n],\ \rho \geq 0.$$

It can be shown that, under conditions **I**, **A**, **T**, **E** and $\mathbf{C_1}$, $\mathbf{C_2}$ there exists $\varepsilon_2 \geq \varepsilon_3 > 0$ such that **(i)** $\omega_{ij}^{(\varepsilon)}[\rho, n] < \infty$ for $\rho \leq \beta$, $i, j \neq 0$, $n = 0, 1, \ldots$ and $\varepsilon \leq \varepsilon_3$, and **(j)** the following formula takes place for the quasi-stationary distribution, for every $\varepsilon \leq \varepsilon_3$ and state $j \neq 0$,

$$\pi_l^{(\varepsilon)}(\rho^{(\varepsilon)}) = \frac{\omega_{jjl}^{(\varepsilon)}[\rho^{(\varepsilon)}, 0]}{\omega_{jj}^{(\varepsilon)}[\rho^{(\varepsilon)}, 0]},\ l \neq 0. \tag{4}$$

Note that this formula defines the quasi-distribution even if condition **N** does not hold.

Finally, let us introduce the following nonlinear perturbation conditions:

P: $p_{ij}^{(\varepsilon)} = p_{ij}^{(0)} + \varepsilon e_{ij}[1] + \cdots + \varepsilon^k e_{ij}[k] + o(\varepsilon^k)$ for $i, j \neq 0$, where $|e_{ij}[r]| < \infty, r = 1, \ldots, k, i, j \neq 0$.

and, for $\rho < \beta$,

P$[\rho]$: $\psi_{ij}^{(\varepsilon)}[\rho, n] = \psi_{ij}^{(0)}[\rho, n] + \varepsilon v_{ij}[\rho, 1, n] + \cdots + \varepsilon^{k+1-n} v_{ij}[\rho, k+1-n, n] + o(\varepsilon^{k+1-n})$ for $n = 0, \ldots, k+1, i \neq 0, j \in X$, where $|v_{ij}[\rho, r, n]| < \infty, r = 1, \ldots, k+1-n, n = 0, \ldots, k+1, i \neq 0, j \in X$.

The following theorem presents the asymptotic expansions for quasi-stationary distributions for nonlinearly perturbed semi-Markov processes.

Theorem 1. *Let conditions* **I**, **A**, **T**, **E**, $\mathbf{C_1}$, $\mathbf{C_2}$, **P**, *and* $\mathbf{P}[\rho^{(\mathbf{0})}]$ *hold. Then, the quasi-stationary probabilities* $\pi_l^{(\varepsilon)}(\rho^{(\varepsilon)}), l \neq 0$, *which do not depend on the choice of the state* $j \neq 0$ *in formula* (4), *have the following asymptotic expansion, for every* $l, j \neq 0$:

$$\pi_l^{(\varepsilon)}(\rho^{(\varepsilon)}) = \frac{\omega_{jjl}^{(0)}[\rho^{(0)}, 0] + \varepsilon f_{jjl}[\rho^{(0)}, 1] + \cdots + \varepsilon^k f_{jjl}[\rho^{(0)}, k] + o(\varepsilon^k)}{\omega_{jj}^{(0)}[\rho^{(0)}, 0] + \varepsilon f_{jj}[\rho^{(0)}, 1] + \cdots + \varepsilon^k f_{jj}[\rho^{(0)}, k] + o(\varepsilon^k)} \tag{5}$$
$$= \pi_l^{(0)}(\rho^{(0)}) + \varepsilon g_l[\rho^{(0)}, 1] + \cdots + \varepsilon^k g_l[\rho^{(0)}, k] + o(\varepsilon^k),$$

where the coefficients $f_{jjl}[\rho^{(0)}, r], f_{jj}[\rho^{(0)}, r], g_l[\rho^{(0)}, r], r = 1, \ldots, k$ *are given by explicit recurrence formulas as functions of coefficients in the expansions penetrating the perturbation conditions* **P** *and* $\mathbf{P}[\rho^{(\mathbf{0})}]$.

3. Conclusion

In conclusion, I would like to note that this paper presents the result from a new book written in cooperation with Professor Mats Gyllenberg. The algorithm for evaluation of the coefficients in the asymptotic expansions (5) and the proof of Theorem 1 is given in this book.

The book mentioned above is devoted to studies of quasi-stationary phenomena in nonlinearly perturbed stochastic systems. The methods based on exponential asymptotics for nonlinearly perturbed renewal equation are used. Mixed ergodic and large deviation theorems are presented for nonlinearly perturbed regenerative processes, semi-Markov processes and Markov chains. Applications to nonlinearly perturbed population dynamics and epidemic models, queueing systems and risk processes are considered. The book also includes an extended bibliography of works in the area.

Bibliography

1. Gyllenberg, M and Silvestrov, D.S. *Quasi-stationary Phenomena in Nonlinearly Perturbed Stochastic Systems* (submitted).

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