Theory of Stochastic Processes
Vol.13 (29), no.1-2, 2007, pp.243-250

YU. S. MISHURA AND G. M. SHEVCHENKO

ON DIFFERENTIABILITY OF SOLUTION TO STOCHASTIC DIFFERENTIAL EQUATION WITH FRACTIONAL BROWNIAN MOTION

Stochastic differential equation with pathwise integral with respect to fractional Brownian motion is considered. For solution of such equation, under different conditions, the Malliavin differentiability is proved. Under infinite differentiability and boundedness of derivatives of the coefficients it is proved that the solution is infinitely differentiable in the Malliavin sense with all derivatives bounded.

1. Introduction

Let $\left\{B_t^H = (B_t^{1,H}, \dots, B_t^{m,H}),\ t \geq 0\right\}$ be m-dimensional fractional Brownian motion (fBm in short) of Hurst parameter $H > \frac{1}{2}$ on a filtered probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t, t \geq 0\},\ P)$. That is, B_t^H is an $\mathcal{F}_t$-adapted Gaussian process, whose components are independent and have the covariance function

$$\mathsf{E}\left[B_t^{i,H} B_s^{i,H} \right] = R_H(t,s) := \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big).$$

There are different ways to define stochastic integrals with respect to fBm. We choose in this paper the approach of Zähle [10], that is, the Riemann–Stiltjes integral defined in pathwise sense.

In this paper we consider the following equation

$$\begin{aligned} X_t &= X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s^H \\ &= X_0 + \int_0^t b(X_s)ds + \sum_{j=1}^m \int_0^t \sigma^j(X_s)dB_s^{j,H}, \quad t \geq 0, \end{aligned} \tag{1}$$

or

$$X_t^i = X_0^i + \int_0^t b_i(X_s)ds + \sum_{j=1}^m \int_0^t \sigma_i^j(X_s)dB_s^{j,H}, \quad i = 1, ..., d.$$

2000 *Mathematics Subject Classifications.* Primary: 60H05, Secondary: 60H07

Key words and phrases. Fractional Brownian motion, pathwise integral, stochastic differential equation, Malliavin derivative.

Many authors studied existence and uniqueness of solution of (1), see [3], [4], [5], [9], [7].

In this paper we investigate stochastic differentiability of the solution of (1). This question was already studied in the paper of Nualart and Saussereau [8], where they have proved that under the conditions that the coefficients are infinitely differentiable and bounded together with their derivatives, the solution will be infinitely differentiable (in a local sense). From a point of view of stochastic derivatives of Nelson's type this problem was studied by Darses and Nourdin [2]. In this paper we establish strong (non-local) differentiability results in two cases: for the diffusion coefficient is linear and for one-dimensional equation with infinitely differentiable coefficients. In the latter case we also prove the uniform boundedness of stochastic derivatives.

1. Stochastic derivative w.r.t. fBm

We briefly recall the notion of the stochastic derivative with respect to the fBm, the detailed description can be found in [1]. First we define the Hilbert space $\mathcal{H}$ associated to the fBm as the closure of the space $\mathbb{R}^m$-valued step function with respect to the scalar product

$$\left\langle(\mathbf{1}_{[0,t_1]},\dots,\mathbf{1}_{[0,t_m]}),((\mathbf{1}_{[0,s_1]},\dots,\mathbf{1}_{[0,s_m]})\right\rangle_{\mathcal{H}} := \sum_{i=1}^m R_H(t_i,s_i).$$

The space $\mathcal{H}$ contains not only usual functions, but also distributions. For $\varphi,\psi\in L^{\frac{1}{H}}([0,T];\mathbb{R}^m)$ one has

$$\langle\varphi,\psi\rangle_{\mathcal{H}} = H(2H-1)\int_0^T\int_0^T \phi(r)\psi(u)\,|r-u|^{2H-2}\,dr\,du.$$

The mapping

$$B:(\mathbf{1}_{[0,t_1]},\dots,\mathbf{1}_{[0,t_m]})\longmapsto\sum_{i=1}^m B_{t_i}^{i,H}$$

can be extended to the isometry between $\mathcal{H}$ and the Hilbert space $H_1(B^H)$ associated with B^H.

For a smooth variable of the form $F = f(B(\varphi_1),\dots,B(\varphi_n))$, where $f\in C_b^\infty(\mathbb{R}^n)$, $\varphi_i\in\mathcal{H}$ the stochastic derivative, or Malliavin derivative, is defined as the $\mathcal{H}$-valued random variable

$$DF := \sum_{i=1}^n \partial_{x_i} f(B(\varphi_1),\dots,B(\varphi_n))\varphi_i.$$

This operator is closable from $L^p(\Omega)$ to $L^p(\Omega;\mathcal{H})$. It is also convenient to write $DF = \{D_sF, s\geq 0\}$ in many cases when DF has usual, not generalized, meaning. The space $\mathbb{D}^{k,p}$ is defined as the closure of the space of smooth random variables with respect to the norm

$$\|F\|_{k,p} := \left(\mathsf{E}\left[\, |F|^p \,\right] + \sum_{j=1}^{k} \mathsf{E}\left[\, \left\|D^j F\right\|^p_{\mathcal{H}^{\otimes j}} \,\right] \right)^{\frac{1}{p}}.$$

We denote by $\mathbb{D}^{k,p}_{\text{loc}}$ the corresponding local domain, i.e. the set of random variables F such that there exists a sequence $\{(\Omega_n, F_n), n \geq 1\} \subset \mathcal{F} \times \mathbb{D}^{k,p}$ satisfying $\Omega_n \uparrow \Omega$, $n \to \infty$ and $F = F_n$ on Ω_n. In [8], the following fact is proved.

Theorem 1. *Let $H > 1/2$ and assume that the coefficients b and σ are infinitely differentiable functions which are bounded together with all their derivatives, then the solution of the SDE* (1) *belongs to $\mathbb{D}^{k,p}_{\text{loc}}(\mathcal{H})$, for any $p > 0$, $k \geq 1$.*

Remark 2. It is easily seen from the argument of paper [8] that for a given $k \geq 1$ one needs no infinite differentiability of b and σ to prove the fact $X_t \in \mathbb{D}^{k,p}_{\text{loc}}$, it is only enough that $b, \sigma \in C^{k+1}(\mathbb{R}^d)$.

We will consider two cases when the global differentiability can be proved for the solution of (1).

2. Differentiability of the solution to quasilinear SDE

Consider equation (1), in which $H > 3/4$ and the coefficient σ is linear, that is $\sigma^j(x) = \sigma^j x$ is some linear operator:

$$X_t = X_0 + \int_0^t b(X_s)ds + \sum_{j=1}^{m} \int_0^t \sigma^j X_s dB^{j,H}_s, \quad i = 1, ..., d. \tag{2}$$

We will assume that the coefficient $b \in C^1_b(\mathbb{R}^d)$ and that X_0 is $\mathcal{F}_0$-measurable bounded random variable. Together with the linearity of σ, these conditions is enough to assure that equation (2) has unique solution, which belongs to all $L^p(\Omega)$, see [7].

Theorem 3. *Under the above assumptions the solution of* (2) *belongs to $\mathbb{D}^{1,p}$ for any $p > 0$.*

Proof. We will assume for simplicity that $d = 1$, all argumentation transfers easily to arbitrary dimension. Throughout the proof we will denote by C all constants which may depend on the coefficients b and σ, but are independent of everything else. We remind that the unique solution of (2) can be constructed as the limit of successive approximations $\left\{X^{(n)}_t, n \geq 1\right\}$, where $X^{(0)}_t \equiv X_0$. Now we are going to prove by induction that for every $p > 0$ $X^{(n)}_t \in \mathbb{D}^{1,p}$ and $D_s X^{(n)}_t$ is Hölder continuous of order $1 - \alpha$ for some $\alpha \in (1 - H, 1/2)$. This is, of course, obvious for $n = 0$. Assume this is true for n. Then the integrals $\int_s^t \sigma D_s X_r^{(n)} dB_r^H$ and $\int_s^t b(X_r^{(n)}) dB_r^H$ are well-defined and due to the closedness of the stochastic derivative we can write

$$D_s X_r^{(n+1)} = \sigma X_s^{(n)} + \int_s^t b'(X_r^{(n)}) D_s X_r^{(n)} dr + \int_s^t \sigma D_s X_r^{(n)} dB_r^H.$$

Now we can write, using the Hölder continuity assumption and well-known estimates for an integral with respect to the fBm [7],

$$\begin{aligned}\left|D_s X_t^{(n+1)}\right| \le C_1(\omega) + C_2(\omega) \int_s^t \frac{\left|D_s X_r^{(n)}\right|}{(r-s)^\alpha} dr \\ + C_2(\omega) \int_s^t \int_s^r \frac{\left|D_s X_r^{(n)} - D_s X_u^{(n)}\right|}{(r-u)^{1+\alpha}} du\, dr,\end{aligned}$$

where $C_1(\omega) = C \exp\{C G_\alpha^\varkappa\}$, $C_2(\omega) = C G_\alpha$, $\varkappa = 1/(1-2\alpha)$, G_α is certain random variable s.t. $\mathsf{E}\left[\exp\left\{p G_\alpha^\delta\right\}\right] < \infty$ for all $p > 0$, $\delta \in (0,2)$. Similarly,

$$\begin{aligned}&\left|D_s X_r^{(n+1)} - D_s X_u^{(n+1)}\right| \\ &\quad \le C_2(\omega) \int_u^r \frac{\left|D_s X_z^{(n)}\right|}{(z-u)^\alpha} dz + C_2(\omega) \int_u^r \int_u^z \frac{\left|D_s X_z^{(n)} - D_s X_v^{(n)}\right|}{(z-v)^{1+\alpha}} dv\, dz.\end{aligned}$$

Define

$$\begin{aligned}&\varphi_n^1(t,s) = \left|D_s X_t^{(n)}\right|,\ \varphi_n^2(t,s) = \int_s^t \frac{\left|D_s X_t^{(n)} - D_s X_u^{(n)}\right|}{(t-u)^{1+\alpha}} du, \\ &\varphi_n(t,s) = \varphi_n^1(t,s) + \varphi_n^2(t,s),\end{aligned}$$

so that we can write

$$\begin{aligned}&\varphi_{n+1}^1(t,s) \le C_1(\omega) + C_2(\omega) \int_s^t \varphi_n^1(r,s)(r-s)^{-\alpha} dr + C_2(\omega) \int_s^t \varphi_n^2(r,s) dr, \\ &\varphi_{n+1}^2(t,s) \le C_2(\omega) \int_s^t \varphi_n^1(v,s)(t-v)^{-2\alpha} dv + C_2(\omega) \int_s^t \varphi_n^2(v,s)(t-v)^{-\alpha} dv,\end{aligned}$$

whence

$$\varphi_{n+1}(t,s) \le C_1(\omega) + C_2(\omega) \int_s^t \left((u-s)^{-2\alpha} + (t-u)^{-2\alpha}\right) \varphi_n(u,s) du$$

and one gets easily by induction that

$$\varphi_n(t,s) \le C_1(\omega) \exp\left\{C_\alpha \left(C_2(\omega)\right)^\varkappa (t-s)\right\},$$

where $C(\alpha) = \big(4\Gamma(1-2\alpha)\big)^{\varkappa}$. Since $H > 3/4$, we can choose $\alpha > 1-H$ so that $\varkappa < 2$. Then we have $\mathsf{E}\left[\,|\varphi_n|^p\,\right] \le C_p$ due to the properties of the random variable G_α. Therefore, we have for any $p>0$ $\left\|X_t^{(n)}\right\|_{1,p} \le C_p$ with constant independent of n.

Further,

$$\left|X_t^{(n+1)} - X_t\right| \le C_2(\omega)\int_0^t \frac{\left|X_s^{(n)} - X_s\right|}{s^\alpha}ds \\ + C_2(\omega)\int_0^t\int_0^r \frac{\left|X_r - X_r^{(n)} - X_u + X_u^{(n)}\right|}{(r-u)^{1+\alpha}}du\,dr,$$

$$\int_0^t \frac{\left|X_t^{(n+1)} - X_t - X_u^{(n+1)} + X_u\right|}{(t-u)^{1+\alpha}}du \\ \le \int_0^t \frac{du}{(t-u)^{1+\alpha}}\Big(C_2(\omega)\int_u^t \left|X_s^{(n)} - X_s\right| s^{-\alpha}ds \\ + C_2(\omega)\int_u^t\int_u^r \frac{\left|X_r - X_v - X_r^{(n)} + X_v^{(n)}\right|}{(r-v)^{1+\alpha}}dv\,dr\Big) \\ \le C_2(\omega)\int_0^t \left|X_s^{(n)} - X_s\right| s^{-\alpha}(t-s)^{-\alpha}ds \\ + C_2(\omega)\int_0^t (t-s)^{-\alpha}\int_0^s \frac{\left|X_s - X_v - X_s^{(n)} + X_v^{(n)}\right|}{(s-v)^{1+\alpha}}ds,$$

or

$$\xi_{n+1}^1(t) \le C_2(\omega)\int_0^t \xi_n^1(s)s^{-\alpha}ds + C_2(\omega)\int_0^t \xi_n^2(s)ds,$$
$$\xi_{n+1}^2(t) \le C_2(\omega)\int_0^t \xi_n^1(s)s^{-\alpha}(t-s)^{-\alpha}ds + C_2(\omega)\int_0^t \xi_n^2(s)(t-s)^{-\alpha}ds,$$

where

$$\xi_n^1(t) = \left|X_t^{(n)} - X_t\right|,\quad \xi_n^2(t) = \int_0^t \frac{\left|X_t^{(n)} + X_u^{(n)} - X_t + X_u\right|}{(t-u)^{1+\alpha}}du.$$

Define $\xi_n = \xi_n^1 + \xi_n^2$ and write

$$\xi_{n+1}(t) \le C_2(\omega)\int_0^t s^{-\alpha}(t-s)^{-\alpha}\xi_n(s)ds \\ \le C_2(\omega)t^{2\alpha}\int_0^t s^{-2\alpha}(t-s)^{-2\alpha}\xi_n(s)ds.$$

It is easily proved by induction that

$$\sup_{0\le t\le T}|\xi_n(t)| \le \frac{C^{n+1}G_\alpha^{n+1}C_3(\omega)}{\Gamma(n(1-2\alpha))},$$

where

$$C_3(\omega)=\sup_{0\le t\le T}|\xi_0(t)|=\sup_{0\le t\le T}\left(|X_0|+|X_t|+\int_0^t\frac{|X_t-X_r|}{(t-r)^{1+\alpha}}drdu\right).$$

It is well-known that for all $p>0$ $\mathsf{E}\left[(C_3(\omega))^p\right]<\infty$. Further, we can write for some $\delta\in(\varkappa,2)$ the inequality $g^n e^{-g^\delta}\le (n/(e\delta))^{\frac{n}{\delta}}$ (the right-hand side is the maximal value for the left-hand side for $g\ge 0$) and apply that $\Gamma(\alpha)\ge C^\alpha\alpha^\alpha$ for α large enough to obtain

$$\sup_{0\le t\le T}|\xi_n(t)|\le \left(C_{\alpha,\delta}\right)^n n^{n(1/\delta-1/\varkappa)}\exp\left\{G^\delta\right\}. \tag{3}$$

Since for all $p>0$ $\mathsf{E}\left[\exp\left\{pG^\delta\right\}\right]<\infty$, inequality (3) yields

$$\mathsf{E}\left[\sup_{0\le t\le T}|\xi_n(t)|^p\right]\to 0,\ n\to\infty.$$

Consequently, $X_t^{(n)}\to X_t$ in $L^p(\Omega)$. Moreover, we can show similarly that for any $p>0$

$$\mathsf{E}\left[\sup_{s,t}\left|D_sX_t^{(n)}-D_sX_t^{(m)}\right|^p\right]\to 0,\quad m,n\to\infty.$$

Thus, the derivatives $DX_t^{(n)}$ converge in $L^p(\Omega;\mathcal{H})$, but the closedness of the stochastic derivative then gives that the limit is DX_t and that $\|X_t\|_{1,p}<\infty$, which concludes the proof. □

Remark 4. Theorem 1 requires boundedness of σ and provides only local differentiability of solution. Theorem 3 above gives global differentiability for an unbounded σ, but the price we pay is the assumptions that $H>3/4$ and that σ is linear.

2. Differentiability of the solution in one-dimensional case

Consider one-dimensional equation

$$X_t=X_0+\int_0^t b(X_s)ds+\int_0^t\sigma(X_s)dB_s^H. \tag{4}$$

We assume that $b,\sigma\in C_b^\infty(\mathbb{R})$. We remind that these assumptions guarantee existence and uniqueness of solution, and moreover by Theorem 1 the solution will be in $\mathbb{D}^{k,p}_{\mathrm{loc}}$ for any $p > 0$, $k \geq 1$. We are going to prove that X_t is globally differentiable and all derivatives are bounded under additional assumption of non-degeneracy of σ.

Theorem 5. *Assume that the coefficients of equation* (4) *satisfy* $b, \sigma \in C_b^\infty(\mathbb{R})$, $\inf_{\mathbb{R}} |\sigma| > 0$. *Then for all* $k \geq 1$ *the solution* X_t *of this equation belongs to* $\mathbb{D}^{k,\infty}$, *i.e.,* X_t *is infinitely stochastically differentiable and all its derivatives are essentially bounded.*

Proof. We assume without loss of generality that $\sigma(x) > 0$. It is proved in [8] that the stochastic derivative of X_t satisfies the linear equation

$$D_r X_t = \sigma(X_r) + \int_r^t b'(X_s) D_r X_s ds + \int_r^t \sigma'(X_s) D_r X_s dB_s^H, \quad t \geq r \qquad (5)$$

which is easily solved, and the solution is

$$D_r X_t = \sigma(X_r) \exp\left\{ \int_r^t b'(X_s) ds + \int_r^t \sigma'(X_s) dB_s^H \right\}, \quad t \geq r.$$

According to Itô formula

$$\log \sigma(X_t) = \log \sigma(X_r) + \int_r^t \sigma'(X_s)\sigma^{-1}(X_s)\big[b(X_s)ds + \sigma(X_s) dB_s^H\big],$$

whence

$$\sigma(X_r) \exp\left\{ \int_r^t \sigma'(X_s) dB_s^H \right\} = \sigma(X_t) \exp\left\{ \int_r^t \sigma'(X_s)\sigma^{-1}(X_s) b(X_s) ds \right\},$$

and thus the derivative $D_r X_t$ is uniformly bounded, which already means that X_t is differentiable in usual sense rather then local. Moreover, we can write

$$D_r X_t = \sigma(X_t) \exp\left\{ \int_r^t b'(X_s) ds + \int_r^t \sigma'(X_s)\sigma^{-1}(X_s) b(X_s) ds \right\} = \sigma(X_t)\mathcal{E}$$

and differentiate this equation, getting for $u \vee r \leq t$

$$\begin{aligned} D_u D_r X_t = \mathcal{E} \times \Big(& \sigma'(X_t) D_u X_t \\ & + \int_{r\vee u}^t b''(X_s) D_u X_s \, ds + \int_{r \vee u}^t \big[\sigma''(X_s)\sigma^{-1}(X_s) b(X_s) \\ & - (\sigma'(X_s))^2 \sigma^{-2}(X_s) b(X_s) + \sigma'(X_s)\sigma^{-1}(X_s) b'(X_s)\big] D_u X_s \, ds \Big). \end{aligned}$$

Then $D_u D_r X_t$ exists and is uniformly bounded. Going on, we can easily prove by induction that

$$D_{s_1} \dots D_{s_k} X_t = \mathcal{E}\Big(P_k + \int_{\bigvee_{i=1}^k s_i}^t Q_k \, ds \Big),$$

where P_k and Q_k are polynomials of $D_{s_{i_1}} \dots D_{s_{i_l}} X_t$, $l < k$, $\sigma^{(j)}(X_t)$, $j \le k$ and $D_{s_{i_1}} \dots D_{s_{i_l}} X_s$, $l < k$, $b^{(j)}(X_s)$, $\sigma^{(j)}(X_s)$, $j \le k$, $\sigma^{-1}(X_s)$ respectively. Then existence and boundedness of all derivatives of X_t can be proved by induction. $\square$

Bibliography

1. Alòs, E. and Nualart, D. *Stochastic integration with respect to the fractional Brownian motion,* Stoch. Stoch. Rep. **75**, no. 3 (2002), 129–152.
2. Darses, S. and Nourdin, I. *Stochastic derivatives for fractional diffusions,* Arxiv preprint math.PR/0604315 (2006).
3. Klepstyna, M. L., Kloeden, P. E., and Anh, V. V. *Existence and uniqueness theorems for fBm stochastic differential equations,* Problems Inform. Transmission **34** (1999), 332–341.
4. Kubilius, K. *The existence and uniqueness of the solution of the integral equation driven by fractional Brownian motion,* Lit. Math. J. **40** (2000), 104–110.
5. Mishura, Yu. S., *Quasilinear stochastic differential equations with fractional Brownian component,* Teor. Imovirn. Mat. Stat., no. 68 (2003), 95–106.
6. Nualart, D. *The Malliavin Calculus and Related Topics.* Springer Verlag, Berlin, (1996).
7. Nualart, D. and Răşcanu, A., *Differential equations driven by fractional Brownian motion,* Collect. Math. **53** (2000), 55–81.
8. Nualart, D. and Saussereau, B. *Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion,* Mathematics Preprint Series of the IMUB, no. 371 (2005).
9. Ruzmaikina, A. A. *Stieltjes integral of Holder continuous functions with applications to fractional Brownian motion,* J. Statist. Phys. **100** (2000), 1049–1069.
10. Zähle, M. *Integration with respect to fractal functions and stochastic calculus, I,* Probab. Theory Related Fields **111** (1998), 333–374.

Department of Probability Theory and Mathematical Statistics, Kyiv National Taras Shevchenko University, Kyiv, Ukraine
E-mail address: myus@univ.kiev.ua

Department of Probability Theory and Mathematical Statistics, Kyiv National Taras Shevchenko University, Kyiv, Ukraine
E-mail address: zhora@univ.kiev.ua