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PRECISE ASYMPTOTICS OVER A SMALL PARAMETER FOR A SERIES OF LARGE DEVIATION PROBABILITIES

We obtain the asymptotics of the series

$$\sum_{k=1}^{\infty} w_k \mathbf{P}\left(|S_k| \geq \varepsilon\varphi_k\right)$$

as $\varepsilon \downarrow 0$, where S_k are partial sums of independent and identically distributed random variables in the domain of attraction of a non-degenerate stable law, w and φ are regularly varying functions (in Karamata's sense).

1. Introduction

Let X, $\{X_n, n \geq 1\}$ be independent, identically distributed (i.i.d.) random variables with distribution function F and let $\{S_n, n \geq 1\}$ denote the sequence of their partial sums.

Let w and φ be two given positive functions and put $w_k = w(k)$ and $\varphi_k = \varphi(k)$. We study the convergence and asymptotics over a small parameter of the series

$$Q(\varepsilon) = \sum_{k=1}^{\infty} w_k \mathbf{P}(|S_k| \geq \varepsilon\varphi_k), \qquad \varepsilon > 0. \tag{1}$$

We deal with the case of regularly varying functions w and φ in this paper. The definitions of and necessary results for regularly varying functions can be found in [24] or [4]. We use the notation $f \in \mathcal{RV}(a)$ to say that f is a measurable regularly varying function of order a. We treat the case of large deviation probabilities in (1) (in the sense that $S_k/\varphi_k \to 0$ in probability as

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$k \to \infty$), however there is a number of other papers devoted to moderate and small deviation probabilities.

Series (1) is useful for various applications of limit theorems in probability theory. We mention only one of them related to the topic of the paper.

Hsu and Robbins [17] introduced the so-called *c*omplete convergence of sequences of random variables. According to the definition in [17], the sequence S_n/n converges completely to zero if

$$\sum_{k=1}^{\infty} \mathbf{P}\left(\left|\frac{S_k}{k}\right| \ge \varepsilon\right) < \infty \qquad \text{for all } \varepsilon > 0.$$

Obviously the Hsu–Robbins series is nothing else but $Q(\varepsilon)$ with $w_k = 1$ and $\varphi_k = k$, $k \ge 1$. It was proved in [17] that $Q(\varepsilon) < \infty$ for all $\varepsilon > 0$ if

$$\mathbf{E}X = 0, \qquad \mathbf{E}X^2 < \infty. \tag{2}$$

Erdös [8] was able to show that the converse is also true for the Hsu–Robbins series. After the papers of Hsu and Robbins [17] and Erdös [8], many results have been obtained concerning the series (1) for various functions w and φ. Below is a list of cases studied in earlier papers:

c_1) $w_k = 1$, $\varphi_k = k$ (Hsu and Robbins [17] and Erdös [8]);

c_2) $w_k = 1/k$, $\varphi_k = k$ (Spitzer [26]);

c_3) $w_k = k^r$, $\varphi_k = k^{1/p}$ (Katz [20] and Baum and Katz [3]);

c_4) $w \in \mathcal{RV}(r)$, $w(t)/t^r$ is increasing, $\varphi_k = k^{1/p}$ (Heyde and Rohatgi [16]).

The above list is incomplete; it contains only the cases we touch in this paper.

The Spitzer series c_2) is remarkable, since it corresponds to the "boundary" case in the range of sequences $w_k = k^r$: namely, the series $\sum w_k$ converges if $r < -1$, while it diverges otherwise. The necessary and sufficient condition for the convergence of the Spitzer series c_2) is

$$\mathbf{E}X = 0 \tag{3}$$

(see [26]).

C. C. Heyde initiated the investigations of the asymptotics of $Q(\varepsilon)$ as $\varepsilon \downarrow 0$. He proved in [14] that *if* (2) *holds, then*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 Q(\varepsilon) = \sigma^2 \tag{4}$$

in the case of the Hsu–Robbins series c_1) *where* $\sigma^2 = \mathbf{var}\,[X]$.

The asymptotic behavior as $\varepsilon \downarrow 0$ of the Spitzer series c_2) is studied in Chow and Lai [6]. In particular, they proved for the case of c_2) that

$$\lim_{\varepsilon \downarrow 0} \frac{Q(\varepsilon)}{\ln(1/\varepsilon)} = 2 \tag{5}$$

if (2) is satisfied. Recall that condition (3) is sufficient in this case for the convergence of $Q(\varepsilon)$ for all $\varepsilon > 0$, so that condition (2) seems to be too strong for the asymptotics (5).

Spatăru [25] used another assumption to study the asymptotics of the Spitzer series c_2), namely

$$\begin{gathered} F \text{ belongs to the domain of attraction of a } \textit{nondegenerate} \\ \alpha\text{-stable law with } 1 < \alpha \le 2. \end{gathered} \tag{6}$$

Spatăru [25] proved under conditions (3) and (6) that

$$\lim_{\varepsilon \downarrow 0} \frac{Q(\varepsilon)}{\ln(1/\varepsilon)} = \frac{\alpha}{\alpha - 1}. \tag{7}$$

If (2) holds, then the distribution function F belongs to the domain of attraction of the Gaussian law, that is $\alpha = 2$ and (5) and (7) coincide in this case. Note further that (6) implies $\mathbf{E}|X|^\eta < \infty$ for all $1 < \eta < \alpha$ and this is weaker than (2) but still is stronger than (3), so that condition (6) seems also to be too strong for the asymptotics.

In fact, the assumption that the limit law is nondegenerate is missed in [25]. This, however, is an important restriction, since if the limit law is concentrated at the origin, then we consider the random variable $X = 0$ which obviously is attracted to that stable law. In this case, $\mathbf{P}(|S_k| \ge \varepsilon k) = 0$ for all $\varepsilon > 0$ and asymptotics (7) fails.

The Baum–Katz series c_3) (including the case of $r = -1$) is studied by Gut and Spatăru [11] under condition (6). Gut and Steinebach [12] showed that one can take $0 < \alpha < 1$ in assumption (6), too.

More general results concerning the asymptotics of series (1) under condition (6) are obtained by Rozovskiĭ [22] for $\alpha \ne 2$.

A weaker assumption than (6) is used by Scheffler [23] for the Spitzer series c_2) and for Baum–Katz series c_3), namely

$$\begin{gathered} F \text{ belongs to the domain of } \textit{semistable} \text{ attraction} \\ \text{of a } \textit{semistable} \text{ law with index } \alpha > 0. \end{gathered} \tag{8}$$

Further asymptotics of $Q(\varepsilon)$ as $\varepsilon \downarrow 0$ have been obtained by Chen [5] for the Baum–Katz series c_3) with $r \ge 0$ (this case includes Heyde's result (4)). The moment condition used in Chen [5] to obtain the asymptotics of $Q(\varepsilon)$ is the same as that for its convergence.

The proof in Chow and Lai [6] has its roots in the area of the central limit theorem and uses the truncation, symmetrization, Berry–Esseen estimate, and desymmetrization. The proof in Chen [5] follows the lines of the proof of Heyde [14], which consists of first obtaining the result for normally distributed summands and then approximating the general case with the Gaussian one. Spatăru [25] adapted the Heyde [14] method for the case of attraction to a stable law. Scheffler [23] used special properties of distributions partially attracted to semi-stable laws. Rozovskiĭ [22] applied a large deviation principle in the case of attraction to stable laws (this principle is due to Heyde [15] but Rozovskiĭ [22] does not cite that paper; instead he refers to another his own paper).

We should however mention that the proof in Spatăru [25] is not complete.

There is a remarkable difference between the series c_2) and c_3) with $r > -1$, namely the Heyde [14] result (4) says that the asymptotics of the Hsu–Robbins series c_1) (as a "representative" of series c_3)) is "almost" independent of the distribution function F: it depends on F only via its variance σ^2. Roughly speaking, the result is independent of the distribution up to a scale parameter. This is not the case for the Spitzer series c_2) whose asymptotic behavior depends on F via the index α of the stable law to which F is attracted (we discuss the case of a simpler condition (6)). Therefore the limit result is irrelevant of other properties of the distribution function F, since the index α does not completely determine the distribution.

A "complete" description of a distribution function F attracted to an α-stable law can be given in terms of the normalizing sequence in the corresponding attraction to a stable law. Namely, it follows from (6) that there are two sequences of real numbers $\{a_n, n \geq 1\}$ and $\{c_n, n \geq 1\}$ such that the distributions of $S_n/c_n - a_n$ converge weakly to a stable law of index $\alpha > 1$. Assumption (3) reduces the consideration to the case of $a_n = 0$, $n \geq 1$. Nevertheless, the sequence $\{c_n, n \geq 1\}$ is still present in the weak convergence and one expects that its properties should somehow be reflected in (7), too. It is known that $c_n = n^\alpha h(n)$ for some slowly varying function h. While α is involved in (7), h is not. In other words, there is no difference in (7) between the case of normal attraction ($h(x) \equiv \text{const}$) and the general case of non-normal attraction ($h(x) \not\equiv \text{const}$).

The latter observation raises two questions: fix $w \in \mathcal{RV}(-1)$;

- is there any difference in the asymptotics of $Q(\varepsilon)$ between the cases of normal attraction (where $c_n = cn^{1/\alpha}$, $c > 0$) and general attraction (where $c_n = n^\alpha h(n)$, h a slowly varying function)?

- If this phenomenon is *not* a general law, why does it occur in the case of $w_k = 1/k$, $k \geq 1$?

We answer both questions in Remark 6 below by applying our Theorem 1. It turns out that the phenomenon mentioned above appears only in the case of the Spitzer series.

Our main aim in this paper is to obtain the asymptotic of the series (1) where $w \in \mathcal{RV}(r)$, $r \geq -1$, and $\varphi \in \mathcal{RV}(1/p)$, $0 < p < \alpha$ (note that this case is even wider than the case of series c_4)).

We keep condition (6) and show that it applies not only for power functions but also for the general case of regularly varying functions.

The proof of the main result consists in two standard steps: first we get the asymptotic behavior for the limiting stable law, and then we approximate the general case by this particular one. Nevertheless, we develop new methods for both steps. Earlier papers make use of the Euler–MacLaurin summation formula for the first step, while we apply an approach based on some Abelian theorems for slowly varying functions due to Aljanĉić et al. [1]. Our approach works for stable laws but for other distribution functions, too.

The proof for the second step is based on some new large deviation results for distributions in the domain of attraction to a stable law. The cases $\alpha \neq 2$ and $\alpha = 2$ are different and require different tools in the proof. The case of $\alpha = 1$ is also obtained: it differs from the other cases by centering constants explicitely involved into the result.

The proof of the large deviation result is an extension of the method used in Heyde [15] but our result holds for the domain $x \geq 0$ instead of $x \geq x_n$ as in Heyde [15] where a sequence $\{x_n\}$ is such that $x_n \to \infty$, $n \to \infty$. The price we pay for this generalization is that we obtain an upper bound and do not get the precise asymptotics as in Heyde [15].

Rozovskiĭ [22] also used an asymptotic result similar to the Heyde [15] large deviation principle and dealt with series like (1) but he restricted himself to the case of $\alpha \neq 2$.

The paper is organized as follows. In Section 2, our main result, Theorem 1, is stated together with some corollaries exhibiting the possible range of asymptotics in the simple cases of either $w(x) = x^r \ln^q(n)$, or $\varphi(x) = x^{1/p} \ln^q(x)$, or $b(x) = x^{1/\alpha} \ln^q(x)$, or a mixture of these cases for various r and q. We also give the negative solution of the conjecture of [11] that the asymptotics for the normal and non-normal attractions coincide if $r > -1$.

The proof of Theorem 1 will be given elsewhere. Nevertheless we prove all other results, since they are of their own interest.

2. Notation, main result, and corollaries

Throughout the paper we assume that X, $\{X_n, n \geq 1\}$ are independent and identically distributed random variables and let S_n be the partial sums of X_k's.

We write $X \in \mathcal{DA}\left(\alpha, \{c_n\}, \{a_n\}\right)$ if there exists a *nondegenerate* α-stable random variable Z_α such that $S_n/c_n - a_n \Rightarrow Z_\alpha$, i.e. if X belongs to the domain of attraction of the nondegenerate random variable Z_α with normalizing sequence $\{c_n\}$ and centering sequence $\{a_n\}$. We write $X \in \mathcal{DA}\left(\alpha, \{c_n\}, \{0_n\}\right)$ if $a_n = 0$, $n \geq 1$.

We consider weight functions for the series (1) such that

$$w \in \mathcal{RV}(r), \qquad w_k = w(k). \tag{9}$$

The normalizing function for the large deviation probabilities in the series (1) is such that

$$\varphi \in \mathcal{RV}(1/p), \qquad \varphi_k = \varphi(k). \tag{10}$$

If $X \in \mathcal{DA}\left(\alpha, \{c_n\}, \{a_n\}\right)$ and $p < \alpha$, then there exists a mormalizing sequence $\{b_n\}$ for which $X \in \mathcal{DA}\left(\alpha, \{b_n\}, \{a_n\}\right)$ and

$$b(x) = x^{1/\alpha} h(x), \qquad b_k = b(k), \tag{11}$$

where h is a slowly varying function such that $\varphi(x)/b(x)$ is continuous and increasing. In what follows we assume that $\{b_n, n \geq 1\}$ is chosen according to (11), so that all the inverse functions considered below exist.

Put

$$\psi(x) = \varphi(x)/b(x), \qquad \psi_k = \psi(k). \tag{12}$$

Note that the inverse ψ^{-1} exists and $\psi^{-1} \in \mathcal{RV}\left(\alpha p/(\alpha - p)\right)$ (see, e.g., Seneta [24], Section 1.5, Proposition 5°). With $W(t) = \int_1^t w(s)\,ds$, we set

$$U(x) = W\left(\psi^{-1}(x)\right). \tag{13}$$

It is easy to see that $U \in \mathcal{RV}\left((r+1)\alpha p/(\alpha - p)\right)$.

In what follows we use at least one of the following conditions:

$$X \in \mathcal{DA}\left(\alpha, \{b_n\}, \{a_n\}\right) \tag{14}$$

or

$$X \in \mathcal{DA}\left(\alpha, \{b_n\}, \{0_n\}\right). \tag{15}$$

Recall that each of them means, in particular, that the limit law is nondegenerate.

Now we are ready to state our main result.

Theorem 1. *Let X, $\{X_n, n \geq 1\}$ be independent identically distributed random variables. Assume that condition* (14) *holds. In addition, we assume* (15) *if $\alpha = 1$ or* (3) *if $\alpha > 1$. Denote by Z_α an α-stable random variable to which X is attracted. Let conditions* (9) *and* (10) *hold with $r \geq -1$ and $0 < p < \alpha$. Let the functions U and Q be as defined in* (13) *and* (1), *respectively. If the series $\sum w_k$ diverges and $\alpha > p(r+2)$, then*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{U(1/\varepsilon)} \sum_{k=1}^{\infty} w_k \mathbf{P}\left(|S_k| \geq \varepsilon \varphi_k\right) = \mathbf{E}|Z_\alpha|^{(r+1)\alpha p/(\alpha - p)}. \tag{16}$$

Remark 1. As seen from conditions of Theorem 1, the cases $\alpha = 1$ and $\alpha \neq 1$ are different, since we assume (15) if $\alpha = 1$ instead of (14) if $\alpha \neq 1$. The reason is that (15) holds automatically if $\alpha < 1$; it also holds for $\alpha > 1$ under condition (3). We prefer to deal with a nicer series $\sum w_k \mathbf{P}(|S_k| \geq \varepsilon\varphi_k)$, thus we assume (15) if $\alpha = 1$. By the way, if $\mathbf{E}X = \mu \neq 0$ for $\alpha > 1$, then the asymptotics is evaluated for the series $\sum w_k \mathbf{P}(|S_k - k\mu| \geq \varepsilon\varphi_k)$.

Remark 2. Note that $(r+1)\alpha p/(\alpha - p) < \alpha$ if $\alpha > p(r+2)$, so that the right hand side of (16) is finite. If $\alpha \neq 2$ and $\alpha \leq p(r+2)$, then the limit value in (16) becomes infinite, so that the function U does not appropriately describes the asymptotics in this case.

Remark 3. If $\alpha < p(r+2)$ and $\alpha \neq 2$, then not only the right hand side of (16) is infinite but the series on the left hand side of (16) diverges for all $\varepsilon > 0$, so that there is no nice asymptotics in this case. The divergence of the series on the left hand side of (16) can easily be shown for $\alpha < \min\{2, p(r+2)\}$. Indeed, fix $\varepsilon > 0$ and put $x_k = \varphi_k/b_k$. Since $p < \alpha$, $x_k \to \infty$ as $k \to \infty$ and we apply the Heyde [14] result

$$\mathbf{P}(|S_k| \geq \varepsilon\varphi_k) \sim k\mathbf{P}(|X_1| \geq \varepsilon\varphi_k), \qquad k \to \infty,$$

to prove that the convergence of $\sum kw_k \mathbf{P}(|S_k| \geq \varepsilon\varphi_k)$ is equivalent to the convergence of $\sum kw_k \mathbf{P}(|X_1| \geq \varepsilon\varphi_k)$. The latter series converges if and only if

$$\sum_{k=1}^{\infty} \frac{kw_k g(\varepsilon\varphi_k)}{\varphi_k^{\alpha}} < \infty.$$

This condition fails if $\alpha < p(r+2)$, since g is a slowly varying function. In the boundary case of $\alpha = p(r+2)$, both convergence and divergence of the series is possible (see [22]).

Remark 4. The convergence of series (1) for $\alpha > p(r+2)$ and all $\varepsilon > 0$ can be proved in a similar manner. To avoid the repetition and to consider the case of $\alpha = 2$ together with $\alpha \neq 2$ we use another method based on the result of Baum and Katz [3]. Indeed, choose $r' > r$ and $p' > p$ such that $\alpha > p'(r'+2)$. Then $w_k \leq \text{const}\, k^{r'}$ and $\varphi_k \geq \text{const}\, k^{1/p'}$ for all $k \geq 1$. Thus the convergence of series (1) for all $\varepsilon > 0$ follows from the convergence of the series

$$\sum_{k=1}^{\infty} k^{r'} \mathbf{P}\left(|S_k| \geq \varepsilon k^{1/p'}\right)$$

for all $\varepsilon > 0$. The latter series converges for all $\varepsilon > 0$ if and only if $\mathbf{E}|X|^{p'(r'+2)} < \infty$ (see [3]). This moment is finite. This reasoning makes it clear that we assumed more in Theorem 1 than what is needed for just the convergence of series (1).

Remark 5. Note also that our restriction $\alpha > p(r+2)$ coincides with that used in Theorem 1 in [11] (the difference between the restrictions is a matter of the different notation here and in [11]).

Remark 6. One can answer both questions posed in the Introduction by using Theorem 1. To be more specific, let $p = 1$. Then, Theorem 1 in the case of $w_k = 1/k$, $k \geq 1$, implies that $Q(\varepsilon)$ is equivalent to $\ln(\psi^{-1}(1/\varepsilon))$ as $\varepsilon \downarrow 0$ where ψ^{-1} is the inverse to the function defined by (12). The function h in (11) determines the *non-normal* attraction (if $h(t) \equiv c$, then the attraction is *normal*).

It is a general fact about regularly varying functions that $\ln(f(t)) \sim \beta \ln(t)$ if $f \in \mathcal{RV}(\beta)$ (see, e.g; Proposition 2°, p. 18, in Seneta [24]). Since the logarithm "kills" any slowly varying function in the asymptotic sense and $\psi^{-1} \in \mathcal{RV}(\alpha/(\alpha-1))$, the function h disappears in the asymptotics (7) and only the part corresponding to the *normal* attraction is left there. This answers the second question in the Introduction.

To answer the first question we again consider the simplest case of $\varphi(t) = t$. Put $\beta = \alpha/(\alpha-1)$ and consider the function w such that $w(t) = 1$ for $0 \leq t < 2$ and

$$w(t) = \frac{e^{\ln^{1/2}(t)}}{t \ln^{1/2}(t)} \quad \text{for } t \geq 2.$$

Then $W(t) \sim 2e^{\ln^{1/2}(t)}$ as $t \to \infty$, where $W(t) = \int_1^t w(s)\,ds$. Note that $w \in \mathcal{RV}(-1)$ and thus $W \in \mathcal{RV}(0)$. In the case of normal attraction, $b(t) = ct^{1/\alpha}$ for some $c > 0$, whence $\psi^{-1}(t) = (ct)^\beta$ and $Q(\varepsilon)$ is equivalent to $U(1/\varepsilon) \sim 2e^{\beta^{1/2} \ln^{1/2}(1/\varepsilon)}$ as $\varepsilon \downarrow 0$ by Theorem 1.

Now consider a special case of non-normal attraction, where $\psi^{-1}(t) = t^\beta e^{\ln^a(t)}$, $0 < a < 1$. (Formally, the case of normal attraction corresponds to the case of $a = 0$.) Then $\psi^{-1} \in \mathcal{RV}(\beta)$ and the normalization b is easy to evaluate from ψ^{-1}. Theorem 1 implies that the asymptotics of $Q(\varepsilon)$ as $\varepsilon \downarrow 0$ is given by $2e^{(\beta \ln(1/\varepsilon) + \ln^a(1/\varepsilon))^{1/2}}$. To distinguish between the functions U for the normal attraction and non-normal attraction we use the notation U_0 and U_a, respectively. Then it is easy to see that

$$\frac{U_a(x)}{U_0(x)} \sim \exp\left\{ \frac{\ln^a(x)}{\sqrt{\beta \ln(x) + \ln^a(x)} + \sqrt{\beta \ln(x)}} \right\}$$

and thus $U_0 = o(U_a)$ if $a > 1/2$, while $U_a \sim U_0$ if $0 < a < 1/2$ and $U_a \sim e^{1/2\sqrt{\beta}} U_0$ if $a = 1/2$. The cas eof $a > 1/2$ means that the asymptotics for the normal and non-normal attractions do not coincide in general. This answers the first question above.

One can notice that U_a "dominates" U_0 in this example. This could produce an impression that this is a general fact. The example of $\psi^{-1}(t) = t^{\beta} e^{-\ln^a(t)}$, $\frac{1}{2} < a < 1$, shows that $U_a = o(U_0)$ and the impression is wrong.

Now we show how Theorem 1 can be applied in particular cases. We start with the case of $r > -1$.

Corollary 1. *Let X, $\{X_n, n \ge 1\}$ be independent identically distributed random variables. Assume that condition* (14) *holds for $b_n = cn^{1/\alpha}$, $c > 0$* (*thus we deal with the normal attraction*). *In addition, we assume* (15) *if $\alpha = 1$ or* (3) *if $\alpha > 1$. Denote by Z_α an α-stable random variable to which X is attracted. Let $r > -1$, $0 < p < \alpha$, and $\alpha > p(r+2)$. Set $\nu = (r+1)\alpha p/(\alpha - p)$. Then*

$$\sum_{k=1}^{\infty} k^r \mathbf{P}\left(|S_k| \ge \varepsilon k^{1/p}\right) \sim \left(\frac{1}{\varepsilon}\right)^{\nu} \cdot \frac{c^{\nu}}{r+1} \cdot \mathbf{E}|Z_\alpha|^{\nu}.$$

Corollary 1 is proved in [11], Theorem 1. Note however that the constant $c^{(r+1)\alpha p/(\alpha-p)}$ is missed in [11]. The proof of Corollary 1 is simple: $\psi(t) = t^{(\alpha-p)/\alpha p}/c$ in this case, whence $\psi^{-1}(t) = (ct)^{\alpha p/(\alpha-p)}$. Since $W(t) \sim t^{r+1}/(r+1)$, we derive Corollary 1 from Theorem 1.

Further we give a negative solution of the conjecture in [11] that the asymptotics obtained in Corollary 1 for the normal attraction holds also in the case of the non-normal attraction. In fact, any slowly varying function may appear in the asymptotic of the underlying series. Below is the corresponding result for the function $\ln^q(t)$.

Corollary 2. *Let X, $\{X_n, n \ge 1\}$ be independent identically distributed random variables. Assume that condition* (14) *holds for $b(t) = t^{1/\alpha}\ln^q(t)$. In addition, we assume* (15) *if $\alpha = 1$ or* (3) *if $\alpha > 1$. Denote by Z_α an α-stable random variable to which X is attracted. Let $r > -1$, $0 < p < \alpha$, and $\alpha > p(r+2)$. Set $\nu = (r+1)\alpha p/(\alpha - p)$. Then*

$$\sum_{k=1}^{\infty} k^r \mathbf{P}\left(|S_k| \ge \varepsilon k^{1/p}\right) \sim \left(\frac{\ln^q(1/\varepsilon)}{\varepsilon}\right)^{\nu} \cdot \left(\frac{\alpha p}{\alpha - p}\right)^{\nu q} \frac{1}{r+1} \cdot \mathbf{E}|Z_\alpha|^{\nu}.$$

First we note that the normalization $b(t) = t^{1/\alpha}\ln^q(t)$ appears in the attraction to an α-stable law for the following distribution

$$\mathbf{P}(|X| \ge x) \sim \text{const}\left(\frac{\ln^q(x)}{x}\right)^{\alpha}.$$

Corollary 2 easily follows from Theorem 1. Indeed, $\psi(t) = t^{\frac{\alpha-p}{\alpha p}} \ln^{-q}(t)$ in this case, whence $\psi^{-1}(t) \sim \left(\frac{\alpha p}{\alpha-p}\right)^{q\alpha p/(\alpha-p)} t^{\alpha p/(\alpha-p)} \ln^{q\alpha p/(\alpha-p)}(t)$. Using the asymptotics of W found in the proof of Corollary 1 we complete the proof of Corollary 2.

Now we turn to the case of $r = -1$.

Corollary 3. *Let X, $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables. Assume that condition* (14) *holds. In addition, we assume* (15) *if $\alpha = 1$ or* (3) *if $\alpha > 1$. If $0 < p < \alpha$, then*

$$\sum_{k=1}^{\infty} \frac{1}{k} \mathbf{P}(|S_k| \geq \varepsilon k^{1/p}) \sim \ln(1/\varepsilon) \cdot \frac{\alpha p}{\alpha - p}.$$

Corollary 3 is proved in [11] (also see [23]). Corollary 3 easily follows from Theorem 1. Indeed, $W(t) \sim \ln(t)$ and $\psi^{-1} \in \mathcal{RV}(\alpha p/(\alpha - p))$ in this case. Therefore, $U(x) \sim \frac{\alpha p}{\alpha-p} \ln(x)$ (see, Proposition 2° in Seneta [24], p. 18).

Below we give two more corollaries corresponding to the case of $r = -1$.

Corollary 4. *Let X, $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables. Assume that condition* (14) *holds. In addition, we assume* (15) *if $\alpha = 1$ or* (3) *if $\alpha > 1$. If $0 < p < \alpha$, then, for $q > -1$,*

$$\sum_{k=2}^{\infty} \frac{\ln^q(k)}{k} \mathbf{P}\left(|S_k| \geq \varepsilon k^{1/p}\right) \sim \ln^{q+1}(1/\varepsilon) \cdot \frac{1}{q+1} \left(\frac{\alpha p}{\alpha - p}\right)^{q+1}.$$

Indeed, $W(t) \sim \frac{1}{q+1} \ln^{q+1}(t)$ and $\psi \in \mathcal{RV}((\alpha - p)/\alpha p)$ in this case. Thus $\psi^{-1} \in \mathcal{RV}(\alpha p/(\alpha - p))$, whence Corollary 4 follows.

For $q = -1$, the right-hand side of the preceding equality becomes meaningless. It turns out that the asymptotic behavior of the series changes in this case.

Corollary 5. *Let X, $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables. Assume that condition* (14) *holds. In addition, we assume* (15) *if $\alpha = 1$ or* (3) *if $\alpha > 1$. If $0 < p < \alpha$, then*

$$\sum_{k=2}^{\infty} \frac{1}{k \ln(k)} \mathbf{P}\left(|S_k| \geq \varepsilon k^{1/p}\right) \sim \ln \ln(1/\varepsilon).$$

Indeed, $W(t) \sim \ln \ln(t)$ in this case, whence Corollary 5 follows in the same way as above.

The case of $q < -1$ is special, since $\sum \ln^q(k)/k$ converges and thus

$$\lim_{\varepsilon\downarrow 0}\sum_{k=2}^{\infty}\frac{\ln^q(k)}{k}\mathbf{P}(|S_k|\ge\varepsilon k^{1/p})=\sum_{k=2}^{\infty}\frac{\ln^q(k)}{k}\mathbf{P}(S_k\ne 0)$$

by the dominated convergence theorem.

The above corollaries dealt with the case of $\varphi(x) = x^{1/p}$. The results below show how does the change of the function φ influence the asymptotics of Q.

Corollary 6. *Let X, $\{X_n, n \ge 1\}$ be a sequence of independent identically distributed random variables. Assume that condition* (14) *holds with $b_n = cn^{1/\alpha}$ for some $c > 0$* (*thus we assume the normal attraction*). *In addition, we assume* (15) *if $\alpha = 1$ or* (3) *if $\alpha > 1$. Put $\nu = (r+1)\alpha p/(\alpha - p)$. If $r > -1$, $p > 0$, and $\alpha > p(r+2)$, then*

$$\sum_{k=1}^{\infty}k^r\mathbf{P}\left(|S_k|\ge\varepsilon k^{1/p}\ln^q(k)\right)\sim\left(\frac{1}{\varepsilon\ln^q(1/\varepsilon)}\right)^{\nu}\cdot\left[c\left(\frac{\alpha-p}{\alpha p}\right)^q\right]^{\nu}\frac{1}{r+1}\cdot\mathbf{E}|Z_\alpha|^{\nu}.$$

Indeed, $W(t) \sim \frac{1}{r+1}t^{r+1}$ and $\psi(t) = \frac{1}{c}x^{(\alpha-p)/\alpha p}\ln^q(t)$ in this case. Thus $\psi^{-1}(t) \sim c't^{\alpha p/(\alpha-p)}\ln^{-q\alpha p/(\alpha-p)}(t)$ for $c' = \left[c\left(\frac{\alpha-p}{\alpha p}\right)^q\right]^{\alpha p/(\alpha-p)}$, whence Corollary 6 follows.

Corollary 7. *Let X, $\{X_n, n \ge 1\}$ be a sequence of independent identically distributed random variables. Assume that condition* (14) *holds. In addition, we assume* (15) *if $\alpha = 1$ or* (3) *if $\alpha > 1$. If $0 < p < \alpha$, then*

$$\sum_{k=2}^{\infty}\frac{1}{k}\mathbf{P}\left(|S_k|\ge\varepsilon k^{1/p}\ln^q(k)\right)\sim\ln(1/\varepsilon)\cdot\frac{\alpha p}{\alpha-p}.$$

Indeed, $W(t) \sim \ln(t)$ in this case and Corollary 7 follows in the same way as above.

A different asymptotic behavior of the series (1) appears if $p = \alpha$. One of possible results is obtained by Scheffler [23] in this case. We will consider the case of $p = \alpha$ in detail elsewhere.

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