A LIMIT THEOREM FOR SEMI-MARKOV PROCESS

A limit theorem for the strongly regular semi-Markov process is proved under conditions C1 – C3.

1. Introduction

This article deals with the asymptotic behavior of the strongly regular semi-Markov process $\xi(t)$ as $t \to \infty$. It may be considered as continuation of the article [1] motivated by the book by A. N. Korlat, V. N. Kuznetsov, M. M. Novikov and A. F. Turbin (1991). Let us introduce basic notations and necessary results from [1], [2].

Let $\xi(t)$ be a strongly regular semi-Markov process with the phase space $\{X, B\}$ and semi-Markov kernel $Q(t, x, B)$, $t \geq 0$, $x \in X$, $B \in B$. Let $H(t, x, B)$, $t \geq 0$, $x \in X$, $B \in B$ be the Markov renewal function of $\xi(t)$. Define $D(X)$ as Banach space of $B$-measurable bounded functions with values in $\mathbb{R}$ with the norm $\|f\| = \sup_{x \in X} |f(x)|$. Consider two operator family $Q(t)$ and $H(t)$, $t \geq 0$, in $D(X)$, defined for all $f \in D(X)$:

$$
\begin{align*}
[Q(t)f](x) &= \int_X Q(t, x, dy)f(y), \\
[H(t)f](x) &= \int_X H(t, x, dy)f(y).
\end{align*}
$$

Suppose that $\xi(t)$ satisfies the following conditions:

C1. Markov chain $\xi_n$, $n \geq 0$, embedded in the $\xi(t)$, is uniformly recurring;

C2. $\|M_l\| < \infty$ for $l = 1, k+2$, $k \geq 1$, where $M_l = \int_0^\infty t^lQ(dt)$;

C3. Semi-Markov kernel of the process $\xi(t)$ is absolutely continuous in $t$:

$$Q(t, x, B) = \int_0^t q(s, x, B)ds, \quad t \geq 0, \quad x \in X, \quad B \in B,$$

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or in the operator form:

\[ Q(t) = \int_0^t q(s) ds, \quad t \geq 0. \]

Condition C3 guarantees existence of the density of the Markov renewal function \( h(t, x, B) \):

\[ H(t, x, B) = I_B(x) + \int_0^t h(s, x, B) ds, \quad t \geq 0, \quad x \in X, \quad B \in \mathcal{B}, \]

or in the operator form

\[ H(t) = I + \int_0^t h(s) ds, \quad t \geq 0, \]

where \( I \) is the identity operator, \( I_B(x) \) is the indicator function.

Let \( \Pi_0 \) be the stationary projector of the embedded Markov chain \( \xi_n \) defined under condition C1 as follows:

\[ [\Pi_0 f](x) = \int_X \rho(dy)f(y)\mathbb{I}(x), \quad \forall f \in \mathcal{D}(X) \]

where \( \rho(x) \) is the stationary distribution of the Markov chain \( \xi_n \), \( \mathbb{I}(x) \equiv 1 \) \( \forall x \in X \). Denote

\[ h_*(t) = h(t) - \frac{1}{\tilde{m}_1} \Pi_0, \quad (1) \]

where

\[ \tilde{m}_1 = \int_X \rho(dx)m_1(x), \quad m_1(x) = \int_0^\infty tQ(dt, x, X). \]

Let \( T_n, \quad n = 0, k \) be bounded operators in \( \mathcal{D}(X) \), introduced in the book [2, p. 1.4], and let \( P = Q(\infty) \) be the operator of transient probabilities of Markov chain \( \xi_n \). The following result was proved for \( n = 0 \) in [2] and for \( n = \overline{1, k} \) in [1]:

**Theorem 1.** Let a strongly regular semi-Markov process satisfies conditions C1 – C3. Then there exists the limit

\[ U_n = \lim_{p \to 0} \frac{(-1)^n}{n!} \int_0^\infty e^{-pt^n} h_*(t) dt, \quad n = \overline{0, k} \quad (2) \]
and the following relations hold:

\[ U_n = \sum_{r=0}^{n} \frac{(-1)^r}{r!} M_r U_{n-r} + \frac{(-1)^n}{n!} M_n + \frac{(-1)^{n+1}}{(n+1)! m_1} M_{n+1}\Pi_0, \quad n = 0, k, \quad (3) \]

\[ U_n = \begin{cases} T_0 - I, & \text{for } n = 0; \\ T_n, & \text{for } n = 1, k; \end{cases} \quad (4) \]

where \( M_0 = P \).

2. Basic results.

In this paper we present a theorem, which is proved by means of the above mentioned results and the Markov renewal theorem.

Let’s introduce a family of operators

\[ U_0(t) = \int_0^t h_*(s)ds, \quad U_n(t) = \int_0^t (U_{n-1}(s)-U_{n-1})ds, \quad t \geq 0, \quad n = 1, k. \quad (5) \]

The following result holds true:

**Theorem 2.** Let a strongly regular semi-Markov process satisfies conditions C1 – C3. Then there exists the limit

\[ \lim_{t \to \infty} U_n(t) = U_n, \quad t \geq 0, \quad n = 0, k. \quad (6) \]

**Proof.** 1. Consider the case \( n = 0 \). Under condition C3 the operator renewal equation holds true [3]:

\[ h(t) = q(t) + \int_0^t q(s)h(t-s)ds. \]

Hence, subject to (1)

\[ h_*(t) = q(t) - \frac{1}{m_1} (I - Q(t))\Pi_0 + \int_0^t q(s)h_*(t-s)ds. \quad (7) \]

Taking integral of (7) and using the Fubbini theorem [4] we will get

\[ \int_0^t h_*(s)ds = Q(t) - \frac{1}{m_1} \int_0^t (I - Q(s))ds \Pi_0 + \int_0^t ds q(s) \int_0^{t-s} h_*(l)dl, \]
or
\[
U_0(t) = Q(t) - \frac{1}{m_1} \int_0^t (I - Q(s)) ds \Pi_0 + \int_0^t q(s)U_0(t - s)ds. \tag{8}
\]

In the case \(n = 0\) from (3) we get
\[
U_0 = P + PU_0 - \frac{1}{m_1} M_1 \Pi_0. \tag{9}
\]

Taking into account the property of stationary projector \(\Pi_0\):
\[
P \Pi_0 = \Pi_0 = \Pi_0 P, \tag{10}
\]
consider the difference between (8) and (9):
\[
U_0(t) - U_0 = V_0(t) + \int_0^t q(s)(U_0(t - s) - U_0)ds, \tag{11}
\]
where
\[
V_0(t) = \int_t^\infty (P - Q(s)) ds \frac{\Pi_0}{m_1} + Q(t) - P - (P - Q(t))U_0. \tag{12}
\]

**Lemma 1.** Let conditions of Theorem 2 be satisfied. Then there exists the limit
\[
\lim_{t \to \infty} (U_0(t) - U_0) = \frac{\Pi_0}{m_1} \int_0^\infty V_0(s)ds. \tag{13}
\]

**Proof.** To prove the operator equation (13) it is sufficient to verify it for functions \(I_B(x), x \in X, B \in \mathcal{B}\), generating \(D(X)\). Define \(V_0(t, x, B), U_0(t, x, B), U_0(x, B)\) as action of operators \(V_0(t), U_0(t), U_0\) on function \(I_B(x)\). Consider positive and negative parts of the function \(V_0(t, x, B)\):

\[
V_0^1(t, x, B) := \max \{V_0(t, x, B), 0\}, \quad V_0^2(t, x, B) := -\min \{V_0(t, x, B), 0\}.
\]

Similarly \(U_0^1(x, B)\) and \(U_0^2(x, B)\) are defined as positive and negative parts of function \(U_0(x, B)\). From (12) it follows that for \(t \geq 0, x \in X, B \in \mathcal{B}\)

\[
V_0^1(t, x, B) = \frac{\rho(B)}{m_1} \int_t^\infty \int_0^\infty q(s, x, X)ds + \int s \int_X q(s, x, dy)U_0^2(y, B),
\]

\[
V_0^2(t, x, B) = \int_t^\infty q(s, x, B)ds + \int s \int_X q(s, x, dy)U_0^1(y, B).
\]
Functions $V_1^0(t, x, B)$ and $V_2^0(t, x, B)$ are bounded. It follows from condition C2 for $l = 1$ and boundedness of the operator $T_0 = U_0$. Besides, for any $x \in X$, $B \in \mathcal{B}$ functions $V_1^0(t, x, B)$, $V_2^0(t, x, B)$ are non-negative, monotone decreasing and integrable in $t$ functions on $[0, \infty)$. Thus for any $B \in \mathcal{B}$ they are directly Riemann integrable [5], so that $\int_{X}^{\infty} \rho(dx) \int_{0}^{\infty} dt V_j^0(t, x, B) < \infty$, $j = 1, 2$. So for a fixed $B \in \mathcal{B}$ the above point and conditions C1 – C3 give a possibility to apply the Markov renewal theorem ([5, p. 107], [6, p. 31]) to the following Markov renewal equation:

$$Z_j^2(t, x, B) = V_j^0(t, x, B) + \int_{0}^{t} ds \int_{X} q(s, x, dy) Z_j^1(t - s, y, B), \quad j = 1, 2. \quad (14)$$

By the Markov renewal theorem there exists

$$\lim_{t \to \infty} Z_j^2(t, x, B) = \frac{1}{\hat{m}_1} \int_{X} \rho(dx) \int_{0}^{\infty} dt V_j^0(t, x, B), \quad x \in X, \; B \in \mathcal{B}. \quad (15)$$

As by definition $V_1^0(t, x, B) - V_2^0(t, x, B) = V_0(t, x, B)$, then from (11) and (14) it follows that $U_0(t, x, B) - U_0(x, B) = Z^1(t, x, B) - Z^2(t, x, B)$. Hence, from (15) follows statement of the lemma. 

$$E_0(t) = \int_{t}^{\infty} dt_0 \int_{t_0}^{\infty} q(s) ds, \quad E_1(t) = \int_{t}^{\infty} dt_1 \int_{t_1}^{\infty} dt_0 \int_{t_0}^{\infty} q(s) ds,$$

$$E_n(t) = \int_{t}^{\infty} dt_n \int_{t_n}^{\infty} \int_{t_{n-1}}^{\infty} dt_{n-1} \cdots \int_{t_1}^{\infty} dt_0 \int_{t_0}^{\infty} q(s) ds, \quad n = 2, k.$$ 

It is easy to see that for $t = 0$ the following equalities hold true:

$$E_0(0) = \int_{0}^{\infty} (P - Q(t)) dt = M_1.$$
\[ E_n(0) = \int_0^\infty E_{n-1}(t)dt = \frac{M_{n+1}}{(n+1)!}, \quad n = 1, k + 1. \] (16)

Transform (3) to the form
\[
U_n = \sum_{r=0}^{n-1} (-1)^{(r+1)} E_r(0) U_{n-r-1} +
\]
\[ + (-1)^{(n+1)} E_n(0) \frac{\Pi_0}{\hat{m}_1} + PU_n + (-1)^n E_{n-1}(0). \] (17)

**Lemma 2.** Let conditions of Theorem 2 be satisfied. Then the following relations hold true:
\[
U_n(t) - U_n = V_n(t) + \int_0^t q(s)(U_n(t-s) - U_n)ds, \quad t \geq 0, \quad n = 1, k, \] (18)

where
\[
V_n(t) = \sum_{r=0}^{n-1} (-1)^r E_r(t) U_{n-r-1} +
\]
\[ + (-1)^n E_n(t) \frac{\Pi_0}{\hat{m}_1} + (-1)^{n-1} E_{n-1}(t) - \int_0^t q(s)ds U_n. \] (19)

**Proof.** The lemma is proved by means of mathematical induction method. From (5) and (11) we have
\[
U_1(t) = -E_0(t)U_0 + E_1(t) \frac{\Pi_0}{\hat{m}_1} - E_0(t) + \int_0^t q(s)U_1(t-s)ds. \] (20)

From (17) for \( n = 1 \) and (20) we obtain statement of the lemma for the case \( n = 1 \). So we have the base of induction. Suppose that statement of the lemma is true for some \( n, \quad n = 1, k - 1 \) and show that it is also true for \( n + 1 \). Indeed, let us integrate (18) and apply the Fubbini theorem. We get
\[
U_{n+1}(t) = \int_0^t V_n(s)ds + \int_0^t ds q(s) \int_0^{t-s} (U_n(l) - U_n)dl \pm \int_0^\infty V_n(s)ds,
\]

or
\[
U_{n+1}(t) = -\int_t^\infty V_n(s)ds + \int_0^\infty V_n(s)ds + \int_0^t q(s)U_{n+1}(t-s)ds. \] (21)
Since by definition \( \int_{t}^{\infty} E_n(s)\,ds = E_{n+1}(t) \), we have

\[
\int_{t}^{\infty} V_n(s)\,ds = \sum_{r=0}^{n-1} (-1)^r E_{r+1}(t)U_{n-r-1} +
\]

\[
+(-1)^n E_{n+1}(t) \frac{\Pi_0}{m_1} + (-1)^{n-1} E_n(t) - E_0(t)U_n =
\]

\[
= \sum_{r=0}^{n} (-1)^{(r+1)} E_r(t)U_{n-r} + (-1)^n E_{n+1}(t) \frac{\Pi_0}{m_1} + (-1)^{n-1} E_n(t). \tag{22}
\]

It follows from (22) for \( t = 0 \) and (17) that

\[
\int_{0}^{\infty} V_n(s)\,ds = U_{n+1} - PU_{n+1}. \tag{23}
\]

So if relation (18) is true for some \( n = \overline{1,k-1} \), then, as follows from (21), (22) and (23), it is also true for \( n + 1 \).

Prove the next lemma in the way similar to one of Lemma 1.

**Lemma 3.** Let conditions of Theorem 2 are satisfied. Then there exists the limit

\[
\lim_{t \to \infty} (U_n(t) - U_n) = \frac{\Pi_0}{m_1} \int_{0}^{\infty} V_n(s)\,ds, \quad n = \overline{1,k}. \tag{24}
\]

**Proof.** To prove the operator equation (24) it is sufficient to verify it for the indicator functions \( I_B(x), x \in X, B \in \mathcal{B} \), generating \( D(X) \). Define \( V_n(t,x,B) \) and \( U_r(x,B), r = 0, n \) as action of operators \( V_n(t), U_r \) on function \( I_B(x) \). From (19) it follows

\[
V_n(t,x,B) = (-1)^n E(B) \int_{t}^{\infty} dt_n \int_{t_1}^{\infty} dt_{n-1} \cdots \int_{t_1}^{\infty} dt_0 \int_{t_0}^{\infty} q(s,x,X)\,ds +
\]

\[
+ \sum_{r=0}^{n-1} (-1)^r \int_{t}^{\infty} dt_r \int_{t_1}^{\infty} dt_{r-1} \cdots \int_{t_1}^{\infty} dt_0 \int_{t_0}^{\infty} ds \int_{X} q(s,x,dy)U_{n-r-1}(y,B) +
\]

\[
+ (-1)^{n-1} \int_{t}^{\infty} dt_{n-1} \cdots \int_{t_1}^{\infty} dt_0 \int_{t_0}^{\infty} q(s,x,B)\,ds - \int_{t}^{\infty} ds \int_{X} q(s,x,dy)U_n(y,B).
\]
Consider positive and negative parts of the function $V_n(t, x, B)$:

$$V_n^1(t, x, B) := \max \{V_n(t, x, B), 0\}, \quad V_n^2(t, x, B) := -\min \{V_n(t, x, B), 0\}.$$ 

Represent functions $U_r(x, B), r = 0, 1, n$ in (25) as $U_r^1(x, B) - U_r^2(x, B)$ where $U_r^1(x, B)$ and $U_r^2(x, B)$ are its positive and negative parts. Then from (25) it follows that $V_n(t, x, B)$ is a sum of functions of constant signs. It is easy to see from (25) the structure of functions $V_n^+$ and $V_n^-$ and make a conclusion, that for any fixed $x \in X, \ B \in \mathcal{B}$ functions $V_n^+(t, x, B), V_n^-(t, x, B)$ are non-negative, monotone decreasing and integrable in $t$ functions on $[0, \infty)$. Boundedness of this functions follows from the condition C2, (4) and boundedness of operators $T_i, \ i = 0, n$. Thus $V_n^1$ and $V_n^2$ are directly Riemann integrable, so that $\int_X \rho(dx) \int_0^t dt V_n^j(t, x, B) < \infty, \ j = 1, 2$. Hence the above point and conditions C1 – C3 give a possibility to apply the Markov renewal theorem to the next equation:

$$Z^j(t, x, B) = V_n^j(t, x, B) + \int_0^t ds \int_X q(s, x, dy) Z^j(t - s, y, B), \ j = 1, 2. \quad (26)$$

By the Markov renewal theorem there exists

$$\lim_{t \to \infty} Z^j(t, x, B) = \frac{1}{\Pi_0} \int_X \rho(dx) \int_0^\infty dt V_n^j(t, x, B), \ x \in X, \ B \in \mathcal{B}. \quad (27)$$

Note that by definition $V_n^1(t, x, B) - V_n^2(t, x, B) = V_n(t, x, B)$. So from (26) and Lemma 2 it follows that $U_n(t, x, B) - U_n(x, B) = Z^1(t, x, B) - Z^2(t, x, B)$. From (27) statement of the lemma follows.

From (23) and (10) it follows that

$$\Pi_0 \int_0^\infty V_n(s) ds = \Pi_0 (U_{n+1} - PU_{n+1}) = 0.$$ 

Using Lemma 3, we get statement of the theorem for $n = \overline{1, k}$.

**Conclusion.**

In Theorem 2 the asymptotic equality (6) is proved for a strongly regular semi-Markov process which satisfies conditions C1 – C3. In the case $n = 0$ this asymptotic equality follows from results of [2] but under two
additional conditions. Note, that (6) is more weak result than existence of
\[ \int_0^\infty t^n h_\ast(t) dt, \quad n = \overline{0, k}. \]
Indeed, if such integral exists, then
\[ U_n = \frac{(-1)^n}{n!} \int_0^\infty t^n h_\ast(t) dt, \quad n = \overline{0, k}, \]
and according to formula of integration by parts, we get
\[ \int_0^\infty t^n h_\ast(t) dt = n! \int_0^\infty dt_n \int_{t_n}^\infty dt_{n-1} \ldots \int_{t_1}^\infty h_\ast(t) dt, \quad n = \overline{1, k}. \]
from which asymptotic equality (6) follows. However, as far as the author
knows, at the present moment existence of \( \int_0^\infty t^n h_\ast(t) dt \) for the general
semi-Markov process is not proved. It is known that such integral is con-
vergent for the renewal process under conditions that are a particular case
of the conditions C1-C3 for the renewal process ([7], [8]).

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