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ARBITRAGE WITH FRACTIONAL BROWNIAN MOTION?

In recent years fractional Brownian motion has been suggested to replace the classical Brownian motion as driving process in the modelling of many real world phenomena, including stock price modelling. In several papers seemingly contradictory results on the existence or absence of a riskless gain (arbitrage) in such stock models have been stated. This survey tries to clarify this issue by pointing to the importance of the chosen class of admissible trading strategies.

1. Introduction

Absence of arbitrage, i.e. the impossibility of receiving a riskless gain by trading into a market, is a basic equilibrium condition at the heart of financial mathematics: Suppose there is a strategy that is feasible for investors and promises a riskless gain. Then investors would like to buy (and not at all sell) this strategy. Hence, by the law of supply and demand, the price of this strategy would increase immediately, showing that the market prices have not been in an equilibrium. Therefore, absence of arbitrage has become a minimum requirement for a sensible pricing model.

Mathematically, the first fundamental theorem of asset pricing, (see [9] for a version relevant in the context of the present paper), links the no-arbitrage property to the martingale property of the discounted stock price process under a suitable pricing measure. Since fractional Brownian motion is not a semimartingale (except in the Brownian motion case), stock price processes (fully or partially) driven by a fractional Brownian motion can typically not be transformed into a martingale by an equivalent change of measure. So, at first glance, the fundamental theorem rules out these models as sensible pricing models. However, this kind of arbitrage does not make the hedging problem irrelevant.

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Note that existence of an arbitrage crucially depends on the class of investment strategies that is at the investor's disposal. The fundamental theorem presupposes a class of admissible strategies which is – in some sense – as big as possible from a mathematical point of view. Apart from being self-financing and a condition excluding doubling strategies, any predictable and integrable process (w.r.t. the stock price process) is admissible.

In the other extreme, it is of course always possible to construct a class of strategies that is small enough to exclude arbitrage. (For instance one can forbid all trading.) So the problem to be discussed can be posed as: Is it possible to construct a class of economically meaningful strategies, that does not contain arbitrage and is at the same time sufficiently rich to be interesting from the perspective of pricing? By 'interesting from the perspective of pricing' we mean that practically relevant options can be priced via (approximative) replication arguments within the class of strategies.

The main purpose of this paper is to discuss several results on existence or absence of arbitrage in models driven by a fractional Brownian motion for different classes of strategies in the light of the above criterion. After explaining the self-financing condition in Section 2 we consider models purely driven by a fractional Brownian motion and different subclasses of self-financing strategies in Section 3. In Section 4 we discuss the notion of Wick-self-financing portfolios. Models simultaneously driven by a Brownian motion and a fractional Brownian motion are treated in Section 5. Finally, Section 6 summarizes our findings.

2. Self-financing portfolios

Throughout this paper a *discounted market model* is a 6-tuple

$$\mathcal{M} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P, S, \mathcal{A}),$$

where $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ is a filtered probability space, $S = (S_t)_{0\le t\le T}$ is an adapted stochastic process modelling discounted stock prices, and $\mathcal{A}$ is a set of admissible trading strategies. For simplicity we shall assume that the process S takes values in R and has continuous trajectories. Hence we consider a market with two tradable assets: the discounted stock S and a discounted money market account $B_t \equiv 1$.

A *portfolio* is a pair of adapted stochastic processes $\varphi = (\beta_t, \gamma_t)$. The processes β_t and γ_t denote the amount of money stored on the bank account at time t and number of stock shares held at time t respectively. Thus, the corresponding wealth process is given by

$$V_t(\varphi) = \beta_t + \gamma_t S_t.$$

Definition 1. *(i) A portfolio φ is an* arbitrage*, if $V_0(\varphi) = 0$, $V_T(\varphi) \ge 0$ P-almost surely, and $P(V_T(\varphi) > 0) > 0$.*
(ii) A market $\mathcal{M}$ is free of arbitrage*, if no portfolio $\varphi \in \mathcal{A}$ is an arbitrage.*

Obviously, the notion of an arbitrage-free market crucially depends on the chosen class of admissible portfolios. A standard restriction is, that only self-financing portfolios can be admissible: Suppose, under absence of transaction costs, the investor's portfolio is constant between times t_1 and t_2, and she rearranges her portfolio at time t_2. Then her wealth immediately before rearranging is $\beta_{t_1} + \gamma_{t_1} S_{t_2}$ and her wealth immediately after rearranging is $\beta_{t_2} + \gamma_{t_2} S_{t_2}$. Now 'self-financing' means, by definition, that these two values coincide, i.e. that neither money is added nor withdrawn when rearranging the portfolio. Note that, by elementary manipulations, this self-financing condition is equivalent to

$$V_{t_2}(\varphi) = V_{t_1}(\varphi) + \gamma_{t_1}(S_{t_2} - S_{t_1}).$$

So, a natural extension of the self-financing condition to a more general class of portfolios can be given in terms of the forward integral. We briefly review the pathwise approach due to Föllmer [13] and refer the reader to [16], [21], and [26] for different approaches.

Definition 2. *Suppose a sequence (π^n) of partitions of $[0,T]$ is given such that mesh$(\pi^n) \to 0$. Then X is said to have a* forward integral *w.r.t. S, if*

$$\lim_{n\to\infty} \sum_{t_i \in \pi^n; t_i \le t} X_{t_{i-1}}(S_{t_i} - S_{t_{i-1}}) =: \int_0^t X_u dS_u$$

exists P-almost surely for all $0 \le t \le T$ and defines a continuous function in t on almost every path.

For the remainder of the paper we will consider an appropriate sequence of partitions π^n fixed and suppress the dependence on π^n in all definitions below.

Definition 3. *A portfolio $\varphi = (\beta_t, \gamma_t)$ is said to be self-financing, if γ_t has a forward integral w.r.t. to S and, for all $0 \le t \le T$,*

$$V_t(\varphi) = V_0(\varphi) + \int_0^t \gamma_u dS_u.$$

We can now introduce some classes of admissible portfolios.

Definition 4. *(i) A portfolio $\varphi_t = (\beta_t, \gamma_t)$ is* simple*, if there exists a finite number of non-decreasing stopping times $\tau_0, \dots, \tau_K$, such that the portfolio is constant on $(\tau_k, \tau_{k+1}]$. The set of simple and self-financing portfolios is denoted $\mathcal{A}^{si}$.*
(ii) An ϵ-simple *portfolio is a simple one such that $\tau_{k+1} - \tau_k > \epsilon$ for all k. The set of ϵ-simple and self-financing portfolios is denoted $\mathcal{A}^{\epsilon,si}$.*
(iii) A self-financing portfolio $\varphi_t = (\beta_t, \gamma_t)$ is nds-admissible*, if there is a constant $a \ge 0$ such that for all $0 \le t \le T$,*

$$V_t(\varphi) \ge a \quad P\text{-almost-surely}.$$

The set of nds-admissible portfolios is denoted $\mathcal{A}^{nds}$. Here nds stands for 'no doubling strategy'.

As a direct consequence of the fundamental theorem of asset pricing (in the version of Delbaen and Schachermeyer [9]), we obtain:

Theorem 1. *If S is not a semimartingale, then there exists an approximative arbitrage in the class $\mathcal{A}^{si}$.*

Here, an approximative arbitrage is meant in the sense of a 'free lunch with vanishing risk'. We refer the reader to [9] for the exact definition of this notion. The previous theorem implies that one needs to restrict the class of self-financing strategies, if one wants to consider non-semimartingale models, for instance the models (fully or partially) driven by a fractional Brownian motion that we consider in the following sections.

3. Arbitrage in fractional Black-Scholes model

Recall that *fractional Brownian motion* B_t^H with Hurst parameter $0 < H < 1$ is a continuous centered Gaussian process with covariance structure

$$E[B_t^H B_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

We state some well known properties of the fractional Brownian motion:

1. For $H = 1/2$ fractional Brownian motion is a classical Brownian motion.
2. If $H \neq 1/2$ fractional Brownian motion is not a semimartingale.
3. If $H > 1/2$, fractional Brownian motion has zero pathwise quadratic variation along appropriate sequences of partitions, i.e. for all $0 \leq t \leq T$,

$$\lim_{n\to\infty} \sum_{t_i \in \pi^n;\, t_i \leq t} |B_{t_i}^H - B_{t_{i-1}}^H|^2 =: \langle B^H \rangle_t = 0; \ P\text{-almost surely.}$$

4. For $H > 1/2$ fractional Brownian motion has a long memory, in the sense that the covariance function

$$\rho_H(n) := \mathrm{Cov}(B_k^H - B_{k-1}^H, B_{k+n}^H - B_{k+n-1}^H)$$

satisfies $\sum_{n=1}^{\infty} |\rho_H(n)| = \infty$.

From now on B^H will always denote a fractional Brownian motion with $H > 1/2$. Since Itô's formula carries over to the forward integral, with the quadratic variation interpreted in the above pathwise sense, (see [13]), we obtain

$$f(B_t^H) = f(0) + \int_0^t f'(B_u^H)\,dB_u^H,$$

(including the existence of the forward integral). Hence, the *pathwise fractional Black-Scholes model* with parameters $\sigma > 0$ and $\mu \in R$

$$\tilde{S}_t^H = s_0 \exp\left\{\sigma B_t^H + \mu t\right\}$$

satisfies the pathwise SDE

$$d\tilde{S}_t^H = \mu \tilde{S}_t^H dt + \sigma \tilde{S}_t^H dB_t^H.$$

Note that the forward integral does not have zero expectation in general. Hence, we cannot interpret the second term in the SDE as an additive noise. A remedy is to consider the SDE in the Wick-Itô sense (see the next Section for the definition of the Wick-Itô integral). Its solution

$$S_t^H = s_0 \exp\left\{\sigma B_t^H - 1/2\sigma^2 t^{2H} + \mu t\right\}$$

is called the *Wick-fractional Black-Scholes model.* It coincides with the classical Black-Scholes model for $H = 1/2$. Actually, the term $-1/2\sigma^2 t^{2H}$ is a drift term that does not have any influence on the arbitrage issue. So we will consider the model S^H only.

Since fractional Brownian motion with $H > 1/2$ is not a semimartingale we conclude from Theorem 1 that there exists approximative arbitrage for S_t^H in the class $\mathcal{A}^{si}$. The following example, essentially due to Shiryaev [23] and Dasgupta and Kallianpur [10], gives a simple explicit construction of an arbitrage in the class of nds-admissible strategies $\mathcal{A}^{nds}$. It only relies on the fact that S_t^H inherits the zero quadratic variation property from B^H.

Example 1. Let g be continuously differentiable nonnegative function satisfying $g(0) = 0$ and $g(x_0) > 0$ for some $x_0 \in R$. Define $\varphi = (\beta_t, \gamma_t)$ with

$$\beta_t = g(S_t^H - s_0) - g'(S_t^H - s_0)S_t^H, \quad \gamma_t = g'(S_t^H - s_0).$$

Then,

$$V_t(\varphi) = g(S_t^H - s_0) - g'(S_t^H - s_0)S_t^H + g'(S_t^H - s_0)S_t^H = g(S_t^H - s_0).$$

Consequently, φ is an arbitrage that is bounded from below by 0. To show that $\varphi \in \mathcal{A}^{nds}$, the self-financing condition must be verified. However, by Itô's formula applied to the zero quadratic variation process $S_t^H - s_0$ we get,

$$V_t(\varphi) = g(S_t^H - s_0) = \int_0^t g'(S_t^H - s_0)dS_u^H = \int_0^t \gamma_u dS_u^H.$$

We now consider the function $g(x) := (x)_+ := \max\{x, 0\}$, and approximate it by a sequence of continuously differentiable nonnegative functions $g_m(x)$ satisfying $g_m(0) = 0$. From the previous example we obtain a sequence of nds-admissible arbitrages φ^m such that $V_0(\varphi^m) = 0$ and

$V_T(\varphi^m) = g_m(S_T^H - s_0) \approx (S_T^H - s_0)_+$ for large m. Hence, these arbitrages constitute, for large m, approximate hedges for an at-the money European call option with initial capital 0.

We close this section by a remark summarizing some interesting results on existence and absence of arbitrage for different classes of portfolios.

Remark. (i) Cheridito proves in [8] that the fractional Black-Scholes model is free of arbitrage with the class $\mathcal{A}^{\epsilon,si}$ of ϵ-simple strategies for every $\epsilon > 0$. On the other hand, the cheapest super-replicating portfolio for a European call option in the class $\mathcal{A}^{\epsilon,si}$ is to buy one share of the stock at time 0 and to hold it until time T. Hence, only a trivial price bound for European call options can be obtained that way.

(ii) A portfolio is called *almost simple*, if there is a sequence of nondecreasing stopping times $(\tau_k)_{k\in N}$ such that $P(\tau_k = T \text{ infinitely often}) = 1$ and the portfolio is constant on $(\tau_k, \tau_{k+1}]$. This means that the number of rearranging times is finite on almost every path, but not necessarily bounded as function on Ω. Existence of a self-financing almost simple arbitrage has been shown by Rogers [20], making use of the history of a fractional Brownian motion starting at $-\infty$, and subsequently by Cheridito [8], taking only the history starting from 0 into account. Both constructions rely on the long memory property of the fractional Brownian motion.

(iii) A very intuitive construction of an arbitrage in fractional Cox-Ross-Rubinstein models, exploiting the memory of a binary version of a fractional Brownian motion, can be found in [24] and [4]. In [24] the discrete approximation of the fractional Black-Scholes model is based on ordinary products, in [4] on discrete Wick products.

(iv) Under proportional transaction costs, absence of arbitrage for the fractional Black-Scholes models is proved in [14]. The introduction of proportional transaction costs requires, however, that only processes of bounded variation can be considered as portfolios.

We conclude that there seems to be no subclass of self-financing strategies that is sufficiently small to be free of arbitrage and, at the same time, sufficiently rich to induce a sensible price for European call options by hedging arguments.

4. 'Wick self-financing portfolios'

It has been first suspected by Hu and Øksendal [15], that the existence of arbitrage in the fractional Black-Scholes model is – mathematically – due to the non-zero expectation property of the forward integral w.r.t. fractional Brownian motion. They suggested to build a financial calculus for fractional Brownian models on the Wick-Itô integrals instead, an approach which was later extended by Elliott and van der Hoek [12]. The Wick-Itô integral is based on a renormalization operator, called the Wick product. Given a Gaussian random variable ξ, we define its Wick exponential by $\mathcal{E}(\xi) := \exp\{\xi - \frac{1}{2}E[|\xi|^2]\}$. For a Brownian motion $B^{1/2}$ we obtain, with $t_1 < t_2 < t_3$,

$$\mathcal{E}(B^{1/2}_{t_3} - B^{1/2}_{t_2})\mathcal{E}(B^{1/2}_{t_2} - B^{1/2}_{t_1}) = \mathcal{E}(B^{1/2}_{t_3} - B^{1/2}_{t_1}).$$

This identity does not hold for any other choice of Hurst parameter due to the correlated increments. To enforce a similar property, and – in some sense – to kill a part of the memory by ignoring the correlation of two Gaussian random variables, one can introduce the *Wick product* for Wick exponentials by

$$\mathcal{E}(\xi) \diamond \mathcal{E}(\eta) := \mathcal{E}(\xi + \eta).$$

The Wick product can then be extended to a larger class of random variables by different equivalent procedures, see e.g. [2], [11], [12], [15].

The simplest way to define the fractional Wick-Itô integral is to replace the ordinary products in the definition of the forward integral by Wick products (see [11]), i.e.

$$\int_0^t X_u d^\diamond B^H_u := \lim_{n\to\infty} \sum_{t_i \in \pi^n; t_i \le t} X_{t_{i-1}} \diamond (B^H_{t_i} - B^H_{t_{i-1}}),$$

if the Wick products and the limit exist in $L^2(\Omega)$. Different approaches to the fractional Wick-Itô integral can be found in [18] via Malliavin calculus, [12] and [15] via white noise analysis, and [2] via a more elementary S-transform approach. We list some properties of the fractional Wick-Itô integral:

1. For $H = 1/2$ it coincides with the Itô integral (under suitable integrability condition on the integrand).
2. It has zero expectation.
3. Its Itô formula has a correction term, precisely

$$f(B^H_t) = f(0) + \int_0^t f'(B^H_u) d^\diamond B^H_u + H \int_0^t f''(B^H_u) u^{2H-1} du.$$

Since S^H satisfies,

$$S^H_t = s_0 + \int_0^t \mu S^H_u du + \int_0^t \sigma S^H_u d^\diamond B^H_u,$$

we define the Wick-Itô integral w.r.t. S^H by

$$\int_0^t X_u d^\diamond S^H_u := \int_0^t \mu X_u S^H_u du + \int_0^t \sigma X_u S^H_u d^\diamond B^H_u.$$

We now recall the notion of a Wick-self-financing portfolio from [12] (cf. [15] for a similar definition).

Definition 5. *A portfolio* $\varphi = (\beta_t, \gamma_t)$ *is said to be Wick-self-financing, if* γ_t *has a Wick-Itô integral w.r.t. to* S^H *and, for all* $0 \le t \le T$,

$$V_t(\varphi) = V_0(\varphi) + \int_0^t \gamma_u d^\diamond S_u.$$

The following Theorem is due to [12], cf. [15] for a variant of this result. The proof is based upon a fractional version of the Girsanov Theorem.

Theorem 2. *The Wick-fractional Black-Scholes model is free of arbitrage with the class of Wick-self-financing portfolios (satisfying an appropriate integrability condition).*

Although this result is a mathematically nice analogue of the no-arbitrage result for the classical Black-Scholes model, it has been noted later by several authors (e.g. [6], [25], and [4] in a discrete setting), that the Wick-self-financing condition does not admit an easy economic interpretation, (and most likely does not have any). In particular, the no-arbitrage result for Wick-self-financing portfolios still holds, if the investor has full information about the future stock prices. In such situation, an arbitrage opportunity should, of course, exist in a realistic setup.

The difference between a self-financing and a Wick-self-financing strategy with the same number of shares γ_t can be calculated in terms of the Malliavin derivative, see [25]. As a special case of the results in [25], we state:

Theorem 3. *Suppose* $\varphi = (\beta_t, \gamma_t)$ *is a Wick-self-financing strategy such that* $\gamma_t = g(t, S_t^H)$ *for some continuously differentiable function* $g(t,x)$. *Then* $\tilde{\varphi} = (\tilde{\beta}_t, \gamma_t)$ *is self-financing for*

$$\tilde{\beta}_t = \beta_t + H \int_0^t u^{2H-1} \sigma^2 (S_u^H)^2 \frac{\partial}{\partial x} g(u, S_u^H) du$$

Remark. (i) If one is only interested in hedging then the Wick-self-financing point of view is safe, but expensive. Indeed, the Wick-self-financing Δ-hedge for a claim $F(S_T^H)$ is $\gamma_t = \frac{\partial}{\partial x} f(t, S_t^H)$, where f solves the *fractional Black-Scholes differential equation*

$$\frac{\partial}{\partial t} f(t,x) = -H\sigma^2 x^2 t^{2H-1} \frac{\partial^2}{\partial x^2} f(t,x),$$

with $f(T,x) = F(x)$ (we refer to [25] for more details). So, if the claim is convex, it follows from Theorem 3 that in the pathwise self-financing sense the agent is super hedging, or consuming.

(ii) The capital needed to hedge a claim $F(S_T^H)$ in the Wick-self-financing sense is $E_Q[F(S_T^H)]$, where Q is the so-called *average risk neutral measure*, i.e. a measure characterized by the properties: $E_Q[S_t^H] = s_0$ and S_t^H is log-normal. These properties would be satisfied if Q were an equivalent martingale measure. In the fractional Black-Scholes model there are of course no martingale measures. However, curiously these minimal requirements provide us with a unique measure. For further details on average risk neutral measure we refer to [25].

Remark. In [19] the notion of a market observer was introduced by Øksendal in order to justify the use of the Wick product in the self-financing condition. Roughly speaking, all formulae containing Wick products are interpreted as abstract quantities which become real prices, wealth processes, etc., by an observation (which mathematically can be thought of as a dual pairing). This approach is discussed in Chapter 6.5 of [3]. In particular, it is shown, that there are Wick-self-financing portfolios that become an arbitrage under *some* observations (weak arbitrages), while the no-arbitrage result of Theorem 2 means that there are no Wick-self-financing strategies that become arbitrages under *all* observations (strong arbitrages). Due to the existence of weak arbitrages and the rather complex concept of market observations, it is doubtful that the market observer interpretation can become a successful approach.

To summarize, it appears that the notion of a Wick-self-financing portfolio is a purely mathematical concept, that should better not be used in financial engineering.

5. Absence of 'regular' arbitrage in mixed models

The previous sections suggest that a Black-Scholes type model driven by a fractional Brownian motion ($H > 1/2$) can most likely not be equipped with a economically sensible class of arbitrage free portfolios that is rich enough for pricing purposes. Therefore, we now consider a model that is driven by a fractional Brownian motion B^H and a Brownian motion W, namely,

$$S_t^{H,\epsilon} = s_0 \exp\left\{\sigma B_t^H + \epsilon W_t + \mu t - \frac{1}{2}\sigma^2 t^{2H} - \frac{1}{2}\epsilon^2 t\right\}.$$

We will assume that $\sigma B_t^H + \epsilon W_t$ is a Gaussian process, which is, for instance, satisfied, if B^H and W are independent as in [26] or if B^H is constructed from W via convolution with a suitable kernel as in [17].

We note that, for independent W and B^H the process $\sigma B_t^H + \epsilon W_t$ is not a semimartingale, if $H \leq 3/4$. However, it is equivalent to ϵW_t, if $H > 3/4$ and $\epsilon > 0$. Both results are proved in [7]. Hence, in the latter case $S^{H,\epsilon}$ inherits absence of arbitrage with nds-admissible strategies from the standard Black-Scholes models. For the general case we now introduce the class of regular portfolios.

Definition 6. *An nds-admissible portfolio* $\varphi = (\beta_t, \gamma_t)$ *is* regular*, if there is a continuously differentiable function* $g : [0,T] \times R_+^4 \to R$ *such that*

$$\gamma_t = g\left(t, S_t^{H,\epsilon}, \max_{0\le u\le t} S_u^{H,\epsilon}, \min_{0\le u\le t} S_u^{H,\epsilon}, \int_0^t S_u^{H,\epsilon} du\right).$$

We know from Section 4 that there exist regular arbitrage opportunities in the pure fractional model, i.e. for $\epsilon = 0$. The following theorem, due to [5], shows that the regularization via the Brownian motion removes this type of arbitrage.

Theorem 4. *The mixed model* $S^{H,\epsilon}$ *is arbitrage-free with regular portfolios, if* $\epsilon > 0$.

This theorem generalizes significantly a result from [1], where only the case $\beta_t = b(t, S_t^{H,\epsilon})$ and $\gamma_t = g(t, S_t^{H,\epsilon})$ is treated by PDE arguments. The main idea to prove Theorem 4 is, to construct continuous functionals $v(t, \cdot)$ on the space of continuous functions such that $V_t(\varphi) = v(t, S^{H,\epsilon})$. Since $S_t^{H,\epsilon}$ has the same pathwise quadratic variation as a standard Black-Scholes model with volatility ϵ along appropriate partitions, one can show that v also induces a wealth functional for the Black-Scholes model. Then, via continuity, absence of arbitrage can be transferred from the Black-Scholes model to the mixed one.

Moreover, it can be shown that European, Asian, and lookback options can be hedged with regular portfolios (up to a regularity issue at $t = T$, which is resolved in [5]). The hedges for these products are given by the same functionals as in the classical Black-Scholes model. This robustness of hedges was first observed in [22] for European options and is extended to exotic options, such as Asian and lookback ones, in [5]. Moreover, the thus obtained hedging prices coincide with those in the classical Black-Scholes model. In conclusion, in the mixed fractional Black-Scholes model the class of regular portfolios constitutes an arbitrage-free class of strategies that is sufficiently large to cover hedges for practically relevant options.

Remark. The results on no-arbitrage and robust hedges in [5] cover a wider class of models than the mixed fractional Black-Scholes model and a larger class of portfolios. They actually indicate, that pricing of a large class of options does basically only depend on the pathwise quadratic variation. Moreover, one can include a so-called volatilty smile by introducing a local volatility structure instead of the constants σ and ϵ. The reader is referred to [5] for the details.

6. Conclusion

1. In the purely fractional Black-Scholes models there is no subclass of self-financing strategies known that is arbitrage-free and sufficiently large to hedge relevant options (and most likely there does not exist such a class).

2. The purely fractional Black-Scholes models become arbitrage-free with Wick-self-financing strategies. But the notion of a Wick self-financing portfolios seems to be void of a sound economic interpretation, if it is to be interpreted in a real world sense. If one sticks to the concept of market observation and thus regards the Wick-self-financing property in an abstract world, arbitrage will appear again in a weak sense, i.e. under some observations.
3. From 1. and 2. we conclude that it is not quite sensible to use the purely fractional models as pricing models.
4. However, if one adds a Brownian component and considers mixed models, the class of regular portfolios is arbitrage free and includes hedges for many practically relevant options. Hence, the mixed models are more promising as candidates for sensible pricing models.

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