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RUIN PROBABILITY FOR GENERALIZED φ-SUB-GAUSSIAN FRACTIONAL BROWNIAN MOTION[1]

In this paper we investigate the ruin problem for the generalized φ-sub-Gaussian fractional Brownian motion (FBM). Such random process has the same covariation function as FBM but its trajectories belong to the space of φ-sub-Gaussian random variables (i.e. not necessarily Gaussian). For this risk process we obtain estimate of the ruin probability.

1. Introduction

Such properties of fractional Brownian motion as long-range dependence and self-similarity make it natural choice in modeling real processes from financial mathematics and queueing theory.

Recall, that the fractional Brownian motion with index $H \in (0, 1)$ is Gaussian centered process Z_H with stationary increments and continuous paths and covariance function

$$R_H(t, s) = \mathbf{E}Z^H(s)Z^H(t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |s - t|^{2H}\right).$$

One of actual tasks of the theory of random processes is finding the estimates of probability that trajectories of a random process exceed the level specified by some curve. It finds an application in risk theory as classical problem of the investigation of the ruin probability

$$\mathbf{P}\left\{\sup_{t>0}(X(t) - f(t)) > x\right\}$$

for various types of risk process $X = (X(t), t \geq 0)$ and functions $f(t)$. The similar problem of finding the buffer overflow probability appears in the queuing theory for different communication network models.

The tasks of such type were solved for many types of processes, including Gaussian ones and aforementioned FBM (see, for example, Norros [1], Michna [2], Baldi and Pacchiarotti [3], etc.). But since in many cases real processes are Gaussian only asymptotically or not Gaussian at all, there arises a problem of introduction of more general class of random processes than Gaussian one. From the such viewpoint the classes of φ-sub-Gaussian and strictly φ-sub-Gaussian random processes are of significant interest as

[1]Invited lecture.

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a natural extension of the class of Gaussian random processes. Detailed overview of their properties one can found in [4] and [5].

In this paper we investigate the properties of generalized φ-sub-Gaussian fractional Brownian motion process which has the same covariation function as fractional Brownian motion but its trajectories are not necessarily Gaussian. This process was introduced firstly in [7] under the name of weakly self-similar stationary increment processes from the space $SSub_\varphi(\Omega)$.

The plan of the paper is as follows. In §2 the general definitions and some properties of random variables and processes from spaces $Sub_\varphi(\Omega)$ and $SSub_\varphi(\Omega)$ are considered. In §3 we give the definition of generalized φ-sub-Gaussian fractional Brownian motion process (φ-GFBM). In §4 the results from in [8, 9] are used to study the sampling distributions for the ruin problem for the generalized fractional Brownian motion and for $f(t)$ of the form $f(t) = ct^\alpha$, $c > 0$, $\alpha \in [0, 1]$. We obtain the following estimates of the ruin probability (and of the buffer overflow probability for corresponding queueing model) for φ-GFBM risk process Z_H from the class $\Psi^q_{x_0}$ which also includes class of sub-Gaussian random processes ($q = 2$) and therefore (Gaussian) FBM.

$$(i)\ \mathbf{P}\left\{\sup_{a\le t\le b}(Z_H(t) - ct^\alpha) > x\right\} \le$$
$$\le 2\left(\frac{e}{p}\right)^{\frac{1}{H}} K_b(p, x)\exp\left\{-\frac{(q-1)x_0^{\frac{q}{q-1}}(Cb^\alpha + x)^{\frac{q}{q-1}}}{q^{\frac{q}{q-1}}b^{\frac{qH}{q-1}}}\right\},$$
$$(ii)\ \mathbf{P}\left\{\sup_{t>0}(Z_H(t) - ct) > x\right\} \le L(\gamma, x)x^{\frac{q(1-H)}{(q-1)H}}\exp\left\{-\kappa(\gamma)x^{\frac{q(1-H)}{q-1}}\right\},$$

where $K_b(p, x)$, $L(\gamma, x)$ are known bounded on x expressions.

2. Space of $Sub_\varphi(\Omega)$ random variables: necessary definitions and some useful properties

2.1. Orlicz N-functions

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space.

Definition 1. A continuous even convex function φ is an Orlicz N-function if it is strictly increasing for $x > 0$, $\varphi(0) = 0$ and

$$\frac{\varphi(x)}{x} \to 0 \quad \text{as} \quad x \to 0 \qquad \text{and} \qquad \frac{\varphi(x)}{x} \to \infty \quad \text{as} \quad x \to \infty.$$

Condition Q. An N-function φ satisfies condition Q if

$$\liminf_{x\to 0}\frac{\varphi(x)}{x^2} = c > 0. \qquad (1)$$

Remark. It may happen that $c = \infty$.

2.2. φ-sub-Gaussian random variables and processes

Definition 2. [5] Let φ be an Orlicz N-function satisfying condition Q. The random variable ξ belongs to the space $Sub_\varphi(\Omega)$ if $\mathbf{E}\xi = 0$, $\mathbf{E}\exp\{\lambda\xi\}$ exists for all $\lambda \in \mathbf{R}$ and there exists a constant $a > 0$ such that the following inequality holds for all $\lambda \in \mathbf{R}$

$$\mathbf{E}\exp(\lambda\xi) \le \exp(\varphi(a\lambda)). \tag{2}$$

Theorem 1. [4] *The space $Sub_\varphi(\Omega)$ is a Banach space with respect to the norm*

$$\tau_\varphi(\xi) = \inf\Big\{a \ge 0 \;:\; \mathbf{E}\exp(\lambda\xi) \;\le\; \exp\big(\varphi(a\lambda)\big),\; \lambda \in \mathbf{R}\Big\}. \tag{3}$$

When $\varphi(x) = \frac{x^2}{2}$ the space $Sub_\varphi(\Omega)$ is called the space of sub-Gaussian random variables and is denoted by $Sub(\Omega)$.

Examples. 1). Centered Gaussian random variable $\xi = N(0, \sigma^2)$ belongs to space $Sub(\Omega)$ and $\tau(\xi) = (\mathbf{E}\xi^2)^{\frac{1}{2}}$. 2). Let ξ be a centered bounded random variable, i.e. $\mathbf{E}\xi = 0$ and there exists number $c > 0$ that $|\xi| \le c$ almost surely. Then $\xi \in Sub(\Omega)$ and $\tau(\xi) \le c$.

Let T be a parameter set.

Definition 3. Random process $X = (X(t), t \in T)$ is a φ-sub-Gaussian process if for all $t \in T$ $X(t) \in Sub_\varphi(\Omega)$.

A random φ-sub-Gaussian process belongs to $\Psi^q_{x_0}$ if

$$\varphi(x) = \begin{cases} \frac{x^q}{x_0^q}, & |x| > x_0, \\ \frac{x^2}{x_0^2}, & |x| \le x_0, \end{cases} \tag{4}$$

where $x_0 > 0$ and $q \ge 2$ are some constants.

2.3. Strictly φ-sub-Gaussian random variables and processes

Theorem 2. [4] *Let φ be an Orlicz N-function satisfying condition Q and suppose that function $\varphi(\sqrt{\cdot})$ is convex. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent random variables from the space $Sub_\varphi(\Omega)$. Then*

$$\tau^2_\varphi\left(\sum_{i=1}^n \xi_i\right) \le \sum_{i=1}^n \tau^2_\varphi(\xi_i). \tag{5}$$

Definition 4. [6] A family of random variables Δ from the space $Sub_\varphi(\Omega)$ is called strictly $Sub_\varphi(\Omega)$, if there exists a constant $C_\Delta > 0$ such that for arbitrary finite set $I : \xi_i \in \Delta, i \in I$, and for any $\lambda_i \in \mathbf{R}$ the following inequality takes place

$$\tau_\varphi\bigg(\sum_{i\in I} \lambda_i\xi_i\bigg) \le C_\Delta\bigg(\mathbf{E}\bigg(\sum_{i\in I} \lambda_i\xi_i\bigg)^2\bigg)^{\frac{1}{2}}. \tag{6}$$

If Δ is a family of strictly $Sub_\varphi(\Omega)$ random variables, then linear closure $\overline{\Delta}$ of the family Δ in the space $L_2(\Omega)$ also is strictly $Sub_\varphi(\Omega)$ family of random variables. Linearly closed families of strictly $Sub_\varphi(\Omega)$ random variables form a space of strictly φ-sub-Gaussian random variables. This space is denoted by $SSub_\varphi(\Omega)$.

When $\varphi(x) = \frac{x^2}{2}$ the space $SSub_\varphi(\Omega)$ is called the space of strictly sub-Gaussian random variables and is denoted by $SSub(\Omega)$.

The space of jointly Gaussian random variables belongs to space $SSub(\Omega)$.

Definition 5. A random process $X = (X(t), t \in T)$ is a strictly φ-sub-Gaussian process if the corresponding family of random variables belongs to the space $SSub_\varphi(\Omega)$.

Example. [6] Let φ be such an Orlicz N-function that the function $\varphi(\sqrt{\cdot})$ is convex and

$$X(t) = \sum_{k=1}^{\infty} \xi_k \phi_k(t),$$

where series $\sum_{k=1}^{\infty} \xi_k \phi_k(t)$ converges in mean square sense for all $t \in T$ and family $\{\xi_k, k \geq 1\}$ belongs to the space $SSub_\varphi(\Omega)$, for instance $\{\xi_k, k \geq 1\}$ are independent random variables from $SSub_\varphi(\Omega)$. Then $X(t)$ is a strictly φ-sub-Gaussian random process.

2.4. Probability of overrunning for φ-sub-Gaussian random process

Let (T, ρ) be a pseudometrical (metrical) compact space with pseudometric (metric) ρ.

Suppose there exists such continuous monotonically increasing function $\sigma = \{\sigma(h), h > 0\}$, that $\sigma(h) \to 0$, as $h \to 0$, and the following inequality is true

$$\sup_{\rho(t,s) \leq h} \tau_\varphi(Y(t) - Y(s)) \leq \sigma(h). \tag{7}$$

Let $\beta > 0$ be some number such that $\beta \leq \sigma\left(\inf_{s \in T} \sup_{t \in T} \rho(t,s)\right)$, $\gamma(u) = \tau_\varphi(Y(u))$, $N_T(u)$ denotes the least number of closed ρ-balls with radius u needed to cover T.

Theorem 3. [9] *Let $Y = \{Y(t), t \in T\}$ be a separable random process from the space $Sub_\varphi(\Omega)$ and $f = \{f(t), t \in T\}$ be such a continuous function that $|f(u) - f(v)| \leq \delta(\rho(u,v))$, where $\delta = \{\delta(s), s > 0\}$ is some monotonically increasing nonnegative function, and $X(t) = Y(t) - f(t)$. Let $r = \{r(u) : u \geq 1\}$ be such a continuous function that $r(u) > 0$ as $u > 1$ and the function $s(t) = r(\exp\{t\}), t \geq 0$, is convex. If*

$$\int_0^\beta r(N_T(\sigma^{(-1)}(u)))du < \infty,$$

then for all $p \in (0;1)$ and $x > 0$ the following inequalities hold true

$$\mathbf{P}\left\{\sup_{t\in T} X(t) > x\right\} \leq \inf_{\lambda>0} Z_r(\lambda,p,\beta), \tag{8}$$

$$\mathbf{P}\left\{\inf_{t\in T} X(t) < -x\right\} \leq \inf_{\lambda>0} Z_r(\lambda,p,\beta), \tag{9}$$

$$\mathbf{P}\left\{\sup_{t\in T} |X(t)| > x\right\} \leq 2\inf_{\lambda>0} Z_r(\lambda,p,\beta), \tag{10}$$

where

$$Z_r(\lambda,p,\beta) = = \exp\left\{\theta_\varphi(\lambda,p) + p\varphi\left(\frac{\lambda\beta}{1-p}\right) + \lambda\left(\sum_{k=2}^{\infty}\delta(\sigma^{(-1)}(\beta p^{k-1})) - x\right)\right\} \times \times r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p} r(N_T(\sigma^{(-1)}(u)))du\right),$$

$$\theta_\varphi(\lambda,p) = \sup_{u\in T}\left((1-p)\varphi\left(\frac{\lambda\gamma(u)}{1-p}\right) - \lambda f(u)\right).$$

3. Process of φ-sub-Gaussian generalized fractional Brownian motion

Definition 6. [7] We call the process $Z_H = (Z^H(t), t \in T)$ φ-sub-Gaussian generalized fractional Brownian motion (φ-GFBM) with Hurst index $H \in (0,1)$ if Z_H is φ-sub-Gaussian process with stationary increments and covariance function

$$R_H(t,s) = \mathbf{E}Z_H(s)Z_H(t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |s-t|^{2H}\right).$$

Example. Let $\{\eta_n, n = 1,2,\ldots\}$ be a sequence of independent random variables such that $\mathbf{E}\eta_n = 0$, $\mathbf{E}\eta_n^2 = 1$ and $\eta_n \in SSub_\varphi(\Omega)$, where φ is such an N-function that function $\varphi(\sqrt{\cdot})$ is convex and $\tau_\varphi(\eta_n) \leq \tau < +\infty$. Then the process

$$Z_H(t) = \sum_{n=1}^{\infty} \lambda_n \eta_n \psi_n(t)$$

is a centered strictly φ-sub-Gaussian random process with covariance function R_H, where λ_n are eigenvalues and ψ_n are corresponding eigen-functions of the following integral equation

$$\psi(s) = \frac{1}{\lambda^2}\int_0^T R_H(t,s)\psi(t)dt.$$

4. Main results

It is easy to obtain the following corollary for the process of φ-GFBM from theorem 3 (see also [8,9]).

Theorem 4. *Let $Z_H = (Z_H(t), t \in [a,b])$ be a process of strictly φ-GFBM with Hurst parameter $H \in (0,1)$, $C > 0$ be some constant. Then for all $x > 0$, numbers a, b such that $0 \le a < b < \infty$, $p \in (0,1)$, $\beta \in \left(0, \left(\frac{b-a}{2}\right)^H\right]$ and $\lambda > 0$ the following inequality holds true*

$$\mathbf{P}\left\{\sup_{a\le t\le b}(Z_H(t) - ct^\alpha) > x\right\} \le (b-a)\left(\frac{e}{\beta p}\right)^{\frac{1}{H}} \times$$

$$\times \exp\left\{\frac{\lambda c(\beta p)^{\frac{\alpha}{H}}}{C_\Delta(1-p^{\frac{\alpha}{H}})} + p\varphi\left(\frac{\lambda\beta}{1-p}\right) + (1-p)\theta_\varphi(\lambda,p) - \frac{\lambda x}{C_\Delta}\right\}, \tag{11}$$

where $\theta_\varphi(\lambda,p) = \sup\limits_{a\le u\le b}\left(\varphi\left(\frac{\lambda u^H}{1-p}\right) - \frac{\lambda c u^\alpha}{C_\Delta(1-p)}\right)$, and C_Δ is the constant from definition 4 of the space $SSub_\varphi(\Omega)$.

Theorem 5. *Let $Z_H = (Z_H(t), t \in [a,b], 0 \le a < b < \infty)$, be a process of strictly φ-GFBM from the class $\Psi^q_{x_0}$ with Hurst parameter $H \in (\frac{1}{2}, 1)$. Let $C > 0$, $p \in (0,1)$ and $\alpha \in [0,1]$ be some constants and suppose that if $q > 2$ then the following condition holds*

$$\max\left\{v^{-H}; \frac{2^H}{(b-a)^H}\right\} \le \left(\frac{x_0 C(b-a)}{(b^{qH} - a^{qH})}\right)^{\frac{1}{q-1}}, \tag{12}$$

where $v = a$ if $a > 0$, or $v = b$ if $a = 0$.

Then for all $\varepsilon > 0$ the following estimate is true

$$\mathbf{P}\left\{\sup_{a\le t\le b}\left(\frac{1}{C_\Delta}Z_H(t) - Ct^\alpha\right) > \varepsilon\right\} \le 2\left(\frac{e}{p}\right)^{\frac{1}{H}} W_{a,b}(\varepsilon) K_{a,b}(p,\varepsilon), \tag{13}$$

where

$$W_{a,b}(\varepsilon) = \exp\left\{-\left(\frac{Cx_0^q(b^\alpha - a^\alpha)}{(b^{qH} - a^{qH})}\right)^{\frac{1}{q-1}}\left(\varepsilon + Ca^\alpha b^\alpha \frac{b^{qH-\alpha} - a^{qH-\alpha}}{b^{qH} - a^{qH}}\right)\right\},$$

$$K_{a,b}(p,\varepsilon) = \exp\left\{p\left(\frac{Cx_0^q(b^\alpha - a^\alpha)}{(b^{qH} - a^{qH})}\right)^{\frac{1}{q-1}}\left(\varepsilon + Ca^\alpha b^\alpha \frac{b^{qH-\alpha} - a^{qH-\alpha}}{b^{qH} - a^{qH}} + \frac{C(b-a)^\alpha p^{\frac{\alpha-H}{H}}(1-p)}{2(1-p^{\frac{\alpha}{H}})} + \frac{C(b-a)^{qH}(b^\alpha - a^\alpha)}{2^{qH}(b^{qH} - a^{qH})}\right)\right\}.$$

Also if

$$\varepsilon \ge \varepsilon_b = \frac{qC(b^\alpha - a^\alpha)}{b^{qH} - a^{qH}}\left(b^{qH} + \frac{(b-a)^{qH}p}{2^{qH}(1-p)}\right) + \frac{C(b-a)^\alpha p^{\frac{\alpha}{H}}}{2(1-p^{\frac{\alpha}{H}})} - Cb^\alpha, \tag{14}$$

then the following estimate, which is better than (13), holds true

$$\mathbf{P}\left\{\sup_{a\le t\le b}\left(\frac{1}{C_\Delta}Z_H(t)-Ct^\alpha\right)>\varepsilon\right\}\le 2\left(\frac{e}{p}\right)^{\frac{1}{H}}\Gamma_b(\varepsilon)K_b(p,\varepsilon), \tag{15}$$

where

$$\Gamma_b(\varepsilon) = \exp\left\{-\frac{(q-1)x_0^{\frac{q}{q-1}}(Cb^\alpha+\varepsilon)^{\frac{q}{q-1}}}{q^{\frac{q}{q-1}}b^{\frac{qH}{q-1}}}\right\},$$

$$K_b(p,\varepsilon) = \exp\left\{\frac{p\,x_0^{\frac{q}{q-1}}(Cb^\alpha+\varepsilon)^{\frac{1}{q-1}}}{q^{\frac{q}{q-1}}b^{\frac{qH}{q-1}}}\left(\frac{(Cb^\alpha+\varepsilon)(b-a)^{qH}}{2^{qH}b^{qH}(1-p)}+\right.\right.$$
$$\left.\left.+\frac{p^{\frac{\alpha}{H}-1}qC(b-a)^\alpha}{2(1-p^{\frac{\alpha}{H}})}+(q-1)(Cb^\alpha+\varepsilon)\right)\right\}.$$

Remarks. 1) If $q=2$ then condition (12) is unnecessary. 2) Since p can be chosen small enough the expression $K_b(p,\varepsilon)$ can be bounded for any ε.

Proof. For simplicity put $\beta=\frac{(b-a)^H}{2^H}$. In order to unambiguously determine function $\varphi(\cdot)$, consider such $\lambda\ge\lambda_0>0$ that $\frac{\lambda_0(b-a)^H}{2^H(1-p)}\ge x_0$ and $\frac{\lambda_0 a^H}{1-p}\ge x_0$ if $a>0$, or $\frac{\lambda_0 b^H}{1-p}\ge x_0$ if $a=0$. Then we can put

$$\lambda_0=(1-p)x_0\max\left\{v^{-H};\ \frac{2^H}{(b-a)^H}\right\},$$

where $v=a$ for $a>0$ or else $v=b$ if $a=0$.

Since $\varphi(x)$ is strictly convex then

$$\theta_\varphi(\lambda,C,p)=\begin{cases}\frac{\lambda^q a^{qH}}{x_0^q(1-p)^q}-\frac{\lambda C a^\alpha}{1-p}, & \lambda_0\le\lambda\le\lambda^*;\\ \frac{\lambda^q b^{qH}}{x_0^q(1-p)^q}-\frac{\lambda C b^\alpha}{1-p}, & \lambda>\lambda^*,\end{cases} \tag{16}$$

where $\lambda^*=\left(\frac{C(b^\alpha-a^\alpha)}{(b^{qH}-a^{qH})}\right)^{\frac{1}{q-1}}(1-p)x_0^{\frac{q}{q-1}}$.

For $\lambda\ge\lambda_0$ consider exponential part from estimate (11) in the theorem 4.

$$\Gamma(\lambda,p,\varepsilon)=\exp\left\{\lambda^q\left(\frac{d^{qH}}{x_0^q(1-p)^{q-1}}+\frac{(b-a)^{qH}p}{x_0^q2^{qH}(1-p)^q}\right)-\right.$$
$$\left.-\lambda\left(Cd^\alpha+\varepsilon-\frac{C(b-a)^\alpha p^{\frac{\alpha}{H}}}{2(1-p^{\frac{\alpha}{H}})}\right)\right\}:=\exp\left\{\lambda^qA_d-\lambda B_d\right\}, \tag{17}$$

where $d=\begin{cases}a, & \text{if } \lambda\le\lambda^*,\\ b, & \text{if } \lambda>\lambda^*.\end{cases}$

It is obvious that if $\lambda^*\le\left(\frac{B_b}{qA_b}\right)^{\frac{1}{q-1}}$, that is if

$$\varepsilon\ge\varepsilon_b=\frac{qC(b^\alpha-a^\alpha)}{b^{qH}-a^{qH}}\left(b^{qH}+\frac{(b-a)^{qH}p}{2^{qH}(1-p)}\right)+\frac{C(b-a)^\alpha p^{\frac{\alpha}{H}}}{2(1-p^{\frac{\alpha}{H}})}-Cb^\alpha, \tag{18}$$

then the function $\Gamma(\lambda,p,\varepsilon)$ reaches its minimum value at the point $\lambda = \left(\frac{B_b}{qA_b}\right)^{\frac{1}{q-1}}$. Also for the unambiguity of the function φ we need $\lambda_0 \le \lambda^*$, i.e.

$$\max\left\{v^{-H}; \frac{2^H}{(b-a)^H}\right\} \le \left(\frac{x_0 C(b^\alpha - a^\alpha)}{(b^{qH} - a^{qH})}\right)^{\frac{1}{q-1}}.$$

Consider the inequalities

$$(1+x')^{\alpha'} < 1+\alpha' x',\ 0<\alpha'<1,\ x'\ge 0; \tag{19}$$

$$(1-x'')^{\alpha''} \ge 1-\alpha'' x'',\ \alpha''\ge 1,\ 0\le x''\le 1. \tag{20}$$

Let $x' = \dfrac{(b-a)^{qH}p}{2^{qH}b^{qH}(1-p)}$, $\alpha' = \frac{1}{q-1}$ and $x'' = \dfrac{C(b-a)^\alpha p^{\frac{\alpha}{H}}}{2(Cb^\alpha+\varepsilon)(1-p^{\frac{\alpha}{H}})}$, $\alpha'' = \frac{q}{q-1}$. From (18) follows that $x''\le 1$. Applying (19) and (20) to (17), using inequality $\frac{1}{1+z}\le 1$ for $z\ge 0$, and the identity $-\frac{z_1}{z_2(1+z_3)} = -\frac{z_1}{z_2} + \frac{z_2}{1+z_2}$ for some positive z_1, z_3 and z_2, we have

$$\min_{\lambda>0}\Gamma(\lambda,p,\varepsilon) \le \min_{\lambda\ge\lambda_0}\Gamma(\lambda,p,\varepsilon) = \exp\left\{-\frac{(q-1)B_b^{\frac{q}{q-1}}}{q^{\frac{q}{q-1}}A_b^{\frac{1}{q-1}}}\right\} =$$

$$= \exp\left\{-\frac{(q-1)x_0^{\frac{q}{q-1}}(Cb^\alpha+\varepsilon)^{\frac{q}{q-1}}\left(1-\frac{C(b-a)^\alpha p^{\frac{\alpha}{H}}}{2(Cb^\alpha+\varepsilon)\left(1-p^{\frac{\alpha}{H}}\right)}\right)^{\frac{q}{q-1}}}{q^{\frac{q}{q-1}}b^{\frac{qH}{q-1}}\left(1+\frac{(b-a)^{qH}p}{(2b)^{qH}(1-p)}\right)^{\frac{1}{q-1}}}\right\}\times$$

$$\times \exp\left\{\frac{p(q-1)x_0^{\frac{q}{q-1}}\left(Cb^\alpha+\varepsilon-\frac{C(b-a)^\alpha p^{\frac{\alpha}{H}}}{2\left(1-p^{\frac{\alpha}{H}}\right)}\right)^{\frac{q}{q-1}}}{q^{\frac{q}{q-1}}\left(b^{qH}+\frac{(b-a)^{qH}p}{2^{qH}(1-p)}\right)^{\frac{1}{q-1}}}\right\} \le$$

$$\le \exp\left\{-\frac{(q-1)x_0^{\frac{q}{q-1}}(Cb^\alpha+\varepsilon)^{\frac{q}{q-1}}\left(1-\frac{qC(b-a)^\alpha p^{\frac{\alpha}{H}}}{2(q-1)(Cb^\alpha+\varepsilon)\left(1-p^{\frac{\alpha}{H}}\right)}\right)}{q^{\frac{q}{q-1}}b^{\frac{qH}{q-1}}\left(1+\frac{(b-a)^{qH}p}{(q-1)(2b)^{qH}(1-p)}\right)}\right\}\times$$

$$\times \exp\left\{\frac{p(q-1)x_0^{\frac{q}{q-1}}(Cb^\alpha+\varepsilon)^{\frac{q}{q-1}}}{q^{\frac{q}{q-1}}b^{\frac{qH}{q-1}}}\right\} \le$$

$$\le \exp\left\{-\frac{(q-1)x_0^{\frac{q}{q-1}}(Cb^\alpha+\varepsilon)^{\frac{q}{q-1}}}{q^{\frac{q}{q-1}}b^{\frac{qH}{q-1}}}\right\}\exp\left\{\frac{px_0^{\frac{q}{q-1}}(Cb^\alpha+\varepsilon)^{\frac{1}{q-1}}}{q^{\frac{q}{q-1}}b^{\frac{qH}{q-1}}}\times\right.$$

$$\left.\left(\frac{(Cb^\alpha+\varepsilon)(b-a)^{qH}}{2^{qH}b^{qH}(1-p)} + \frac{p^{\frac{\alpha}{H}-1}qC(b-a)^\alpha}{2(1-p^{\frac{\alpha}{H}})} + (q-1)(Cb^\alpha+\varepsilon)\right)\right\}. \tag{21}$$

Alternatively, if $\varepsilon < \varepsilon_b$

$$\min_{\lambda>0} \Gamma(\lambda, p, \varepsilon) \le \Gamma(\lambda^*, p, \varepsilon) = \exp\left\{\left(\frac{C(b^\alpha - a^\alpha)}{(b^{qH} - a^{qH})}\right)^{\frac{q}{q-1}} \times \right.$$
$$\times (1-p)^q x_0^{\frac{q^2}{q-1}} \left(\frac{b^{qH}}{x_0^q (1-p)^{q-1}} + \frac{(b-a)^{qH} p}{x_0^q 2^{qH} (1-p)^q}\right) -$$
$$-\left(\frac{C(b^\alpha - a^\alpha)}{(b^{qH} - a^{qH})}\right)^{\frac{1}{q-1}} (1-p) x_0^{\frac{q}{q-1}} \left(Cb^\alpha + \varepsilon - \frac{C(b-a)^\alpha p^{\frac{\alpha}{H}}}{2(1-p^{\frac{\alpha}{H}})}\right)\Bigg\} =$$
$$= \exp\left\{-\left(\frac{C(b^\alpha - a^\alpha)}{(b^{qH} - a^{qH})}\right)^{\frac{1}{q-1}} x_0^{\frac{q}{q-1}} \left(\varepsilon + Ca^\alpha b^\alpha \frac{b^{qH-\alpha} - a^{qH-\alpha}}{b^{qH} - a^{qH}}\right)\right\} \times$$
$$\times \exp\left\{p\left(\frac{C(b^\alpha - a^\alpha)}{(b^{qH} - a^{qH})}\right)^{\frac{1}{q-1}} x_0^{\frac{q}{q-1}} \left(\varepsilon + Ca^\alpha b^\alpha \frac{b^{qH-\alpha} - a^{qH-\alpha}}{b^{qH} - a^{qH}} + \right.\right.$$
$$\left.\left. + \frac{C(b-a)^\alpha p^{\frac{\alpha-H}{H}}(1-p)}{2(1-p^{\frac{\alpha}{H}})} + \frac{C(b-a)^{qH}(b^\alpha - a^\alpha)}{2^{qH}(b^{qH} - a^{qH})}\right)\right\}. \tag{22}$$

It is obvious that the latter estimate holds true also for $\varepsilon \ge \varepsilon_b$. So from (21) and (22) we have the assertion of the theorem. □

Theorem 6. *Let $Z_H = (Z_H(t), t \ge 0)$ be random processes from class $\Psi^q_{x_0}$ with Hurst parameter $H \in [0.5, 1)$, $qH > 1$. Let $C > 0$ and $\gamma > 1$ be some constants. Then for all*

$$\varepsilon \ge \frac{2^{\frac{H(q-1)}{1-H}} \gamma^M q(1-H)(\gamma^{qH-1} - 1)}{x_0^{\frac{1}{1-H}} C^{\frac{H}{1-H}} (qH-1)} \max\left\{\frac{1}{\gamma^{qH} - 1}; \frac{(\gamma^{qH} - 1)^{\frac{H}{1-H}}}{(\gamma - 1)^{\frac{H(q-1)}{1-H}}}\right\} \tag{23}$$

and

$$\zeta \in \left(0, \varepsilon^{\frac{q(1-H)}{(q-1)H}}\right),$$

where M is such an integer that

$$M \ge 1 + \frac{q-1}{q(1-H)} \log\left(\frac{q-1}{q(1-H)} \frac{(\gamma-1)^{\frac{1}{q-1}}}{(\gamma^{qH} - 1)^{\frac{1}{q-1}}}\right), \tag{24}$$

the following inequality holds true

$$\begin{aligned} &\mathbf{P}\left\{\sup_{t>0}\left(\frac{1}{C_\Delta} Z_H(t) - Ct\right) > \varepsilon\right\} \\ &\quad \le L(\gamma, \varepsilon) \varepsilon^{\frac{q(1-H)}{(q-1)H}} \exp\left\{-\kappa(\gamma) \varepsilon^{\frac{q(1-H)}{q-1}}\right\}, \end{aligned} \tag{25}$$

where

$$\kappa(\gamma) = \frac{x_0^{\frac{q}{q-1}} C^{\frac{qH}{q-1}} (q-1)(\gamma-1)^{\frac{1}{q-1}} (\gamma^{qH}-\gamma)^{\frac{qH-1}{q-1}}}{(q-qH)^{\frac{q-qH}{q-1}} (qH-1)^{\frac{qH-1}{q-1}} (\gamma^{qH}-1)^{\frac{qH}{q-1}}}, \tag{26}$$

$$L(\gamma,\varepsilon) = 2\zeta^{-1} e^{\frac{1}{H}} \big(K_0(\gamma) + K_1(\gamma) S_1(\gamma,\varepsilon) + \gamma^{\frac{q(1-H)M}{(q-1)H}} K_M(\gamma) + K_{M+1}(\gamma) S_2(\gamma,\varepsilon)\big) < \infty, \tag{27}$$

$$S_1(\gamma,\varepsilon) = \sum_{k=1}^{M-1} \gamma^{\frac{q(1-H)k}{(q-1)H}} \exp\left\{-\kappa(\gamma)\varepsilon^{\frac{q(1-H)}{q-1}}\right\}^{S_{k,M}(\gamma)}, \tag{28}$$

$$S_2(\gamma,\varepsilon) = \sum_{k=M+1}^{\infty} \gamma^{\frac{q(1-H)k}{(q-1)H}} \exp\left\{-\kappa(\gamma)\varepsilon^{\frac{q(1-H)}{q-1}}\right\}^{S_{k,M}(\gamma)}, \tag{29}$$

$$S_{k,M}(\gamma) = \frac{q(1-H)}{q-1}\gamma^{\frac{qH-1}{q-1}(M-k)} + \frac{qH-1}{q-1}\gamma^{\frac{q(1-H)}{q-1}(k-M)} - 1, \tag{30}$$

$$K_0(\gamma) = \exp\left\{ x_0^{\frac{q}{q-1}} \zeta^H C^{\frac{qH}{q-1}} \left(\frac{\gamma^M q(1-H)(\gamma^{qH-1}-1)}{(qH-1)(\gamma^{qH}-1)} \right)^{\frac{qH-1}{q-1}} \times \left(1 + \frac{\gamma^{-M}(qH-1)(\gamma^{qH}-1)}{q(1-H)(\gamma^{qH-1}-1)} \left(\frac{1}{2^{qH}} + \frac{H}{2}\right)\right)\right\}, \tag{31}$$

(32)

$$K_k(\gamma) = \exp\left\{ x_0^{\frac{q}{q-1}} \zeta^H C^{\frac{qH}{q-1}} \left(\frac{\gamma-1}{\gamma^{qH}-1}\right)^{\frac{1}{q-1}} \times \right.$$
$$\times \left(\frac{\gamma^M q(1-H)(\gamma^{qH}-\gamma)}{(qH-1)(\gamma^{qH}-1)} \right)^{\frac{qH-1}{q-1}} \times$$
$$\times \left(\gamma^{-k} + \frac{\gamma^{-M}(qH-1)}{q(1-H)} + \frac{\gamma^{-M}(\gamma-1)^{qH+1}(qH-1)}{2^{qH}(\gamma^{qH}-\gamma)q(1-H)} \right.$$
$$\left.\left. + \frac{\gamma^{-M}(\gamma-1)(\gamma^{qH}-1)(qH-1)}{2(\gamma^{qH}-\gamma)q(1-H)\gamma^{\frac{q(1-H)^2k}{(q-1)H}}\left(1-\gamma^{-\frac{q(1-H)k}{(q-1)H}}\right)} \right) \right\}, \ k \geq 1.$$

Remark. The ruin probability can be minimized by appropriate selection of parameters γ and ζ.

Proof. Let us consider the following partition: $[0,\infty) = \bigcup_{k=0}^{\infty} [a_k, b_k]$, where $a_0 = 0,\ b_0 = a,\ b_k = a_{k+1} = \gamma^k a,\ k \geq 1,\ a > 0, \gamma > 1$ and apply for each interval theorem 5. Then for any $k \geq 1$

$$W_{a_k,b_k}(\varepsilon) = W_k(\gamma,\varepsilon) =$$
$$= \exp\left\{ -\frac{C^{\frac{1}{q-1}} x_0^{\frac{q}{q-1}}}{a^{\frac{qH-1}{q-1}} \gamma^{\frac{qH-1}{q-1}(k-1)}} \left(\frac{\gamma-1}{\gamma^{qH}-1}\right)^{\frac{1}{q-1}} \left(\varepsilon + C\gamma^k a \frac{\gamma^{qH-1}-1}{\gamma^{qH}-1}\right) \right\}.$$

Consider the function

$$j(n) = \gamma^{\frac{1-qH}{q-1}(n-1)} \left(\varepsilon + C\gamma^n a \frac{\gamma^{qH-1}-1}{\gamma^{qH}-1} \right)$$

as continuous relative to it's argument n. Then

$$\frac{dj(n)}{dn} = \left(\frac{1-qH}{q-1} \left(\varepsilon + \frac{C\gamma^n a(\gamma^{qH-1}-1)}{\gamma^{qH}-1} \right) + \frac{Ca\gamma^n(\gamma^{qH-1}-1)}{\gamma^{qH}-1} \right) \times \gamma^{\frac{1-qH}{q-1}(n-1)} \log\gamma$$

$$= \left(\frac{\varepsilon(1-qH)}{q-1} + \frac{C\gamma^n a(\gamma^{qH-1}-1)q(1-H)}{(\gamma^{qH}-1)(q-1)} \right) \gamma^{\frac{1-qH}{q-1}(n-1)} \log\gamma,$$

$$\frac{dj(n)}{dn} = 0 \Leftrightarrow a = \frac{\varepsilon\gamma^{-n}}{C} \frac{qH-1}{q(1-H)} \frac{\gamma^{qH}-1}{\gamma^{qH-1}-1}.$$

If the previous equality holds true then $W_n(\gamma,\varepsilon)$ takes maximal value. Choose such an a that $W_k(\gamma,\varepsilon)$ takes maximal values for $k = M$ for some $M \geq 1$. Then

$$a = \frac{\varepsilon\gamma^{-M}}{C} \frac{qH-1}{q(1-H)} \frac{\gamma^{qH}-1}{\gamma^{qH-1}-1}. \tag{33}$$

After substituting a from (33) in $W_k(\gamma,\varepsilon)$ we have

$$W_k^{(M)}(\gamma,\varepsilon) = \exp\Bigg\{ -C^{\frac{qH}{q-1}} x_0^{\frac{q}{q-1}} \varepsilon^{\frac{q(1-H)}{q-1}} \gamma^{\frac{qH-1}{q-1}(M+1-k)} \left(\frac{q(1-H)}{qH-1} \right)^{\frac{qH-1}{q-1}} \times \frac{(\gamma-1)^{\frac{1}{q-1}} (\gamma^{qH-1}-1)^{\frac{qH-1}{q-1}}}{(\gamma^{qH}-1)^{\frac{qH}{q-1}}} \left(1 + \gamma^{k-M} \frac{qH-1}{q(1-H)} \right) \Bigg\}$$

for $k \geq 1$ and

$$W_0^{(M)}(\gamma,\varepsilon) = \exp\Bigg\{ -C^{\frac{qH}{q-1}} x_0^{\frac{q}{q-1}} \varepsilon^{\frac{q(1-H)}{q-1}} \gamma^{\frac{(qH-1)M}{q-1}} \times \left(\frac{q(1-H)}{qH-1} \right)^{\frac{qH-1}{q-1}} \left(\frac{\gamma^{qH-1}-1}{\gamma^{qH}-1} \right)^{\frac{qH-1}{q-1}} \Bigg\}.$$

Let's define by

$$W(\gamma,\varepsilon) = W_M^{(M)}(\gamma,\varepsilon) = \exp\left\{ -\kappa(\gamma)\varepsilon^{\frac{q(1-H)}{q-1}} \right\}, \tag{34}$$

$$\kappa(\gamma) = \frac{x_0^{\frac{q}{q-1}} C^{\frac{qH}{q-1}} (q-1)(\gamma-1)^{\frac{1}{q-1}} (\gamma^{qH}-\gamma)^{\frac{qH-1}{q-1}}}{(q-qH)^{\frac{q-qH}{q-1}} (qH-1)^{\frac{qH-1}{q-1}} (\gamma^{qH}-1)^{\frac{qH}{q-1}}}. \tag{35}$$

It is obvious that $W(\gamma,\varepsilon) \geq W_0^{(M)}(\gamma,\varepsilon)$ if

$$M \geq 1 + \frac{q-1}{q(1-H)} \log\left(\frac{q-1}{q(1-H)} \frac{(\gamma-1)^{\frac{1}{q-1}}}{(\gamma^{qH}-1)^{\frac{1}{q-1}}} \right). \tag{36}$$

Consider the following ratio

$$\begin{aligned}
&\frac{W_k^{(M)}(\gamma,\varepsilon)}{W(\gamma,\varepsilon)} \\
&\quad= \exp\Bigg\{ -C^{\frac{qH}{q-1}} x_0^{\frac{q}{q-1}} \varepsilon^{\frac{q(1-H)}{q-1}} \gamma^{\frac{qH-1}{q-1}} \frac{(q-1)(qH-1)^{\frac{1-qH}{q-1}}}{(q(1-H))^{\frac{q(1-H)}{q-1}}} \\
&\qquad\quad \times \frac{(\gamma-1)^{\frac{1}{q-1}}(\gamma^{qH-1}-1)^{\frac{qH-1}{q-1}}}{(\gamma^{qH}-1)^{\frac{qH}{q-1}}} \\
&\qquad\quad \times \left(\frac{q(1-H)}{q-1}\gamma^{\frac{qH-1}{q-1}(M-k)} + \frac{qH-1}{q-1}\gamma^{-\frac{q(1-H)}{q-1}(M-k)} - 1\right)\Bigg\} \\
&\quad= W(\gamma,\varepsilon)^{S_{k,M}(\gamma)},
\end{aligned}$$

where $S_{k,M}(\gamma)$ are defined in (30). It is easy to see that $S_{k,M}(\gamma) > 0$ for any $\gamma > 1$ and $k \geq 1$. In order to estimate the expressions consider the inequalities

$$e^x \geq 1 + x + x^2$$

and

$$e^{-x} \geq 1 - x$$

for $x \geq 0$. Then

$$S_{k,M}(\gamma) \geq \frac{(qH-1)q^2(1-H)^2\log^2(\gamma)(k-M)^2}{(q-1)^2}, \quad k > M,$$

and the series

$$\begin{aligned}
S_2(\gamma,\varepsilon) &= \sum_{k=M+1}^{\infty} \gamma^{\frac{q(1-H)k}{(q-1)H}} W(\gamma,\varepsilon)^{S_{k,M}(\gamma)} \\
&\leq \sum_{k=M+1}^{\infty} \gamma^{\frac{q(1-H)k}{(q-1)H}} \\
&\quad \times \exp\left\{-\kappa(\gamma)\varepsilon^{\frac{q(1-H)}{q-1}} \frac{(qH-1)q^2(1-H)^2\log^2\gamma}{(q-1)^2}\right\}^{(k-M)^2}
\end{aligned}$$

converges for any $\varepsilon > 0$ and $\gamma > 1$.

Consider $K_{a,b}(p,\varepsilon)$ from theorem 5 in details after substituting a from (33).

$$K_{a_k,b_k}(p,\varepsilon)=\exp\left\{px_0^{\frac{q}{q-1}}\left(\frac{C(b_k-a_k)}{(b_k^{qH}-a_k^{qH})}\right)^{\frac{1}{q-1}}\left(\varepsilon+\right.\right.$$

$$\left.\left.+Ca_kb_k\frac{b_k^{qH-1}-a_k^{qH-1}}{b_k^{qH}-a_k^{qH}}+\frac{C(b_k-a_k)p^{\frac{1-H}{H}}(1-p)}{2(1-p^{\frac{1}{H}})}+\frac{C(b_k-a_k)^{qH+1}}{2^{qH}(b_k^{qH}-a_k^{qH})}\right)\right\}$$

$$=\exp\left\{px_0^{\frac{q}{q-1}}\left(\frac{C(\gamma-1)}{(\gamma^{qH}-1)}\right)^{\frac{1}{q-1}}\left(\frac{\varepsilon\gamma^{-M}(qH-1)(\gamma^{qH}-1)}{Cq(1-H)(\gamma^{qH-1}-1)}\right)^{\frac{1-qH}{q-1}}\times\right.$$

$$\times\gamma^{\frac{(1-qH)(k-1)}{q-1}}\left(\varepsilon+\frac{\varepsilon\gamma^{k-M}(qH-1)}{q(1-H)}+\frac{\varepsilon\gamma^{k-M}(\gamma-1)^{qH+1}(qH-1)}{2^{qH}(\gamma^{qH}-\gamma)q(1-H)}+\right.$$

$$\left.\left.+\frac{\varepsilon\gamma^{k-M}(\gamma-1)(\gamma^{qH}-1)(qH-1)p^{\frac{1-H}{H}}(1-p)}{2(\gamma^{qH}-\gamma)q(1-H)(1-p^{\frac{1}{H}})}\right)\right\}. \tag{37}$$

For $k\geq 0$ let's put $p=p_k=\frac{\zeta^H}{\varepsilon^{\frac{q(1-H)}{q-1}}\gamma^{\frac{q(1-H)k}{q-1}}}$, where ζ is such a positive constant that for all $k\geq 0$ $p_k<1$, i.e. $0<\zeta<\varepsilon^{\frac{q(1-H)}{q-1}}$. Then for $k\geq 1$ $K_{a_k,b_k}(p_k,\varepsilon)\leq K_k(\gamma)$, $K_k(\gamma)$ specified in (32).

It easy to check that fraction $\frac{p^{\frac{1-H}{H}}(1-p)}{1-p^{\frac{1}{H}}}$ monotone increases for $p<1$ and $\frac{p^{\frac{1-H}{H}}(1-p)}{1-p^{\frac{1}{H}}}\nearrow H$ if $p\nearrow 1$. So in the same way for $k=0$

$$\begin{aligned}K_{a_0,b_0}(p_0,\varepsilon)&=\exp\left\{px_0^{\frac{q}{q-1}}C^{\frac{1}{q-1}}a^{\frac{1-qH}{q-1}}\left(\varepsilon+\frac{Ca}{2^{qH}}+\frac{Cap^{\frac{1-H}{H}}(1-p)}{2(1-p^{\frac{1}{H}})}\right)\right\}\\&\leq K_0(\gamma).\end{aligned}$$

Let's investigate fulfilment of the condition (12) from theorem 5 for each interval of the partition. For the first interval $[0,a]$ we have

$$\max\{a^{-H};2^Ha^{-H}\}=\frac{2^H}{a^H}\leq\left(x_0Ca^{1-qH}\right)^{\frac{1}{q-1}}.$$

And after substituting (33) we have the following inequality.

$$\varepsilon\geq\frac{2^{\frac{H(q-1)}{1-H}}\gamma^Mq(1-H)(\gamma^{qH-1}-1)}{x_0^{\frac{1}{1-H}}C^{\frac{H}{1-H}}(qH-1)(\gamma^{qH}-1)}. \tag{38}$$

For the intervals $[a_k,b_k]=[a\gamma^{k-1},a\gamma^k], k\geq 1$ in similar fashion

$$\max\left\{a^{-H}\gamma^{-kH};\frac{2^H}{a^H\gamma^{H(k-1)}(\gamma-1)^H}\right\}$$

$$=\frac{2^H}{a^H\gamma^{H(k-1)}(\gamma-1)^H}\leq\left(\frac{x_0Ca^{1-qH}(\gamma-1)}{q(\gamma^{qH}-1)}\right)^{\frac{1}{q-1}}$$

After simple transformations we have

$$\varepsilon \geq \frac{2^{\frac{H(q-1)}{1-H}}\gamma^{M-k+1}q(1-H)(\gamma^{qH-1}-1)(\gamma^{qH}-1)^{\frac{H}{1-H}}}{x_0^{\frac{1}{1-H}}C^{\frac{H}{1-H}}(qH-1)(\gamma-1)^{\frac{H(q-1)}{1-H}}} \tag{39}$$

From (38) and (39) follows (23).

Then from all the above

$$\begin{aligned}
&\mathbf{P}\left\{\sup_{t>0}\left(\frac{1}{C_\Delta}Z_H(t)-Ct\right)>\varepsilon\right\}\leq \\
&\leq \sum_{k\geq 0}\mathbf{P}\left\{\sup_{t\in[a_k,b_k]}\left(\frac{1}{C_\Delta}Z_H(t)-Ct\right)>\varepsilon\right\}\leq \\
&\leq \sum_{k\geq 0}2\left(\frac{e}{p_k}\right)^{\frac{1}{H}}W_k^{(M)}(\gamma,\varepsilon)K_{a_k,b_k}(p_k,\varepsilon)\leq \\
&\leq 2\zeta^{-1}e^{\frac{1}{H}}\varepsilon^{\frac{q(1-H)}{(q-1)H}}\left(K_0(\gamma)W_0^{(M)}(\gamma,\varepsilon)+\sum_{k=1}^{M-1}\gamma^{\frac{q(1-H)k}{(q-1)H}}W_k^{(M)}(\gamma,\varepsilon)K_k(\gamma)\right. \\
&\qquad \left.+\gamma^{\frac{q(1-H)M}{(q-1)H}}W(\gamma,\varepsilon)K_M(\gamma)+\sum_{k=M+1}^{\infty}\gamma^{\frac{q(1-H)k}{(q-1)H}}W_k^{(M)}(\gamma,\varepsilon)K_k(\gamma)\right)\leq \\
&\leq 2\zeta^{-1}e^{\frac{1}{H}}\varepsilon^{\frac{q(1-H)}{(q-1)H}}W(\gamma,\varepsilon)\left(K_0(\gamma)+K_1(\gamma)\sum_{k=1}^{M-1}\gamma^{\frac{q(1-H)k}{(q-1)H}}\frac{W_k^{(M)}(\gamma,\varepsilon)}{W(\gamma,\varepsilon)}+\right. \\
&\qquad \left.+\gamma^{\frac{q(1-H)M}{(q-1)H}}K_M(\gamma)+K_{M+1}(\gamma)\sum_{k=M+1}^{\infty}\gamma^{\frac{q(1-H)k}{(q-1)H}}\frac{W_k^{(M)}(\gamma,\varepsilon)}{W(\gamma,\varepsilon)}\right)\leq \\
&\leq \varepsilon^{\frac{q(1-H)}{(q-1)H}}W(\gamma,\varepsilon)2\zeta^{-1}e^{\frac{1}{H}}\times \\
&\qquad \times\left(K_0(\gamma)+K_1(\gamma)S_1(\gamma,\varepsilon)+\gamma^{\frac{q(1-H)k}{(q-1)H}}K_M(\gamma)+K_{M+1}(\gamma)S_2(\gamma,\varepsilon)\right). \qquad \square
\end{aligned}$$

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