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EXACT NON-RUIN PROBABILITIES IN INFINITE TIME

Using the Wiener-Hopf method, for the fundamental equation of the risk theory it is obtained an exact solution in terms of the Fourier transforms and factorization.

1. Introduction

Let $F(v)$ be the distribution of claims Z_k, $EZ_k = \mu$, $K(v)$ be the distribution of waiting time T_k, $ET_k = 1/\alpha$, $k \in \mathbf{Z}_+$, random variables Z_k and T_k be independent, and $c > \alpha\mu$ be the gross premium rate. Probability of solvency of an insurance company, $\varphi(u)$, with initial capital u, in *ordinary renewal process*, satisfies the Feller integral equation, [1,2]:

$$\varphi(u) = \int_0^{\infty} dK(v) \int_0^{u+cv} \varphi(u + cv - z)\, dF(z). \tag{1}$$

We are interested by a bounded, monotonically nondecreasing solution of (1), for which

$$\lim_{u \to +\infty} \varphi(u) = 1. \tag{2}$$

The example from the classical model, when T and Z have the exponential distributions, [1], p. 5-6, shows, that the condition (2) is unsufficient for determination of the interesting us solution. Therefore, in general case, the solution of equation (1) should satisfy one more additional condition, for example, a condition in zero or some equivalent one. Note that in the classical model such additional condition is

$$\lim_{u \to +0} \varphi(u) = 1 - \alpha\mu/c. \tag{3}$$

But, unfortunately, in general case, the condition in zero for $\varphi(u)$ is not known, however this condition is known for the solution $\varphi_0(u)$ in the case of accompanying *stationary renewal process*, calculated by the formula:

$$\varphi_0(u) = \alpha \int_0^{\infty} [1 - K(v)] \left\{ \int_0^{u+cv} \varphi(u + cv - z)\, dF(z) \right\} dv, \tag{4}$$

for which

$$\lim_{u \to +0} \varphi_0(u) = 1 - \alpha\mu/c \quad \Longleftrightarrow \quad \lim_{u \to +\infty} \varphi_0(u) = 1. \tag{5}$$

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Therefore hereinafter we shall search for the solution of equation (1) such that for function $\varphi(u)$ two initial conditions (2) and (5) are fulfilled.

At the derivation of the equations (1) and (4) nothing does not prevent us to consider u to be negative. Really, the right side of the equation (1) in this case means the probability for the company, being in the state of ruin, to be found in the state of non-ruin at the epoch of the first claim and later on, [1], p. 4, [2], p. 149. Note also that in the case $u + cv \le z \le 0$

$$\int_0^{u+cv} \varphi(u + cv - z)\, dF(z) = 0,$$

since $F(z) = 0$ for $z < 0$, and

$$\int_0^{u+cv} \varphi(u + cv - z)\, dF(z) = P\{0 < Z_1 \le u + cv\},$$

when $0 \le z \le u + cv$. Therefore hereinafter we shall search for the solution $\varphi(u)$ of the problem (1) for u on all numerical axis, $u \in \mathbf{R}$.

2. One-sided Wiener-Hopf equation

Reducing the equation (1) by the difference method to the system of linear algebraic equations, we observe that the received system is a discrete system of Wiener-Hopf type. This makes us think that the equation (1) is the homogeneous one-sided integral equation of Wiener-Hopf type

$$\varphi(u) = \int_0^{\infty} \varphi(v)\, dk(u - v), \quad -\infty < u < +\infty, \tag{6}$$

with some kernel $k(v)$ defined by $K(v)$ and $F(v)$. Using the Wiener-Hopf method presented in the monographs [3], pp. 55-56, 103, for the symbol $\mathcal{A}(x)$ of the equation (1) we receive the formula

$$\mathcal{A}(x) = 1 - \overline{\mathcal{F}_T(cx)} \cdot \mathcal{F}_Z(x), \quad -\infty < x < \infty,$$

where $\mathcal{F}.(\cdot)$ denotes a characteristic function of a corresponding distribution (Fourier transform of density), and the line denotes the complex conjugation. Returning to the equation (6), we obtain that $k(u)$ can be found with the help of the inverse Fourier transform of $\overline{\mathcal{F}_T(cx)} \cdot \mathcal{F}_Z(x)$ and integration.

The symbol $\mathcal{A}(x)$ is differentiable on the numerical line and has always the zero of the first order at $x = 0$. The first follows from the existence of EZ_k and ET_k, and the latter follows from the condition $c > \alpha\mu$. There is yet the possibility of existence of infinite number of other zeros x_k of $\mathcal{A}(x)$ on the numerical line, which represent not the interest for us since they generate nonmonotonic solutions. Therefore the equation (1) is of nonnormal (nonelliptic) type, for the solution of which we can also apply the Wiener-Hopf factorization method [3].

There exists a factorization

$$\mathcal{A}(x) = x \cdot \mathcal{A}^+(x) \cdot \mathcal{A}^-(x) \cdot \prod_k (x - x_k), \quad -\infty < x_k < +\infty,$$

where the function $\mathcal{A}^+(x)$ ($\mathcal{A}^-(x)$) is assumed to be analytically extendable in the domains $\mathcal{D}_+ = \{z : \operatorname{Im} z > 0\}$, ($\mathcal{D}_- = \{z : \operatorname{Im} z < 0\}$) continuous in $\overline{\mathcal{D}_+}$ ($\mathcal{D}_-$), and $\mathcal{A}^+(z) \neq 0$ in $\overline{\mathcal{D}}_+$ ($1/\mathcal{A}^-(z)$ can have singularity at $z = \infty$). For the simplicity of the presentation, we some digress from the definition of factorization given in the monograph [3] since for us it is sufficient to know only the function $\mathcal{A}^+(x)$. We also leave aside the question about the index of the operator corresponding to the equation (1).

Following the Wiener-Hopf method, the solution of the problem (1), equal to zero on the interval $(-\infty, 0)$, is represented in the form of a linear combination:

$$\varphi_+(u) = C_1\, \mathcal{F}^{-1}\left[\frac{1}{x^+} \cdot \frac{1}{\mathcal{A}^+(x)}\right] + C_2\, \mathcal{F}^{-1}\left[\frac{1}{\mathcal{A}^+(x)}\right] = C_1\, \varphi_1(u) + C_2\, \varphi_2(u), \quad (7)$$

where

$$\frac{1}{x^+} = \frac{i}{x} + \pi\delta(x)$$

is the boundary value of the function $1/z^+$ analytic in the upper half-plane, obtained from the factorization of the function $\frac{1}{x}$ [3], $\delta(x)$ is the Dirac function, and $\mathcal{F}^{-1}[\cdot]$ denotes the inverse Fourier transform in the sense of L. Schwartz. Index '+' at φ means that the function $\varphi_+(u)$ is equal to 0 for $u \in (-\infty, 0)$. Here $\varphi_1(u)$ is a monotonically increasing and bounded function, and $\varphi_2(u)$ is a monotonically decreasing function tending to zero when $u \to +\infty$. The constants C_1 and C_2 are uniquely determined by the conditions (2) and (5).

Calculating

$$\varphi(u) = \mathcal{F}^{-1}\left[\overline{\mathcal{F}_T(cx)} \cdot \mathcal{F}_Z(x) \cdot \mathcal{F}_{\varphi_+}(x)\right],$$

we obtain the solution of the problem (1) on the axis $(-\infty, +\infty)$, non equal 0 on the negative semiaxis, which represents the probability of exit from the state of ruin when $u < 0$, with $\varphi(u) = \varphi_+(u)$ for $u > 0$. Note that

$$\mathcal{F}_\varphi(x) = \overline{\mathcal{F}_T(cx)} \cdot \mathcal{F}_Z(x) \cdot \mathcal{F}_\varphi(x),$$

so that the corresponding $\varphi(u)$ satisfies the problem (1) also, and $\varphi(u) = \varphi_+(u)$ for $u \in (0, +\infty)$. Note that in the literature, in general, the meaning is not attached to the solution of the Wiener-Hopf equation arising in the case when $u < 0$, [3], p. 56.

The solution $\varphi_0(u)$ in (2) can be also calculated in the terms of the Fourier transform

$$\varphi_0(u) = \mathcal{F}^{-1}\left[\overline{\mathcal{F}_{\alpha(1-K)}(cx)} \cdot \mathcal{F}_Z(x) \cdot \mathcal{F}_{\varphi_+}(x)\right], \quad u \in (-\infty, +\infty)$$

or

$$\varphi_0(u) = \mathcal{F}^{-1}\left[\overline{\mathcal{F}_{\alpha(1-K)}(cx)} \cdot \mathcal{F}_Z(x) \cdot \mathcal{F}_\varphi(x)\right], \quad u \in (-\infty, +\infty).$$

Since the characteristic function for Gamma distribution with integer exponent (Erlang distribution) is a rational function, the solution for the problem (1) in the case when T_k and Z_k have such distributions can be found in explicit form.

3. Examples

Example 1. Classical model, exponentially distributed claim. Consider the case when the inter-occurrence times and claims have the exponential distributions with $\alpha = 2$, $\mu = 1/2$, respectively, and $c = 3$. Then

$$\mathcal{F}_T(x) = \frac{2}{2+ix}, \quad \mathcal{F}_Z(x) = \frac{2}{2-ix}, \quad \mathcal{A}(x) = \frac{x(3x+4i)}{(3x-2i)(x+2i)},$$

$$\mathcal{A}^+(x) = \frac{3x+4i}{x+2i}, \quad \mathcal{A}^-(x) = 3x-2i,$$

$$\varphi_1(u) = \mathcal{F}^{-1}\left[\frac{1}{x^+}\cdot\frac{1}{\mathcal{A}^+(x)}\right] = \mathcal{F}^{-1}\left[\left(\frac{1}{x}+\pi\delta(x)\right)\cdot\frac{x+2i}{3x+4i}\right] =$$

$$= \frac{1}{2}\cdot H(u)\cdot\left(1-\frac{1}{3}e^{-\frac{4}{3}u}\right),$$

$$\varphi_2(u) = \mathcal{F}^{-1}\left[\frac{1}{\mathcal{A}^+(x)}\right] = \mathcal{F}^{-1}\left[\frac{x+2i}{3x+4i}\right] = \frac{1}{3}\cdot\delta(u) + \frac{2}{9}H(u)e^{-\frac{4}{3}u},$$

where $H(u)$ is the Heaviside function (nondetermined at $u = 0$). From the condition (2) we obtain $C_1 = 2$. The constant $C_2 = 0$ since

$$\varphi_+(u) = \varphi_{0+}(u) = 2\cdot\varphi_1(u) = H(u)\cdot\left(1-\frac{1}{3}e^{-\frac{4}{3}u}\right)$$

satisfies the condition (3). The obtained solution coincides with solution given by the well known formula for exponentially distributed claims in classical model, [1,2], obtained by reduction (1) to a differential equation.

The solution of the problem on $(-\infty,+\infty)$ is given by the formula

$$\varphi(u) = \varphi_0(u) = \mathcal{F}^{-1}\left[\overline{\mathcal{F}_T(cx)}\cdot\mathcal{F}_Z(x)\cdot\mathcal{F}_{\varphi_+}(x)\right]$$

$$= 4\cdot\mathcal{F}^{-1}\left[\frac{4\pi\delta(x)x - 3i\pi\delta(x)x^2 + 4i + 2x}{(2+3ix)(-2+ix)x(-4+3ix)}\right]$$

$$= \frac{2}{3}H(-u)e^{\frac{2}{3}u} + H(u)(1-\frac{1}{3}e^{-\frac{4}{3}u}).$$

The value of $\varphi(u) = \varphi_0(u)$ at $u = 0$ is determined by the continuity

$$\varphi(0) = \lim_{u\to 0}\varphi(u) = \frac{2}{3}.$$

Example 2. Classical model, Γ-distributed claims. Consider the case when the inter-occurrence times have the exponential distributions with $\alpha = 2$, $c = 3$, and the claims density is given by

$$f_Z(t) = te^{-2t} \quad \text{for} \quad t > 0$$

so that $\mu = 1$. Then

$$\mathcal{F}_T = \frac{2}{2-ix}, \quad \mathcal{F}_Z = \frac{4}{(-2+ix)^2}, \quad \mathcal{A}(x) = \frac{x(3x^2+10ix-4)}{(3x-2i)(x+2i)^2},$$

$$\mathcal{A}^+(x) = \frac{3x^2+10ix-4}{(x+2i)^2}, \quad \mathcal{A}^-(x) = \frac{1}{3x-2i}.$$

Omitting obvious intermediate calculations, we obtain the solution on $(0,+\infty)$

$$\varphi_+(u) = \varphi_{0+}(u) =$$
$$H(u)\cdot\left[1-\frac{1}{39}\left((13-4\sqrt{13})e^{-\frac{5+\sqrt{13}}{3}u}+(13+4\sqrt{13})e^{-\frac{5-\sqrt{13}}{3}u}\right)\right]$$

which coincides with solution obtained by using the Laplace transform, [1], p. 13,

$$\varphi_+(u) = H(u)\left[1-\left(\frac{8}{39}\sqrt{13}\sinh\left(\frac{\sqrt{13}}{3}u\right)-\frac{2}{3}\cosh\left(\frac{\sqrt{13}}{3}u\right)\right)e^{-\frac{5}{3}u}\right].$$

The solution on $(-\infty,\infty)$ is

$$\varphi(u) = \varphi_0(u) = H(u)\times$$
$$\left[1-\frac{1}{39}\left((13-4\sqrt{13})e^{-\frac{5+\sqrt{13}}{3}u}+(13+4\sqrt{13})e^{-\frac{5-\sqrt{13}}{3}u}\right)\right]+\frac{1}{3}H(-u)e^{\frac{2}{3}u},$$
$$\varphi(0) = \varphi_0(0) = \frac{1}{3}.$$

Example 3. *Γ-distributed inter-occurrence times, Γ-distributed claims.* Consider the case when the inter-occurrence times and claims have the Γ-distributions

$$f_T(t) = te^{-t}, \quad f_Z(t) = 4te^{-2t} \quad \text{for} \quad t>0,$$

with $\alpha = 1/2$, $\mu = 1$, respectively, and $c = 1$. Then

$$\mathcal{F}_T = \frac{1}{(1+ix)^2}, \quad \mathcal{F}_Z = \frac{4}{(-2+ix)^2}, \quad \mathcal{A}(x) = \frac{x(x^2+ix+4)(x+i)}{(x+2i)^2(x-i)^2},$$

$$\mathcal{A}^+(x) = \frac{(2x+(1+\sqrt{17})i)(x+i)}{(x+2i)^2}, \quad \mathcal{A}^-(x) = \frac{2x+(1-\sqrt{17})i}{4(x-i)^2}.$$

After obvious calculations, we obtain the solutions of (1) on $(0,+\infty)$

$$\varphi_+(u) = H(u)\cdot\left[1-\frac{1}{32}\left((19-5\sqrt{17})e^{-\frac{1+\sqrt{17}}{2}u}+(9+\sqrt{17})e^{-u}\right)\right],$$

and on $(-\infty,\infty)$

$$\varphi(u) = H(u)\cdot\left[1-\frac{1}{32}\left((19-5\sqrt{17})e^{-\frac{1+\sqrt{17}}{2}u}+(9+\sqrt{17})e^{-u}\right)\right]+$$
$$\frac{1}{32}H(-u)\left((-28+4\sqrt{17})u+4(1+\sqrt{17})\right)e^{u},$$

$$\varphi(0) = \lim_{u\to 0} \varphi(u) = \frac{1+\sqrt{17}}{8}.$$

The solution of (4) on $(-\infty, \infty)$ is as follows

$$\varphi_0(u) = H(u)\left[1 - \frac{1}{64}\left(32 - (27+3\sqrt{17})e^{-u} - (5-3\sqrt{17})e^{-\frac{-1+\sqrt{17}}{2}u}\right)\right]$$
$$+\frac{1}{64}H(-u)\left(-(28-4\sqrt{17})u + 32\right)e^{u}, \quad \varphi_0(0) = \lim_{u\to 0}\varphi_0(u) = \frac{1}{2}.$$

Note that in all these three examples $C_2 = 0$ so that there is not necessity to calculate the function $\varphi_2(u)$.

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