Theory of Stochastic Processes
Vol. 12 (28), no. 1–2, 2006, pp. 38–54

UDC 519.21

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REGULAR VARIATION IN THE BRANCHING RANDOM WALK

Let $\{\mathcal{M}_n, n = 0, 1, \dots\}$ be the supercritical branching random walk starting with one initial ancestor located at the origin of the real line. For $n = 0, 1, \dots$, let W_n be the moment generating function of $\mathcal{M}_n$ normalized by its mean. Denote by AW_n any of the following random variables: maximal function, square function, L_1 and a.s. limit W, $\sup_{n\geq 0} |W - W_n|$, $\sup_{n\geq 0} |W_{n+1} - W_n|$. Under mild moment restrictions and the assumption that $\mathbb{P}\{W_1 > x\}$ regularly varies at ∞, it is proved that $\mathbb{P}\{AW_n > x\}$ regularly varies at ∞ with the same exponent. All the proofs given are non-analytic in the sense that these do not use Laplace–Stieltjes transforms. The result on the tail behaviour of W is established in two distinct ways.

An introduction, notation, and results

Let $\mathcal{M}$ be a point process on $\mathbb{R}$, i.e. a random, locally finite counting measure. Explicitly,

$$\mathcal{M}(A)(\omega) := \sum_{i=1}^{J(\omega)} \delta_{X_i(\omega)}(A),$$

where $J := \mathcal{M}(\mathbb{R})$, $\{X_i : i = \overline{1, J}\}$ are the points of $\mathcal{M}$, A is any Borel subset of $\mathbb{R}$ and δ_x is the Dirac measure concentrated at x. We assume that $\mathcal{M}$ has no atom at $+\infty$, and the J may be deterministic or random, finite or infinite with positive probability.

Let $\{\mathcal{M}_n, n = 0, 1, \dots\}$ be a $\underline{b}$ranching $\underline{r}$andom $\underline{w}$alk (BRW), i.e. the sequence of point processes which, for any Borel set $B \subseteq \mathbb{R}$, are defined as follows: $\mathcal{M}_0(B) = \delta_0(B)$,

$$\mathcal{M}_{n+1}(B) := \sum_r \mathcal{M}_{n,r}(B - A_{n,r}), n = 0, 1, \dots,$$

where $\{A_{n,r}\}$ are the points of $\mathcal{M}_n$, and $\{\mathcal{M}_{n,r}\}$ are independent copies of $\mathcal{M}$. The more detailed definition of the BRW can be found in, for example, [3,17,22].

In the case where $\mathbb{P}\{J < \infty\} = 1$ we assume that $\mathbb{E}J > 1$. In the contrary case, the condition holds automatically. Thus, we only consider the supercritical BRW. As a consequence, $\mathbb{P}\{\mathcal{M}_n(\mathbb{R}) > 0 \text{ for all } n\} > 0$.

In what follows, we use the notation that is generally accepted in the literature on the BRW: A_u denotes the position of a generic point $u = i_1 \dots i_n$ on $\mathbb{R}$; the record $|u| = n$ means that u is a point of $\mathcal{M}_n$; the symbol $\sum_{|u|=n}$ denotes the summation over all points of $\mathcal{M}_n$; $\mathcal{F}_n = \sigma(\mathcal{M}_1, \dots, \mathcal{M}_n)$ denotes the σ-field generated by $\{\mathcal{M}_k, k = 1, \dots, n\}$; $\mathcal{F}_0$ is the trivial σ-field.

Define the function

$$m(y) := \mathbb{E}\int_{\mathbb{R}} e^{yx}\mathcal{M}(dx) = \mathbb{E}\sum_{|u|=1} e^{yA_u} \in (0, \infty], y \in \mathbb{R},$$

2000 *AMS Mathematics Subject Classification.* Primary 60G42, 60J80; Secondary 60E99.
Key words and phrases. Branching random walk, supercritical case, perpetuity.

and assume that there exists a $\gamma > 0$ such that $m(\gamma) < \infty$. Set $Y_u := e^{\gamma A_u}/m^{|u|}(\gamma)$ and

$$W_n := m(\gamma)^{-n} \int_{\mathbb{R}} e^{\gamma x} \mathcal{M}_n(dx) = \sum_{|u|=n} Y_u.$$

As is well known (see, for example, [12]), the sequence $\{(W_n, \mathcal{F}_n), n = 0, 1, \dots\}$ is a non-negative martingale. Notice that $W_0 = \mathbb{E}W_n = 1$.

Let $\{d_n, n = 1, 2, \dots\}$ be the martingale difference sequence, i.e.

$$W_n = 1 + \sum_{k=1}^{n} d_k, n = 1, 2, \dots$$

The square function S and maximal function W^* are defined by

$$S := \left(1 + \sum_{k=1}^{\infty} d_k^2\right)^{1/2} \quad \text{and} \quad W^* := \sup_{n\geq 0} W_n.$$

Set also

$$S_n := \left(1 + \sum_{k=1}^{n} d_k^2\right)^{1/2}, n = 1, 2, \dots \quad \text{and} \quad \Delta := \sup_{n\geq 1} |d_n|.$$

Recall that since W_n is a non-negative martingale all the defined variables are a.s. finite (for the finiteness of S for general L_1-bounded martingales, we refer to [1] or to Theorem 2 on p.390[11]).

When the martingale W_n is uniformly integrable, we denote, by $W_\infty = W$, its L_1 and a.s. limit and then define

$$M := \sup_{n\geq 0} |W - W_n| = \sup_{n\geq 0} \Big|\sum_{k=n+1}^{\infty} d_k\Big|.$$

Lemma 1 [21] (see also [2] for a slightly different proof in the case $J < \infty$ a.s.) states that there exist $r \in (0, 1)$ and $\theta = \theta(r) > 1$ such that, whenever $t > 1$,

$$\mathbb{P}\{W > t\} \leq \mathbb{P}\{W^* > t\} \leq \theta \mathbb{P}\{W > rt\}. \tag{1}$$

This suggests that the tail behaviours of W and W^* are quite similar.

Let now $\{f_n := \sum_{k=0}^{n} g_k, n = 0, 1, \dots\}$ be any martingale. It is well known that the distributions of maximal $f^* := \sup_{n\geq 0} |f_n|$ and square $S(f) := (\sum_{k=0}^{\infty} g_k^2)^{1/2}$ functions are close in many respects. The evidence in favor of such a statement is provided by, for example, the (moment) Burkholder—Gundy—Davis inequality (Theorem 1.1 [10]) or the distribution function inequalities like (34) of this paper. From [9] and [10] and many other subsequent works, it follows that there exist a subset $\mathcal{H}$ of the set of all martingales and a class $\mathcal{A}$ of operators on martingales such that the distributions of $A_1 h$ and $A_2 h$ are close in an appropriate sense whenever $A_i \in \mathcal{A}$ and $h \in \mathcal{H}$. Often, it can be possible to express this closeness via moment or distribution function inequalities like those mentioned above. Keeping this in mind, it would not be an unrealistic conjecture that the regular variation of $\mathbb{P}\{A_1 h > x\}$ is equivalent to that of $\mathbb{P}\{A_2 h > x\}$, where A_i and h belong to some subsets of operators and martingales, respectively, that may be different from $\mathcal{A}$ and $\mathcal{H}$. On the other hand, let us notice that, as far as we know, the conjecture does not follow from previously known results on martingales.

The aim of this paper is to prove a variant of the conjecture for the martingales W_n and operators $A_i, i = \overline{1, 5}$ given as follows: $A_1 W = W^*$, $A_2 W = \Delta$, $A_3 W = S$, $A_4 W = W_\infty$, $A_5 W = M$.

In addition to the notation introduced above, other frequently used notations and conventions include: $L(t)$ denotes a function that slowly varies at infinity; 1_A denotes the indicator function of the set A; $f(t) \sim g(t)$ is the abbreviation of the limit relation $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$; $x^+ := \max(x, 0)$; $x \wedge y = \min(x, y)$; $x \vee y = \max(x, y)$; we write $\mathbb{P}_n\{\cdot\}$ instead of $\mathbb{P}\{\cdot|\mathcal{F}_n\}$ and $\mathbb{E}_n\{\cdot\}$ instead of $\mathbb{E}\{\cdot|\mathcal{F}_n\}$; the record "const" denotes a constant, whose value is of no importance and may be different on different appearances.

Now we are ready to state our result.

Proposition 1.1. *Assume that there exist $\beta > 1$ and $\epsilon > 0$ such that*

$$k_\beta := \mathbb{E} \sum_{|u|=1} Y_u^\beta < 1, \quad \mathbb{E} \sum_{|u|=1} Y_u^{\beta+\epsilon} < \infty \quad \text{and} \tag{2}$$

$$\mathbb{P}\{W_1 > x\} \sim x^{-\beta} L(x). \tag{3}$$

Then
(I) $\mathbb{P}\{W^* > x\} \sim \mathbb{P}\{\Delta > x\} \sim \mathbb{P}\{S > x\} \sim (1 - k_\beta)^{-1}\mathbb{P}\{W_1 > x\}$;
(II) W_n converges almost surely and in mean to a random variable W and

$$\mathbb{P}\{W > x\} \sim (1 - k_\beta)^{-1}\mathbb{P}\{W_1 > x\}; \tag{4}$$

$$\mathbb{P}\{M > x\} \sim (1 - k_\beta)^{-1}\mathbb{P}\{W_1 > x\}.$$

Remark 1.1 We are not aware of any papers on branching processes which investigate the tail behaviour of random variables like Δ, M, or S. [21] is the only paper we know of that deals with the tail behaviour of random variables like W^*.

Remark 1.2 When $\gamma = 0$ and $J < \infty$ a.s., W_n reduces to the (supercritical) normalized Galton—Watson process. In this case, (4) was proved in [5] for non-integer β and in [13] for integer β. When $\gamma > 0$, $J < \infty$ a.s. and $\mathcal{M}(-\infty, -\gamma^{-1} \log m(\gamma)) = 0$ a.s., W can be viewed as a limit random variable in the Crump-Mode branching process. In this case (4) was established in [6] for non-integer β. The technique used in the last three cited works is purely analytic (based on using the Laplace—Stieltjes transforms and the Abel—Tauberian theorems) and completely different from ours. On the other hand, let us notice that the above-mentioned analytic approach was successfully employed and further developed by the second-named author. In 2003, in an unpublished diploma paper, he proved (4) for non-integer β for the general case treated here.

Our desire to find a non-analytic proof of (4) was a starting point for the development of this paper. In the course of writing, two different (non-analytic) proofs were found. One of these proofs given in Section 2 falls within the general scope of the paper. The second given in Section 3 continues a line of research initiated in [17] , [22], [18]. Here, an underlying idea is that the martingale W_n and the so-called *perpetuities* have many features in common. In particular, several non-trivial results on perpetuities (however, it seems, only those related to perpetuities with not all moments finite) can be effectively exploited to obtain similar results on the limiting behaviour of W_n. Maybe, we should recall that, in modern probability, a perpetuity means a random variable

$$B_1 + \sum_{k=2}^{\infty} A_1 A_2 \cdots A_{k-1} B_k,$$

provided the latter series absolutely converges, and where $\{(A_k, B_k) : k = 1, 2, \dots\}$ are independent identically distributed random vectors.

The paper is structured as follows. In Section 2, we prove Proposition 1.1. Here, an essential observation is that, given $\mathcal{F}_n$, W_{n+1} looks like a weighted sum of independent identically distributed random variables. This allows us to exploit the well-known result

[25] on the tail behaviour of such sums under the regular variation assumption. The second key ingredient of the proof is the use of the distribution function inequalities for martingales. In Section 3, we give another proof of (4) which rests on a relation between the BRW and perpetuities. Here, the availability of Grincevičius—Grey [16] result on the tail behaviour of perpetuities is crucial. Finally, in Section 4, we discuss the applicability of Proposition 1.1 to several classes of point processes. The section closes with two remarks which show that (2) and (3) are not the necessary conditions for a regular variation of the tails of W^*, W and a related random variable.

Proof of Proposition 1.1

(I) We will prove the result for W^* and Δ simultaneously. To this end, let Q and $\tilde{Q}$ be independent identically distributed random variables, whose distribution is supported by $(a,\infty), a>-\infty$. Assume that $\mathbb{P}\{Q>x\}\sim x^{-\beta}L(x)$ for $\beta>1$. In particular, this assumption ensures that $\mathbb{E}|Q|<\infty$ and $\mathbb{P}\{|Q|>x\}\sim\mathbb{P}\{Q>x\}$. With a slight abuse of the notation, set $Q^s:=|Q|-|\tilde{Q}|$. Then

$$1-F(x):=\mathbb{P}\{|Q^s|>x\}\sim 2x^{-\beta}L(x). \tag{5}$$

Indeed, $1-F(x)=2\int_0^\infty(1-G(x+y))dG(y)$, where $G(x)=\mathbb{P}\{|Q|\leq x\}, x\geq 0$. Now (5) follows from the monotonicity of $1-G$, the relation $1-G(x+y)\sim 1-G(x), y\in\mathbb{R}$ and the Fatou lemma.

The equality

$$\mathbb{E}t(Z)=\mathbb{E}\sum_{|u|=1}Y_u t(Y_u),$$

which is assumed to hold for all bounded Borel functions t, defines the distribution of a random variable Z. More generally,

$$\mathbb{E}t(Z_1\cdots Z_n)=\mathbb{E}\sum_{|u|=n}Y_u t(Y_u), \tag{6}$$

where $Z_1,Z_2,\ldots$ are independent copies of the Z. Notice that we can permit for (6) to hold for any Borel function t. In that case, we assume that if the right-hand side is infinite or does not exist, the same is true for the left-hand side.

Under the assumptions of the theorem, the function $k_x:=\mathbb{E}\sum_{|u|=1}Y_u^x$ is log-convex for $x\in(1,\beta)$, $k_1=1$ and $k_\beta<1$. Therefore,

$$k_{\beta-\epsilon}<1 \quad\text{for all } \epsilon\in(0,\beta-1). \tag{7}$$

Also we can pick a $\delta\in(0,\beta-1)$ such that $k_{\beta+\delta}<1$. By using these facts and equality (6), we conclude that, with this δ,

$$\mathbb{E}\sum_{|u|=n}Y_u^{\beta-\delta}=k_{\beta-\delta}^n<1 \quad\text{and}\quad \mathbb{E}\sum_{|u|=n}Y_u^{\beta+\delta}=k_{\beta+\delta}^n<1. \tag{8}$$

Let us notice, for later needs, that we can choose δ as small as needed. Among other things, (8) implies that, for $x\in[1,\beta+\delta]$,

$$\sum_{|u|=n}Y_u^x<\infty \quad\text{a.s.} \tag{9}$$

Until a further notice, we fix an arbitrary $n\in\mathbb{N}$. Put

$$T_n:=|\sum_{|u|=n}Y_uQ_u| \quad\text{and}\quad X_n:=\sum_{|u|=n}Y_u|Q_u|.$$

Given $\mathcal{F}_n$, let $\{Q_u : |u| = n\}$ and $\{Q^s_u : |u| = n\}$ be conditionally independent copies of the random variables Q and Q^s, respectively. In view of (9), an appeal to Lemma A3.7[25] allows us to conclude that

$$\mathbb{P}_n\{T_n > x\} \sim \sum_{|u|=n} Y_u^\beta \mathbb{P}\{|Q| > x\} \quad \text{a.s.} \tag{10}$$

The cited lemma assumes that each term of the series on the left-hand side has zero mean, but this condition is not needed in the proof of the result used here.

Denote, by $\mu_n^{\mathcal{F}_n}$, the conditional median of X_n w.r.t. $\mathcal{F}_n$, i.e. $\mu_n^{\mathcal{F}_n}$ is a random variable that satisfies

$$\mathbb{P}_n\{X_n - \mu_n^{\mathcal{F}_n} \geq 0\} \geq 1/2 \leq \mathbb{P}_n\{X_n - \mu_n^{\mathcal{F}_n} \leq 0\} \quad \text{a.s.}$$

Let also μ_n denote the usual median of X_n. Since $\mu_n^{\mathcal{F}_n} \geq 0$ a.s., relation (10) yields

$$\limsup_{x\to\infty} \frac{\mathbb{P}_n\{T_n > x + \mu_n^{\mathcal{F}_n}\}}{\mathbb{P}\{|Q| > x\}} \leq \sum_{|u|=n} Y_u^\beta \quad \text{a.s.}$$

If we could prove that, for large x,

$$\frac{\mathbb{P}_n\{T_n > x + \mu_n^{\mathcal{F}_n}\}}{\mathbb{P}\{|Q| > x\}} \leq U_n \quad \text{a.s.} \quad \text{and} \quad \mathbb{E}U_n < \infty, \tag{11}$$

where U_n is a random variable, then using the Fatou lemma yields

$$\limsup_{x\to\infty} \mathbb{E}\frac{\mathbb{P}_n\{T_n > x + \mu_n^{\mathcal{F}_n}\}}{\mathbb{P}\{|Q| > x\}} = \limsup_{x\to\infty} \frac{\mathbb{P}\{T_n > x + \mu_n\}}{\mathbb{P}\{|Q| > x\}} \leq \mathbb{E}\sum_{|u|=n} Y_u^\beta \overset{(6)}{=} k_\beta^n. \tag{12}$$

Since $\mathbb{P}\{|Q| > x + \mu_n\} \sim \mathbb{P}\{|Q| > x\}$, (12) implied that

$$\limsup_{x\to\infty} \frac{\mathbb{P}\{T_n > x\}}{\mathbb{P}\{|Q| > x\}} \leq k_\beta^n.$$

On the other hand, by using (10) and the Fatou lemma, the reverse inequality for the lower limit follows easily. Therefore, as soon as (11) is established, we get

$$\mathbb{P}\{T_n > x\} \sim k_\beta^n \mathbb{P}\{|Q| > x\}. \tag{13}$$

We now intend to show that (11) holds with

$$U_n = const\left(\sum_{|u|=n} Y_u^{\beta-\delta} + \sum_{|u|=n} Y_u^{\beta+\delta}\right) \tag{14}$$

for an appropriate small δ that satisfies (8). Notice that

$$\mathbb{E}U_n = const(k^n_{\beta-\delta} + k^n_{\beta+\delta}) < \infty. \tag{15}$$

By the triangle inequality and the conditional symmetrization inequality,

$$(1/2)\mathbb{P}_n\{T_n > x + \mu_n^{\mathcal{F}_n}\} \leq (1/2)\mathbb{P}_n\{X_n > x + \mu_n^{\mathcal{F}_n}\} \leq \mathbb{P}_n\{|\sum_{|u|=n} Y_u Q_u^s| > x\}. \tag{16}$$

Let us show that, for $x > 0$,

$$\mathbb{P}_n\{|\sum_{|u|=n} Y_u Q_u^s| > x\} \leq$$

(17)

$$\leq \mathbb{P}_n\{\sup_{|u|=n} Y_u|Q_u^s| > x\} + x^{-2}\mathbb{E}_n\left(\sum_{|u|=n} Y_u^2(Q_u^s)^2 1_{\{Y_u|Q_u^s|\leq x\}}\right) := I_1(n,x) + I_2(n,x).$$

Let $\{Y(k)Q^s(k) : k = 1, 2, \dots\}$ be any enumeration of the set $\{Y_u Q_u^s : |u| = n\}$. The inequality $\mathbb{E}|Q| < \infty$ implies that the series $\sum_{|u|=n} Y_u Q_u$ is absolutely convergent. Therefore, $\sum_{|u|=n} Y_u Q_u = \sum_{k=1}^{\infty} Y(k)Q^s(k)$. Define

$$\tau_x := \begin{cases} \inf\{k \geq 1 : Y(k)|Q^s(k)| > x\}, & \text{if } \sup_{k\geq 1} Y(k)|Q^s(k)| > x; \\ +\infty, \text{ otherwise }. \end{cases}$$

For any fixed $m \in \mathbb{N}$ and $x > 0$,

$$\mathbb{P}_n\{|\sum_{k=1}^{m} Y(k)Q^s(k)| > x\} \leq$$

$$\leq \mathbb{P}_n\{\tau_x \leq m-1\} + \mathbb{P}_n\{|\sum_{k=1}^{m} Y(k)Q^s(k)| > x, \tau_x \geq m\} \leq$$

$$\leq \mathbb{P}_n\{\sup_{1\leq k\leq m-1} Y(k)|Q^s(k)| > x\} + \mathbb{P}_n\{|\sum_{k=1}^{\tau_x\wedge m} Y(k)Q^s(k)| > x\} \leq$$

(by the Markov inequality)

$$\leq \mathbb{P}_n\{\sup_{1\leq k\leq m-1} Y(k)|Q^s(k)| > x\} + x^{-2}\mathbb{E}_n\left(\sum_{k=1}^{m} Y(k)Q^s(k)1_{\{\tau_x\geq k\}}\right)^2 \leq$$

($\mathbb{E}_n Q^s(k) = 0$ and, given $\mathcal{F}_n$, $Q^s(k)$ and $1_{\{\tau_x\geq k\}}$ are independent)

$$\leq \mathbb{P}_n\{\sup_{1\leq k\leq m-1} Y(k)|Q^s(k)| > x\} + x^{-2}\mathbb{E}_n\sum_{k=1}^{m} Y^2(k)(Q^s(k))^2 1_{\{Y(k)|Q^s(k)|\leq x\}}.$$

If the distribution of Q^s is continuous, sending $m \to \infty$ then completes the proof of (17).

Assume now that the distribution of Q^s has atoms. Let R be a random variable with a uniform distribution on $[-1, 1]$ which is independent of Q^s. Given $\mathcal{F}_n$, let $\{R_u : |u| = n\}$ be conditionally independent copies of R which are also independent of $\{Q_u^s : |u| = n\}$. Since, for all $t > 0$,

$$\mathbb{P}\{|Q^s| > t\} \leq 2\mathbb{P}\{|Q^s||R| > t/2\},$$

we have by Theorem 3.2.1[23]

$$\mathbb{P}_n\{|\sum_{|u|=n} Y_u Q_u^s| > t\} \leq 4\mathbb{P}_n\{|\sum_{|u|=n} Y_u Q_u^s R_u| > t/4\}, \tag{18}$$

and the distribution of $Q^s R$ is (absolutely) continuous. Now we can apply the already established part of (17) to the right-hand side of (18). Strictly speaking, when the distribution of Q^s has atoms, (17) should be written in a modified form: additional constants should be added, and Q_u^s should be replaced with $Q_u^s R_u$. On the other hand, a perusal of the subsequent proof reveals that only the regular variation of $\mathbb{P}\{|Q^s| > x\}$ plays a crucial role. Therefore, to simplify the notation, we prefer to keep (17) in its present form. This does not cause any mistakes as $\mathbb{P}\{|Q^s R| > x\} \sim \mathbb{E}|R|^{\beta}\mathbb{P}\{|Q^s| > x\}$.

Assume temporarily that $1-F(x)$ regularly varies with index $-\beta$, $\beta\in(1,2)$. Set $T(x):=\int_0^x y^2dF(y)$. By Theorem 1.6.4[7],

$$T(x)\sim\frac{\beta}{2-\beta}x^2(1-F(x))\sim\frac{\beta}{2-\beta}x^{2-\beta}L_1(x).$$

Also by Theorem 1.5.3[7], there exists a non-decreasing $S(x)$ such that

$$T(x)\sim S(x). \tag{19}$$

For any $A_i>0$ and δ defined in (8), there exists an $x_i>0$ such that, whenever $x\geq x_i, i=1,2,3,$

$$x^{\beta+\delta}(1-F(x))\geq 1/A_1; \tag{20}$$

$$x^{\beta-2+\delta}S(x)\geq 1/A_2; \tag{21}$$

$$T(x)\leq(A_3+\frac{\beta}{2-\beta})x^2(1-F(x)):=Bx^2(1-F(x)). \tag{22}$$

Also for any $A_i>1$ and the same δ as above, there exists an $x_i>0$ such that, whenever $x\geq x_i$ and $ux\geq x_i, i=4,5,6,$

$$\frac{1-F(ux)}{1-F(x)}\leq A_4(u^{-\beta+\delta}\vee u^{-\beta-\delta}); \tag{23}$$

$$\frac{T(ux)}{T(x)}\leq A_5(u^{2-\beta+\delta}\vee u^{2-\beta-\delta}); \tag{24}$$

$$\frac{T(ux)}{T(x)}\leq A_6\frac{S(ux)}{S(x)}. \tag{25}$$

Inequalities (23) and (24) follow from Potter's bound Theorem 1.5.6 (iii)[7]; (25) is implied by (19). Set $x_0:=\max\limits_{1\leq i\leq 6}x_i$ and assume that $x_0>1$.

To check (11) and (14), we consider three cases: (a) $\beta\in(1,2)$; (b) $\beta>2,\beta\neq 2^n,n\in\mathbb{N}$; (c) $\beta=2^n,n\in\mathbb{N}$.

(a) For any fixed $x\geq x_0$ and $y>0$,

$$\frac{I_1(n,x/y)}{1-F(x)}\leq\sum_{|u|=n}\frac{\mathbb{P}_n\{Y_u|Q_u^s|>x/y\}}{1-F(x)}=\sum_{|u|=n}\frac{1-F(x(yY_u)^{-1})}{1-F(x)}=$$

$$=\sum_{|u|=n}\cdots 1_{\{yY_u>x/x_0\}}+\sum_{|u|=n}\cdots 1_{\{yY_u\leq x/x_0\}}=:I_{11}(n,x,y)+I_{12}(n,x,y).$$

Since

$$(yY_u)^{\beta+\delta}\geq(yY_u)^{\beta+\delta}1_{\{yY_u>x/x_0\}}\geq(x/x_0)^{\beta+\delta}1_{\{yY_u>x/x_0\}},$$

we get

$$I_{11}(n,x,y)\leq(x_0y)^{\beta+\delta}\frac{\sum_{|u|=n}Y_u^{\beta+\delta}}{x^{\beta+\delta}(1-F(x))}\overset{(20)}{\leq}A_1x_0^{\beta+\delta}y^{\beta+\delta}\sum_{|u|=n}Y_u^{\beta+\delta}.$$

Further,

$$I_{12}(n,x,y)\overset{(23)}{\leq}A_4\sum_{|u|=n}(yY_u)^{\beta-\delta}\vee(yY_u)^{\beta+\delta}\leq$$

$$\leq A_4 \left(y^{\beta-\delta} \sum_{|u|=n} Y_u^{\beta-\delta} + y^{\beta+\delta} \sum_{|u|=n} Y_u^{\beta+\delta} \right);$$

$$\frac{I_2(n,x/y)}{1-F(x)} = y^2 \sum_{|u|=n} \frac{Y_u^2 \int_0^{x(yY_u)^{-1}} z^2 dF(z)}{x^2(1-F(x))} \overset{(22)}{\leq} By^2 \sum_{|u|=n} \frac{Y_u^2 T(x(yY_u)^{-1})}{T(x)} =$$

$$= By^2 \left(\sum_{|u|=n} \cdots 1_{\{yY_u > x/x_0\}} + \sum_{|u|=n} \cdots 1_{\{yY_u \leq x/x_0\}} \right) =:$$

$$=: By^2(I_{21}(n,x,y) + I_{22}(n,x,y)).$$

$$I_{21}(n,x,y) \overset{(25)}{\leq} A_6 \sum_{|u|=n} \frac{Y_u^2 S(x(yY_u)^{-1})}{S(x)} 1_{\{yY_u > x/x_0\}} \leq$$

$$\leq A_6 (x_0 y)^{\beta-2+\delta} S(x_0) \sum_{|u|=n} \frac{Y_u^{\beta+\delta}}{x^{\beta-2+\delta} S(x)} \overset{(21)}{\leq} A_2 A_6 (x_0 y)^{\beta-2+\delta} S(x_0) \sum_{|u|=n} Y_u^{\beta+\delta};$$

$$I_{22}(n,x,y) \overset{(24)}{\leq} A_5 \sum_{|u|=n} Y_u^2 ((yY_u)^{\beta-2-\delta} \vee (yY_u)^{\beta-2+\delta}) \leq$$

$$\leq A_5 \left(y^{\beta-2-\delta} \sum_{|u|=n} Y_u^{\beta-\delta} + y^{\beta-2+\delta} \sum_{|u|=n} Y_u^{\beta+\delta} \right).$$

Thus, according to (17), we have proved that, for $x \geq x_0$ and $y > 0$,

$$\frac{\mathbb{P}_n\{|\sum_{|u|=n} Y_u Q_u^s| > x/y\}}{\mathbb{P}\{|Q^s| > x\}} \leq const \left(y^{\beta-\delta} \sum_{|u|=n} Y_u^{\beta-\delta} + y^{\beta+\delta} \sum_{|u|=n} Y_u^{\beta+\delta} \right). \tag{26}$$

In particular, since $\mathbb{P}\{|Q^s| > x\} \sim 2\mathbb{P}\{|Q| > x\}$, setting $y = 1$ in (26) and using (16) lead to (11), with U_n being a multiple of the right-hand side of (26).

In the remaining cases, we only investigate the situation where $\beta \in (2,4)$ and $\beta = 2$. For other values of β, inequality (11) with U_n satisfying (14) follows by induction.

(b) $\beta \in (2,4)$. Given $\mathcal{F}_n$, let $\{\tilde{N}, N_u : |u| = n\}$ be conditionally independent copies of a random variable N with normal $(0,1)$ distribution. Using the approach similar to that exploited to obtain (18)(this fruitful argument has come to our attention from [25]) allows us to conclude that, for $x > 0$ and appropriate positive constants c_1, c_2,

$$\mathbb{P}_n\{|\sum_{|u|=n} Y_u Q_u^s| > x\} \leq$$

$$\leq c_1 \mathbb{P}_n\{|\sum_{|u|=n} Y_u N_u Q_u^s| > c_2 x\} = c_1 \mathbb{P}_n \left\{ |\tilde{N}| \left(\sum_{|u|=n} Y_u^2 (Q_u^s)^2 \right)^{1/2} > c_2 x \right\}. \tag{27}$$

Notice that $\mathbb{P}\{(Q^s)^2 > x\}$ regularly varies with index $-\beta/2 \in (-2,-1)$. Also it is obvious that, if needed, we can reduce δ in (8) to ensure that $k_{\beta-2\delta} < 1$ and $k_{\beta+2\delta} < 1$. Therefore, we can use (26) with Y_u replaced with Y_u^2 and Q_u^s replaced with $(Q_u^s)^2$, which gives after a little manipulation that, for $x \geq x_0^{1/2}$,

$$\frac{\mathbb{P}_n\{|\sum_{|u|=n} Y_u Q_u^s| > x\}}{\mathbb{P}\{|Q^s| > x\}} \leq$$

$$\leq const\left(\sum_{|u|=n} Y_u^{\beta+2\delta}\mathbb{E}|N|^{\beta+2\delta} + \sum_{|u|=n} Y_u^{\beta-2\delta}\mathbb{E}|N|^{\beta-2\delta}\right).$$

By using the same argument as in case (a), we can check that (11) and (14) have been proved.

(c) $\beta = 2$. In the same manner as we have established (26), it can be proved that, for $y > 0$ and large x,

$$\frac{\mathbb{P}_n\{\sum_{|u|=n} Y_u^2(Q_u^s)^2 > x/y\}}{\mathbb{P}\{(Q^s)^2 > x\}} \leq const\left(y^{2-2\delta}\sum_{|u|=n} Y_u^{2-2\delta} + y^{2+2\delta}\sum_{|u|=n} Y_u^{2+2\delta}\right).$$

Hence, an appeal to (27) assures that (11) and (14) hold in this case too.

We have a representation

$$W_{n+1} = \sum_{|u|=n} Y_u W_1^{(u)}, \tag{28}$$

where, given $\mathcal{F}_n$, $W_1^{(u)}$ are (conditionally) independent copies of W_1. Each element of the set $\{W_1^{(u)} : |u| = n\}$ is constructed in the same way as W_1, the only exception being that while W_1 is defined on the whole family tree, $W_1^{(u)}$ is defined on the subtree with root u.

We only give a complete proof for Δ. The analysis of W^* is similar but simpler, and hence omitted. From (28), we conclude that $|d_{n+1}|$ is the same as T_n with $Q_u = W_1^{(u)} - 1$. Hence, by (10),

$$1_{\{\max_{1\leq k\leq n}|d_k|\leq x\}}\mathbb{P}_n\{|d_{n+1}| > x\} \sim \sum_{|u|=n} Y_u^\beta \mathbb{P}\{|W_1 - 1| > x\} \sim$$

$$\sim \sum_{|u|=n} Y_u^\beta \mathbb{P}\{W_1 > x\} \quad \text{a.s.,} \tag{29}$$

and, by (13),

$$\mathbb{P}\{|d_{n+1}| > x\} \sim k_\beta^n \mathbb{P}\{|W_1 - 1| > x\} \sim k_\beta^n \mathbb{P}\{W_1 > x\}. \tag{30}$$

Recall that

$$\mathbb{P}\{\Delta > x\} = \mathbb{P}\{|d_1| > x\} + \sum_{n=1}^\infty \mathbb{P}\{\max_{1\leq k\leq n}|d_k| \leq x, |d_{n+1}| > x\} =$$

$$= \mathbb{P}\{|d_1| > x\} + \mathbb{E}\sum_{n=1}^\infty 1_{\{\max_{1\leq k\leq n}|d_k|\leq x\}}\mathbb{P}_n\{|d_{n+1}| > x\}.$$

Using this relation and (29) and applying the Fatou lemma twice allows us to conclude that

$$\liminf_{x\to\infty}\frac{\mathbb{P}\{\Delta > x\}}{\mathbb{P}\{W_1 > x\}} \geq 1 + \sum_{n=1}^\infty \mathbb{E}\liminf_{x\to\infty}\frac{1_{\{\max_{1\leq k\leq n}|d_k|\leq x\}}\mathbb{P}_n\{|d_{n+1}| > x\}}{\mathbb{P}\{W_1 > x\}} \geq$$

$$\geq 1 + \sum_{n=1}^\infty \mathbb{E}\left(\sum_{|u|=n} Y_u^\beta\right) = (1 - k_\beta)^{-1}.$$

To complete the proof for Δ, we must calculate the corresponding upper limit. For this, it suffices to check that, for large x and large $n \in \mathbb{N}$,

$$\frac{\mathbb{P}\{|d_{n+1}| > x\}}{\mathbb{P}\{W_1 > x\}} \le C_n, \tag{31}$$

where C_n is a summable sequence, and use the dominated convergence theorem. Taking the expectation in (11) allows us to conclude that, for $n = 1, 2, \dots$ and large x,

$$\frac{\mathbb{P}\{|d_{n+1}| > x + \mu_n\}}{\mathbb{P}\{W_1 > x\}} \le const\mathbb{E}U_n,$$

where μ_n is the median of $V_n := \sum_{|u|=n} Y_u |W_1^{(u)} - 1|$, and $\mathbb{E}U_n$ is given by (15). The family of distributions of V_n is tight. In view of (30),

$$\mathbb{P}\{|d_{n+1}| > x + y\} \sim \mathbb{P}\{|d_{n+1}| > x\} \ \text{ locally uniformly in } y.$$

Therefore, (31) holds with $C_n = const\,\mathbb{E}U_n$, and the result for Δ has been proved.

For later needs, let us notice here that, in the same way as above, we can prove that, for fixed $n \in \mathbb{N}$,

$$\mathbb{P}\{\sup_{m \ge n} W_m > x\} \sim k_\beta^n (1 - k_\beta)^{-1} \mathbb{P}\{W_1 > x\}. \tag{32}$$

Consider now the square function S. Since $S \ge \Delta$ a.s., and we have already proved that $\mathbb{P}\{\Delta > x\} \sim (1 - k_\beta)^{-1} \mathbb{P}\{W_1 > x\}$,

$$\liminf_{x \to \infty} \frac{\mathbb{P}\{S > x\}}{\mathbb{P}\{W_1 > x\}} \ge \frac{1}{1 - k_\beta}.$$

Therefore, we must only calculate the upper limit. We begin with showing that, for any $n \in \mathbb{N}$,

$$\limsup_{x \to \infty} \frac{\mathbb{P}\{S_n > x\}}{\mathbb{P}\{W_1 > x\}} \le \sum_{m=0}^{n-1} k_\beta^m. \tag{33}$$

We use induction on n.
(1) If $n = 1$, then $S_1 \le W_1$, and (33) is obvious.
(2) Assume that (33) holds for $n = j$ and show that it holds for $n = j + 1$. For every $x > 0$ and $\epsilon \in (0, 1)$,

$$\mathbb{P}\{S_{j+1} > x\} \le \mathbb{P}\{S_j^2 > (1-\epsilon)x^2\} + \mathbb{P}\{d_{j+1}^2 > (1-\epsilon)x^2\} +$$
$$+ \mathbb{P}\{S_j^2 > \epsilon x^2, d_{j+1}^2 > \epsilon x^2\} =$$

(write the latter $\mathbb{P}$ as $\mathbb{E}\mathbb{P}_j$ and use the $\mathcal{F}_j$-measurability of S_j)

$$= \mathbb{P}\{S_j > (1-\epsilon)^{1/2} x\} + \mathbb{P}\{|d_{j+1}| > (1-\epsilon)^{1/2} x\} + \mathbb{E}1_{\{S_j > \epsilon^{1/2} x\}} \mathbb{P}_j\{|d_{j+1}| > \epsilon^{1/2} x\}.$$

According to (10) with Q_u replaced by $W_1^{(u)} - 1$,

$$\lim_{x \to \infty} 1_{\{S_j > \epsilon^{1/2} x\}} \frac{\mathbb{P}_j\{|d_{j+1}| > \epsilon^{1/2} x\}}{\mathbb{P}\{W_1 > x\}} = 0 \ \text{ a.s.},$$

and there exists a $\delta_1 > 0$ such that, for large x,

$$1_{\{S_j > \epsilon^{1/2} x\}} \frac{\mathbb{P}_j\{|d_{j+1}| > \epsilon^{1/2} x\}}{\mathbb{P}\{W_1 > x\}} \le \epsilon^{-\beta/2} \sum_{|u|=n} Y_u^\beta + \delta_1 \ \text{ a.s.}$$

Therefore, by the dominated convergence,

$$\lim_{x\to\infty}\mathbb{E}1_{\{S_j>\epsilon^{1/2}x\}}\frac{\mathbb{P}_j\{|d_{j+1}|>\epsilon^{1/2}x\}}{\mathbb{P}\{W_1>x\}}=0.$$

By the inductive assumption and (30),

$$\limsup_{x\to\infty}\frac{\mathbb{P}\{S_{j+1}>x\}}{\mathbb{P}\{W_1>x\}}\leq(1-\epsilon)^{-\beta/2}\sum_{m=0}^{j-1}k_\beta^m+(1-\epsilon)^{-\beta/2}k_\beta^j=(1-\epsilon)^{-\beta/2}\sum_{m=0}^{j}k_\beta^m.$$

Sending $\epsilon\to 0$ proves (33).

For $m=0,1,\dots$ and fixed $n\in\mathbb{N}$, set $\tilde{W}_m:=W_{m\vee n}$ and $\tilde{\mathcal{F}}_m;=\mathcal{F}_{m\vee n}$. Choose $\rho\in(0,\sqrt{3})$ so small that $\nu:=\frac{2\rho^2}{3-\rho^2}2^{\beta+1}\in(0,1)$. Applying Theorem 18.2[8] (in the notation of that paper, we take $\beta=2$ and $\delta=\rho$) to the non-negative martingale $(\tilde{W}_m,\tilde{\mathcal{F}}_m)$ gives

$$\mathbb{P}\{(\sum_{m=n+1}^{\infty}d_m^2)^{1/2}>2x\}\leq$$

$$\leq\mathbb{P}\{\sup_{m\geq n}\tilde{W}_m>\rho x\}+\mathbb{P}\{(\sum_{m=n+1}^{\infty}d_m^2)^{1/2}>2x,\ \sup_{m\geq n}\tilde{W}_m\leq\rho x\}\leq$$

$$\leq\mathbb{P}\{\sup_{m\geq n}W_m>\rho x\}+\frac{2\rho^2}{3-\rho^2}\mathbb{P}\{(\sum_{m=n+1}^{\infty}d_m^2)^{1/2}>x\}. \tag{34}$$

By Potter's bound, we can take $y>0$ such that $\frac{\mathbb{P}\{W_1>x\}}{\mathbb{P}\{W_1>2x\}}\leq 2^{\beta+1}$ for $x\geq y$. Set $A(y):=\sup_{x\geq y}\frac{\mathbb{P}\{\sup_{m\geq n}W_m>\rho x\}}{\mathbb{P}\{W_1>2x\}}$. In view of (32), $A(y)<\infty$ and $\lim_{x\to\infty}A(x)=\frac{k_\beta^n}{1-k_\beta}\left(\frac{2}{\rho}\right)^\beta$. Now we have, for $x\geq y$,

$$\frac{\mathbb{P}\{(\sum_{m=n+1}^{\infty}d_m^2)^{1/2}>2x\}}{\mathbb{P}\{W_1>2x\}}\leq A(y)+\nu\frac{\mathbb{P}\{(\sum_{m=n+1}^{\infty}d_m^2)^{1/2}>x\}}{\mathbb{P}\{W_1>x\}}.$$

Iterating the latter inequality gives that, for $k=0,1,\dots,$

$$\sup_{x\in[2^ky,2^{k+1}y]}\frac{\mathbb{P}\{(\sum_{m=n+1}^{\infty}d_m^2)^{1/2}>x\}}{\mathbb{P}\{W_1>x\}}\leq$$

$$\leq A(y)(1+\nu+\dots+\nu^{k-1})+\nu^k\sup_{x\in[y,2y]}\frac{\mathbb{P}\{(\sum_{m=n+1}^{\infty}d_m^2)^{1/2}>x\}}{\mathbb{P}\{W_1>x\}}.$$

Let $k\to\infty$ to obtain

$$\limsup_{x\to\infty}\frac{\mathbb{P}\{(\sum_{m=n+1}^{\infty}d_m^2)^{1/2}>x\}}{\mathbb{P}\{W_1>x\}}\leq A(y)(1-\nu)^{-1}.$$

Now sending $y\to\infty$ gives

$$\limsup_{x\to\infty}\frac{\mathbb{P}\{(\sum_{m=n+1}^{\infty}d_m^2)^{1/2}>x\}}{\mathbb{P}\{W_1>x\}}\leq\frac{k_\beta^n}{(1-k_\beta)(1-\nu)}\left(\frac{2}{\rho}\right)^\beta=const\ k_\beta^n. \tag{35}$$

For any $\lambda\in(0,1)$ and any $n\in\mathbb{N}$,

$$\mathbb{P}\{S>x\}\leq\mathbb{P}\{S_n>(1-\lambda)^{1/2}x\}+\mathbb{P}\{(\sum_{k=n+1}^{\infty}d_k^2)^{1/2}>\lambda^{1/2}x\}.$$

Therefore,

$$\limsup_{x\to\infty}\frac{\mathbb{P}\{S>x\}}{\mathbb{P}\{W_1>x\}}\overset{(33),(35)}{\leq}(1-\lambda)^{-\beta/2}\sum_{m=0}^{n-1}k_\beta^m+const\ \lambda^{-\beta/2}k_\beta^n.$$

Let $n\to\infty$ and then $\lambda\to 0$ to get the desired bound for the upper limit:

$$\limsup_{x\to\infty}\frac{\mathbb{P}\{S>x\}}{\mathbb{P}\{W_1>x\}}\leq\frac{1}{1-k_\beta}.$$

This completes the proof for S.
(II) From the already proved relation

$$\mathbb{P}\{W^*>x\}\sim(1-k_\beta)^{-1}\mathbb{P}\{W_1>x\}\sim(1-k_\beta)^{-1}x^{-\beta}L(x),\tag{36}$$

it follows that $\mathbb{E}W^*<\infty$, which ensures, in turn, the uniform integrability of W_n.

Let us now prove (4). Since $W^*\geq W$ a.s., $\mathbb{E}(W^*-x)^+\geq\mathbb{E}(W-x)^+$ for any $x\geq 0$. Relation (36) and Proposition 1.5.10[7] yield

$$\mathbb{E}(W^*-x)^+\sim(\beta-1)^{-1}(1-k_\beta)^{-1}x\mathbb{P}\{W_1>x\}.$$

Therefore,

$$\limsup_{x\to\infty}\frac{\mathbb{E}(W-x)^+}{x\mathbb{P}\{W_1>x\}}\leq\frac{1}{(\beta-1)(1-k_\beta)}.\tag{37}$$

For each $x\geq 1$, we define the stopping time ν_x by

$$\nu_x:=\begin{cases}\inf\{n\geq 1: W_n>x\}, & \text{if } W^*>x;\\ +\infty, & \text{otherwise.}\end{cases}$$

The random variable W closes the martingale W_n. Hence, for each $x\geq 1$,

$$\mathbb{E}(W-x)1_{\{\nu_x<\infty\}}=\mathbb{E}(W_{\nu_x}-x)1_{\{\nu_x<\infty\}},$$

and, hence,

$$\mathbb{E}(W-x)^+\geq\mathbb{E}(W_{\nu_x}-x)^+1_{\{\nu_x<\infty\}}.$$

We now transform the right-hand side into a more tractable form

$$\mathbb{E}(W_{\nu_x}-x)^+1_{\{\nu_x<\infty\}}=\sum_{k=1}^{\infty}\mathbb{E}(W_k-x)^+1_{\{\nu_x=k\}}=\mathbb{E}\sum_{k=1}^{\infty}\mathbb{E}_{k-1}((W_k-x)^+1_{\{\nu_x\geq k\}})=$$

$$=\mathbb{E}\sum_{k=1}^{\infty}1_{\{\nu_x\geq k\}}\mathbb{E}_{k-1}(W_k-x)^+=\mathbb{E}\sum_{k=1}^{\nu_x}\mathbb{E}_{k-1}(W_k-x)^+.$$

From (28) and (10) with Q replaced by W_1, it follows that, for $k=2,3,\dots$,

$$\mathbb{P}_{k-1}\{W_k>y\}\sim\sum_{|u|=k-1}Y_u^\beta\mathbb{P}\{W_1>y\}\ \text{ a.s.}$$

An appeal to Proposition 1.5.10[7] gives that, for $k=2,3,\dots$,

$$\mathbb{E}_{k-1}(W_k-y)^+\sim(\beta-1)^{-1}\sum_{|u|=k-1}Y_u^\beta y\mathbb{P}\{W_1>y\}\ \text{ a.s.}$$

Since $\lim_{x\to\infty}\nu_x=+\infty$ a.s., using the Fatou lemma allows us to conclude that

$$\liminf_{x\to\infty}\frac{\mathbb{E}(W-x)^+}{x\mathbb{P}\{W_1>x\}}\geq\liminf_{x\to\infty}\frac{\mathbb{E}\sum_{k=1}^{\nu_x}\mathbb{E}_{k-1}(W_k-x)^+}{x\mathbb{P}\{W_1>x\}}=$$

$$= \frac{1}{\beta - 1}\left(1 + \sum_{k=2}^{\infty} \mathbb{E}\left(\sum_{|u|=k-1} Y_u^\beta\right)\right) = \frac{1}{(\beta-1)(1-k_\beta)}.$$

Combining the latter inequality with (37) yields

$$\mathbb{E}(W - x)^+ \sim (\beta-1)^{-1}(1-k_\beta)^{-1} x \mathbb{P}\{W_1 > x\},$$

which by the monotone density theorem (see Theorem 1.7.2[7]) implies (4).

The result for M immediately follows from

$$\mathbb{P}\{W > x\} \sim \mathbb{P}\{W^* > x\} \sim (1-k_\beta)^{-1}\mathbb{P}\{W_1 > x\},$$

as $|W - 1| \le M \le W^*$ a.s. The proof of the theorem is finished.

The second proof of (4) in the case where $\beta > 2$

Assume that the assumptions of Proposition 1.1 hold with $\beta > 2$. By (36) and Theorem 1.6.5[7],

$$\mathbb{E}W^*(W^* - x)^+ \sim \beta(\beta-1)^{-1}(\beta-2)^{-1}(1-k_\beta)^{-1}x^{2-\beta}L(x).$$

Since, for each $x > 0$, $\mathbb{E}W^*(W^* - x)^+ \ge \mathbb{E}W(W - x)^+$

$$\limsup_{x\to\infty} \frac{\mathbb{E}W(W-x)^+}{x^{2-\beta}L(x)} \le \frac{\beta}{(\beta-1)(\beta-2)(1-k_\beta)}. \tag{38}$$

Lyons [24] constructed a probability space with probability measure $\mathbb{Q}$ and proved the equality

$$\mathbb{E}_{\mathbb{Q}}(W_n|\mathcal{G}) = 1 + \sum_{k=1}^{n} \Pi_{k-1}(S_k - 1) \qquad \mathbb{Q} \text{ a.s.},$$

where $\mathbb{E}_{\mathbb{Q}}$ is the expectation with respect to $\mathbb{Q}$; $\Pi_0 := 1$, $\Pi_k := M_1M_2\cdots M_k$, $k = 1, 2, \dots$; $\{(M_k, S_k) : k = 1, 2, \dots\}$ are Q-independent copies of a random vector (M, S), whose distribution is defined by the equality

$$\mathbb{E}\sum_{|u|=1} Y_u h(Y_u, \sum_{|v|=1} Y_v) = \mathbb{E}h(M, S) \tag{39}$$

which is assumed to hold for any nonnegative Borel bounded function $h(x, y)$; $\mathcal{G}$ is the σ-field that can be explicitly described (we only note that $\sigma((M_k, S_k), k = 1, 2, \dots) \subset \mathcal{G}$). Moreover, for any Borel function r with the obvious convention when the right-hand side is infinite or does not exist,

$$\mathbb{E}_{\mathbb{Q}} r(W_n) := \mathbb{E}W_n r(W_n) \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}} r(W) := \mathbb{E}W r(W). \tag{40}$$

Lyons explains his clever argument in a quite condensed form. More details clarifying his way of reasoning can be found in [4] and [22].

Since $\mathbb{P}\{W_1 > x\}$ regularly varies with exponent $-\beta$, $\beta > 2$, $\mathbb{E}W_1^2 < \infty$. Also by (7), $\mathbb{E}\sum_{|u|=1} Y_u^2 < 1$. By Proposition 4[17], the last two inequalities together ensure that $\mathbb{E}W^2 < \infty$. In view of (40), $\mathbb{E}_{\mathbb{Q}}W = \mathbb{E}W^2$, and hence $\mathbb{E}_{\mathbb{Q}}W < \infty$. In Lemma 4.1[22], it was proved that (1) holds with $\mathbb{P}$ replaced by $\mathbb{Q}$. This implies that $\mathbb{E}_{\mathbb{Q}}W^* < \infty$ iff $\mathbb{E}_{\mathbb{Q}}W < \infty$. Therefore, we have checked that $\mathbb{E}_{\mathbb{Q}}W^* < \infty$ which implies by the dominated convergence theorem that

$$\mathbb{E}_{\mathbb{Q}}(W|\mathcal{G}) = 1 + \sum_{k=1}^{\infty} \Pi_{k-1}(S_k - 1) =: R \qquad \mathbb{Q} \text{ a.s.}$$

By Jensen's inequality, for any convex function g,

$$\mathbb{E}_{\mathbb{Q}}(g(W)|\mathcal{G}) \geq g(\mathbb{E}_{\mathbb{Q}}(W|\mathcal{G})) = g(R) \qquad \mathbb{Q} \text{ a.s.}$$

Setting $g(u) := (u-x)^+, x > 0$ and taking expectations yields

$$\mathbb{E}W(W-x)^+ \overset{(40)}{=} \mathbb{E}_{\mathbb{Q}}(W-x)^+ \geq \mathbb{E}_{\mathbb{Q}}(R-x)^+. \tag{41}$$

It follows from (39) that $\mathbb{E}_{\mathbb{Q}}M^{\beta-1} = k_\beta < 1$, $\mathbb{E}_{\mathbb{Q}}M^{\beta-1+\epsilon} = k_{\beta+\epsilon} < \infty$ and

$$\mathbb{Q}\{S-1>t\} = \int_{t+1}^{\infty} y d\mathbb{P}\{W_1 \leq y\}.$$

Using the latter equality and Theorem 1.6.5[7] leads to

$$\mathbb{Q}\{S-1>t\} \sim \beta(\beta-1)^{-1}t^{1-\beta}L(t).$$

Therefore, Theorem 1[16] can be applied to the perpetuity R, which gives

$$\mathbb{Q}\{R>t\} \sim \beta(\beta-1)^{-1}(1-\mathbb{E}_{\mathbb{Q}}M^{\beta-1})^{-1}t^{1-\beta}L(t).$$

By Proposition 1.5.10[7],

$$\mathbb{E}_{\mathbb{Q}}(R-x)^+ = \int_x^\infty \mathbb{Q}\{R>t\}dt \sim \frac{\beta}{(\beta-1)(\beta-2)(1-k_\beta)}x^{2-\beta}L(x).$$

An appeal to (41) now results in

$$\liminf_{x\to\infty} \frac{\mathbb{E}W(W-x)^+}{x^{2-\beta}L(x)} \geq \frac{\beta}{(\beta-1)(\beta-2)(1-k_\beta)}.$$

Combining this with (38) yields

$$\mathbb{E}_{\mathbb{Q}}(W-x)^+ \overset{(40)}{=} \mathbb{E}W(W-x)^+ \sim \frac{\beta}{(\beta-1)(\beta-2)(1-k_\beta)}x^{2-\beta}L(x).$$

By the monotone density theorem,

$$\mathbb{Q}\{W>x\} \sim \frac{\beta}{(\beta-1)(1-k_\beta)}x^{1-\beta}L(x).$$

Since $\mathbb{Q}\{W>x\} = \int_x^\infty y d\mathbb{P}\{W \leq y\}$, integrating by parts gives

$$\frac{x\mathbb{P}\{W>x\}}{\mathbb{Q}\{W>x\}} = 1 - \frac{x}{\mathbb{Q}\{W>x\}}\int_x^\infty y^{-2}\mathbb{Q}\{W>y\}dy.$$

By Proposition 1.5.10[7], the right-hand side tends to $(\beta-1)\beta^{-1}$ when $x \to \infty$. Therefore, $\mathbb{P}\{W>x\} \sim (\beta-1)(\beta x)^{-1}\mathbb{Q}\{W>x\} \sim (1-k_\beta)^{-1}x^{-\beta}L(x)$ as desired.

Remark 3.1. This argument seems not to work as just described when $\beta \in (1,2]$. If $\beta \in (1,2)$, we can get a bound for the upper limit

$$\limsup_{x\to\infty} \frac{\mathbb{E}W(W\wedge x)}{x^{2-\beta}L(x)} \leq \frac{\beta}{(\beta-1)(2-\beta)(1-k_\beta)}.$$

However, we do not know how the corresponding lower limit could be obtained. In fact, we have not been able to find any random variable ξ with the appropriate tail behaviour and such that $W \geq \xi$ in some strong or weak sense.

Miscellaneous comments

We begin this section with discussing the following problem: which classes of point processes satisfy both (2) and (3) and which do not.

Let $h : [0, \infty) \to [0, \infty)$ be a nondecreasing and right-continuous function with $h(+0) > 0$.

Example 4.1 Let $\{\tau_0 := 0, \tau_i, i \geq 1\}$ be the renewal times of an ordinary renewal process. In addition to the conditions imposed above on h, assume that $h(0)$ is finite. Consider the point process $\mathcal{M}$ with points $\{A_i = \gamma^{-1} \log h(\tau_i), i = 1, 2, \dots\}$, where $\gamma > 0$ is chosen so that $\mathbb{E} \sum_{i=1}^{\infty} h(\tau_i) = 1$. According to Theorem 1[14], $W_1 = \sum_{i=1}^{\infty} h(\tau_i)$ has exponentially decreasing tail. Thus, while we can find h and $\{\tau_i\}$ such that (2) holds, (3) always fails.

The situation where $\mathbb{P}\{W_1 > x\} \sim x^{-\beta} L(x)$ and $\mathbb{E}W_1^{\beta} < \infty$ is not terribly interesting. However, if this is the case, Proposition 1.1 yields the one-way implication of a well-known moment result (see [17] and [21])

$$\mathbb{E} \sum_{|u|=1} Y_u^{\beta} < 1, \quad \mathbb{E}W_1^{\beta} < \infty \Leftrightarrow \mathbb{E}W^{\beta} < \infty, \quad \mathbb{E}(W^*)^{\beta} < \infty.$$

In the subsequent examples in addition to (2) and (3), we require that $\mathbb{E}W_1^{\beta} = \infty$. Examples 4.2 and 4.3 essentially show that when the number of points in a point process is infinite, and the points are independent or constitute an (inhomogeneous) Poisson flow, $\mathbb{E} \sum_{|u|=1} Y_u^{\beta} < \infty$ implies $\mathbb{E}W_1^{\beta} < \infty, \beta > 1$.

Example 4.2 Assume that $\mathcal{M}$ is a point process with independent points $\{C_i\}$, and $\mathbb{E} \sum_{i=1}^{\infty} Y_i = 1$ and, for some $\beta > 1$, $\mathbb{E} \sum_{i=1}^{\infty} Y_i^{\beta} < \infty$, where $Y_i = e^{\gamma C_i}$ and $\gamma > 0$. Then $\mathbb{E}W_1^{\beta} < \infty$.

In this case, $W_1 = \sum_{i=1}^{\infty} Y_i$. Hence, we must check that $\mathbb{E}(\sum_{i=1}^{\infty} Y_i)^{\beta} < \infty$. By using the c_{β}-inequality, let us write the (formal) inequality

$$\mathbb{E}(\sum_{i=1}^{\infty} Y_i)^{\beta} = \mathbb{E} \sum_{i=1}^{\infty} Y_i (Y_i + \sum_{k \neq i} Y_k)^{\beta-1} \leq$$

$$\leq (2^{\beta-2} \vee 1)(\mathbb{E} \sum_{i=1}^{\infty} Y_i^{\beta} + (\mathbb{E} \sum_{i=1}^{\infty} Y_i) \mathbb{E}(\sum_{i=1}^{\infty} Y_i)^{\beta-1}). \tag{42}$$

For any $\beta > 1$, there exists $n \in \mathbb{N}$ such that $\beta \in (n, n+1]$. We will use induction on n. If $\beta \in (1, 2]$, then $\mathbb{E} \sum_{i=1}^{\infty} Y_i < \infty$ implies $\mathbb{E}(\sum_{i=1}^{\infty} Y_i)^{\beta-1} < \infty$. Hence by (42), $\mathbb{E}(\sum_{i=1}^{\infty} Y_i)^{\beta} < \infty$. Assume that the conclusion is true for $\beta \in (n, n+1]$ and prove it for $\beta \in (n+1, n+2]$. Since $\mathbb{E} \sum_{i=1}^{\infty} Y_i < \infty$ and $\mathbb{E} \sum_{i=1}^{\infty} Y_i^{\beta} < \infty$, we have $\mathbb{E} \sum_{i=1}^{\infty} Y_i^{\beta-1} < \infty$, which, by the assumption of induction, implies $\mathbb{E}(\sum_{i=1}^{\infty} Y_i)^{\beta-1} < \infty$. It remains to apply (42).

Example 4.3 Let $\{\tau_i, i \geq 1\}$ be the arrival times of a Poisson process with intensity $\lambda > 0$. Consider a Poisson point process $\mathcal{M}$ with points $\{B_i\}$ and assume that, for any $a \in \mathbb{R}$, $\mathcal{M}(a, \infty) \geq 1$ a.s. Then there exists a function h, as described at the beginning of this section, that additionally satisfies $h(0) = \infty$, and $\gamma > 0$ such that $h(\tau_i) = e^{\gamma B_i}$ and $\mathbb{E}W_1 = \mathbb{E} \sum_{i=1}^{\infty} h(\tau_i) = \lambda \int_0^{\infty} h(u) du = 1$. The distribution of W_1 is infinitely divisible with zero shift and the Lévy measure ν given as follows: $\nu(dx) = \lambda h^{\leftarrow}(dx)$, where $h^{\leftarrow}$ is a generalized inverse function. Since $\lambda \mathbb{E} \sum_{i=1}^{\infty} h^{\beta}(\tau_i) = \int_0^{\infty} x^{\beta} \nu(dx)$ and, as is well known from the general theory of infinitely divisible distributions, $\int_1^{\infty} x^{\beta} \nu(dx) < \infty$ implies $\mathbb{E}W_1^{\beta} < \infty$, we conclude that $\mathbb{E} \sum_{i=1}^{\infty} e^{\gamma \beta B_i} < \infty$ implies $\mathbb{E}W_1^{\beta} < \infty$. In the next example, we point out a class of point processes which satisfy (2) and (3), and $\mathbb{E}W_1^{\beta} = \infty$.

Example 4.4 Let K be a nonnegative integer-valued random variable with $\mathbb{P}\{K > x\} \sim x^{-\beta}L(x), \beta > 1$, and let $\{D_i, i \geq 1\}$ be independent identically distributed random variables which are independent of K. If $\mathcal{M}$ is a point process with points $\{D_i, i = \overline{1, K}\}$ and there exists a $\gamma > 0$ such that $\mathbb{E}\sum_{i=1}^{K} e^{\gamma D_i} = 1$ and $\mathbb{E}\sum_{i=1}^{K} e^{\gamma\beta D_i} < 1$, then, according to Proposition 4.3[15], we have $\mathbb{P}\{W_1 > x\} \sim \mathbb{E}e^{\gamma\beta D_1}x^{-\beta}L(x)$. We conclude with two remarks that fit the context of the present paper.

Remark 4.1 The tail behaviour of $\mathbb{P}\{W > x\}$ and $\mathbb{P}\{W^* > x\}$ has been investigated in [17] and [21]. In particular, it follows from those works that, when the condition $\mathbb{E}\sum_{|u|=1} Y_u^{\beta} < 1$ fails, the condition $\mathbb{P}\{W_1 > x\} \sim x^{-\beta}L(x)$ is not a necessary one for either $\mathbb{P}\{W > x\} \sim x^{-\beta}L(x)$ or $\mathbb{P}\{W^* > x\} \sim x^{-\beta}L(x)$ to hold.

Remark 4.2 Let $\{X_i\}$ be the points of a point process, and let V be a random variable satisfying the distributional equality

$$V \stackrel{d}{=} \sum_{i=1}^{\infty} X_i V_i,$$

where $V_1, V_2, \dots$ are conditionally independent copies of V on $\{X_i\}$. The distribution of V is called a fixed point of the smoothing transform (see [17] for more details and [19] and [20] for an interesting particular case). It is known and can be easily checked that the distribution of W is a fixed point of the smoothing transform with $\{X_i : i = 1, 2, \dots\} = \{Y_u : |u| = 1\}$. Thus, (4) can be reformulated as a result on the tail behaviour of the fixed points *with finite mean*. The tail behaviour of fixed points with infinite mean deserves a special mention. Typically, their tails regularly vary with index $\alpha \in (0, 1)$, or $\int_0^x \mathbb{P}\{V > y\}dy$ slowly varies. This follows from Proposition 1(b)[17] and Proposition 8.1.7[7].

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