Theory of Stochastic Processes
Vol. 11 (27), no. 3–4, 2005, pp. 104–109

UDC 519.21

SERGEY YA. MAKHNO

THE LIMIT STOCHASTIC EQUATION CHANGES TYPE

We study the weak convergence of solutions of the Itô stochastic equation, whose coefficients depend on a small parameter. Conditions under which the limit process changes the type and will be a solution of the stochastic equation with a local time are obtained.

We study the solutions of Itô stochastic equations, whose coefficients depend on a small parameter $\epsilon > 0$. We investigate their convergence as $\epsilon \to 0$ without assuming the convergence of the coefficients themselves.Conditions under which these solutions converge in the weak sense to a solution of the Itô stochastic equation are known [6,7]. In [11], W.Rosenkrants considered the random processes $x^\epsilon(t)$ as solutions of the Itô stochastic equations

$$x^\epsilon(t) = x + \frac{1}{\epsilon}\int_0^t b\left(\frac{x^\epsilon(s)}{\epsilon}\right)ds + w(t).$$

Let we suppose that $\int_{-\infty}^{\infty} b(x)dx \neq 0$. The results of [11, Theorems 1 and 3] show that the limit process $x(t)$ does not possess the Itô stochastic integral representation. By Le Gall [3 , Corollary 3.3], the limit process is a solution of the stochastic equation with a local time, or it is the skew Brownian motion in the terminology of Walsh. On the other hand, Portenko [10] considered the class of random processes he called as generalized diffusion processes. He proved [10, Theorem 3.4] that the random process $x(t)$ described above is also a generalized diffusion process, and it can be represented in integral form with the Dirac delta function in the drift coefficient [10, Corollary of Theorem 3.5]. Thus, if we have Itô stochastic equations with coefficients unbounded in the parameter ϵ, we may have another type of the equation for the limit processes. For a particular form of coefficients in the Itô equations, the same situation arises in [4,10], and the limit process was classified as a generalized diffusion process. In the present paper, we consider a similar problem in the case where the coefficients are not assumed to be smooth and depend irregularly on a small parameter. Moreover, the coefficients may not be uniformly bounded in ϵ at certain points and tend to infinity as $\epsilon \to 0$ or may not have a limit at all. Let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)$ denote the main probability space with filtration $\mathfrak{F}_t$, $t \in [0, T]$, R is the one-dimensional Euclidian space, $(w(t), \mathfrak{F}_t)$ are one-dimensional Wiener processes, and $(\mathcal{C}[0, T], \mathcal{C}_t)$, $t \in [0, T]$, is the space of continuous functions $f(t)$ on the interval $[0, T]$. We use the following notations: $B_0(R)$ is the space of all measurable bounded functions with compact support on R, and $C_0^\infty(R)$ is a subspace of all infinitely differentiated functions from $B_0(R)$. The notations $L_2([0, T] \times R)$, $L_{2,loc}$, and $W^{1,2}_{2,loc}$ (the Sobolev space) have standard sense [5], and $|| \cdot ||_2$ is the norm in L_2. For the weak convergence in $L_{2,loc}$, we use the symbol $\rightharpoonup$. The different positive constants are denoted by Ł.

Consider the one-dimensional Itô stochastic equations

$$(1) \quad \xi^\epsilon(t) = x + \int_0^t (b^\epsilon(\xi^\epsilon(s)) + g^\epsilon(s, \xi^\epsilon(s)))ds + \int_0^t (a^\epsilon(\xi_\epsilon(s)) + A^\epsilon(s, \xi^\epsilon(s)))^{\frac{1}{2}} dw(s).$$

2000 *AMS Mathematics Subject Classification.* Primary 60H10,60J60.
Key words and phrases. Local time, stochastic equation, limit process.

The limit process for the processes $\xi^\epsilon(t)$ changes the Itô type and is a solution of the stochastic equation with a local time

$$\xi(t) = x + \beta L^\xi(t,0) + \int_0^t g(\xi(s))ds + \int_0^t \sigma(\xi(s))dw(s), \tag{2}$$

where $L^\xi(t,0)$ is the symmetric local time of the process $\xi(t)$ at the level 0. The symmetric local time for the continuous semimartingale can be defined in the following way. Let $X(t)$ be a continuous semimartingale with the canonical decomposition $X(t) = X(0) + M(t) + A(t)$, where M stands for a continuous local martingale, and A is a continuous process of finite variation. Then its symmetric local time at the level a is given by the Tanaka formula

$$L^X(t,a) = |X(t)-a| - |X(0)-a| - \int_0^t \operatorname{sgn}(X(s)-a)dX(s),$$

where

$$sgnx = \begin{cases} 1, & \text{for} \quad x>0, \\ 0, & \text{for} \quad x=0, \\ -1, & \text{for} \quad x<0. \end{cases}$$

Under conditions of the theorem proved below, Eq. (2) has unique weak solution by [2, Theorem 4.35].

Let $Du(x)$ denote the symmetric derivative of the function $u(x)$,

$$Du(x) = \lim_{h\to 0} \frac{u(x+h)-u(x-h)}{2h},$$

and the signed measure $n_u(dx)$ on $(R,\mathcal{R})$ is the second derivative of $u(x)$ in the sense of distributions if, for any $\chi(x)\in C_0^\infty$,

$$\int_R \chi''(x)u(x)dx = \int_R \chi(x)n_u(dx).$$

For every convex real function $u(x)$, the generalized Itô formula

$$u(X(t)) = u(X(0)) + \int_0^t Du(X(s))dX(s) + \frac{1}{2}\int L^X(t,y)n_u(dy) \tag{3}$$

holds. Let

$$u(x) = \begin{cases} u_1(x), & \text{for} \quad x\le 0, \\ u_2(x), & \text{for} \quad x\ge 0. \end{cases}$$

Suppose that $u_1(x)$ and $u_2(x)$ are twice continuously differentiable functions for $x\in R$ such that $u_1(0)=u_2(0)$. Then

$$Du(x) = \frac{u_2'(x)+u_1'(x)}{2} + \frac{u_2'(x)-u_1'(x)}{2} sgnx.$$

$$n_u(dx) = (u_2'(0)-u_1'(0))\delta_0(x)dx + N_u(x)dx,$$

where $\delta_0(x)$ is the Dirac function at point 0 and

$$N_u(x) = \frac{u_2''(x)+u_1''(x)}{2} + \frac{u_2''(x)-u_1''(x)}{2} sgnx.$$

Let l and Ł be constants such that $0<l\le Ł<\infty$. We say that the couple of functions $(r,a)\in\mathcal{L}(Ł,l)$, if the functions $r(x)$ and $a(x)$ are measurable functions, and there are the constants $Ł, l>0$ such that

$$|r(x)| + a(x) \le Ł, \qquad a(x)\ge l.$$

Let μ^ϵ, μ be the measures on $(\mathcal{C}[0,T],\mathcal{C}_t)$ corresponding to the random processes $\xi^\epsilon(t)$ and $\xi(t)$, respectively. To indicate the weak convergence of measures, we use the symbol

$\Longrightarrow$. With the coefficients of Eq. (1), we connected the functions

$$F^\epsilon(x) = \exp\left\{-2\int_0^x \frac{b^\epsilon(y)}{a^\epsilon(y)}dy\right\}, \qquad f^\epsilon(x) = \int_0^x F^\epsilon(y)dy.$$

We introduce the following condition.

Condition (I).

I_1. For any $\epsilon > 0$, the functions $b^\epsilon(x)$, $g^\epsilon(t,x)$, $a^\epsilon(x)$, $A^\epsilon(t,x)$ are measurable functions, $l \leq a^\epsilon(x) \leq Ł$, $A^\epsilon(t,x) \geq 0$, and Eq. (1) has a weak solution.

I_2. For any $x \in R$,

$$\left|\int_0^x \frac{b^\epsilon(y)}{a^\epsilon(y)}dy\right| \leq Ł.$$

I_3. There exist functions $r^\epsilon(x)$, $\alpha^\epsilon(t)$, $\alpha^\epsilon(t)$, and $h^\epsilon(t,x)$ such that $|r^\epsilon(x)| \leq Ł$,

$$|g^\epsilon(t,x) - r^\epsilon(x)| + (|b^\epsilon(x)| + 1)A^\epsilon(t,x) \leq \alpha^\epsilon(t) + h^\epsilon(t,x),$$

and

$$\lim_{\epsilon\to 0}\left(||h^\epsilon||_2 + \int_0^T \alpha^\epsilon(t)dt\right) = 0$$

I_4. There exists

$$\leq f^\epsilon(x) = f(x) = \begin{cases} f_1(x), & \text{for} \quad x \leq 0, \\ f_2(x), & \text{for} \quad x \geq 0, \end{cases}$$

where $f_1(x)$ and $f_2(x)$ are twice continuously differentiable monotonically increasing functions for $x \in R$ such that $f_1(0) = f_2(0)$.

By $\phi^\epsilon(x)$, we denote the function inverse to the function $f^\epsilon(x)$. Then

$$\leq \phi^\epsilon(x) = \phi(x) = \begin{cases} \phi_1(x), & \text{for} \quad x \leq 0, \\ \phi_2(x), & \text{for} \quad x \geq 0, \end{cases}$$

where $\phi_i(x)$ is the inverse function for $f_i(x)$, $i = 1, 2$. Set $\beta_1 = f_1'(0)$, $\beta_2 = f_2'(0)$.

Theorem. *Let condition (I) be satisfied and*

$$i) \quad \leq \int_R \chi(y)\frac{1}{F^\epsilon(y)a^\epsilon(y)}dy = \int_R \chi(y)a(y)dy \quad \textit{for any} \quad \chi \in B_0(R)$$

$$ii) \quad \leq \int_R \chi(y)\frac{r^\epsilon(y)}{a^\epsilon(y)}dy = \int_R \chi(y)r(y)dy \qquad \textit{for any} \quad \chi \in B_0(R),$$

iii) the couple of functions $(r + N_\phi(f), a) \in \mathcal{L}(Ł, l)$.

Then $\mu_\epsilon \Longrightarrow \mu$, *and*

$$\beta = \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2}, \qquad \sigma(x) = \frac{1}{\sqrt{a(x)}}, \qquad g(x) = \frac{r(x)}{a(x)} + N_\phi(f(x)).$$

To prove the theorem, we use Theorem 2.1 from [9]. For convenience, we present a complete formulation of this theorem for d=1.

Lemma (Theorem 2.1 [9]). *Let* $x^\epsilon(t)$ *and* $x(t)$ *be solutions of the stochastic equations*

$$x^\epsilon(t) = x^\epsilon + \int_0^t (\gamma^\epsilon(s, x^\epsilon(s)) + B^\epsilon(s, x^\epsilon(s)))ds + \int_0^t (q^\epsilon(s, x^\epsilon(s)) + Q^\epsilon(s, x^\epsilon(s)))^{\frac{1}{2}}dw(s)$$

and

$$x(t) = x + \int_0^t \gamma(s, x(s))ds + \int_0^t q(s, x(s))dw(s).$$

Suppose that the couple $(\gamma^\epsilon, q^\epsilon) \in \mathcal{L}(L, l)$ *and the functions* $\gamma^\epsilon(t,x)$ *and* $q^\epsilon(t,x)$ *satisfy the following conditions (V) and (N).*

Condition (V): There exists a function $V^\epsilon(t,x) \in W^{1,2}_{2,loc}$ *such that*

$$V_1) \qquad \hat{\gamma}^\epsilon \stackrel{def}{=}: \gamma^\epsilon + \frac{1}{2} q^\epsilon \frac{\partial^2 V^\epsilon}{\partial x^2} \rightharpoonup \gamma;$$

$$V_2) \qquad \lim_{\epsilon\to 0} \sup_{t\in[0,T],x\in D} |V^\epsilon(t,x)| = 0, \text{ for any bounded } D \in R;$$

$$V_3) \qquad \lim_{\epsilon\to 0} \left|\left| \frac{\partial V^e}{\partial t} + \gamma^\epsilon \frac{\partial V^\epsilon}{\partial x} + \hat{\gamma}^\epsilon - \gamma \right|\right|_{2,loc} = 0.$$

Condition (N): There exists a function $N^\epsilon(t,x) \in W^{1,2}_{2,loc}$ *such that*

$$N_1) \qquad \hat{q}^e \stackrel{def}{=}: q^\epsilon + q^\epsilon \frac{\partial^2 N^\epsilon}{\partial x^2} \rightharpoonup q;$$

$$N_2) \qquad \lim_{\epsilon\to 0} \sup_{t\in[0,T],x\in D} |N^\epsilon(t,x)| = 0, \text{ for any bounded } D \in R;$$

$$N_3) \qquad \lim_{\epsilon\to 0} \left|\left| \frac{\partial N^e}{\partial t} + \gamma^\epsilon \frac{\partial N^\epsilon}{\partial x} + \frac{1}{2}(\hat{q}^\epsilon - q) \right|\right|_{2,loc} = 0.$$

Let the functions $(\gamma, q) \in \mathcal{L}(L,l)$*. In addition, we assume that, for any bounded domain* $D \in R$*, the following conditions are satisfied:*

$$V_4) \quad \lim_{\epsilon\to 0} \sup_{t\in[0,T],x\in D} \left| \frac{\partial V^\epsilon(t,x)}{\partial x} \right| = 0;$$

$$V_5) \quad \sup_{t\in[0,T],x\in D} \left| \frac{\partial^2 V^\epsilon(t,x)}{\partial x^2} \right| \le \L;$$

$$N_4) \quad \lim_{\epsilon\to 0} \sup_{t\in[0,T],x\in D} \left| \frac{\partial N^\epsilon(t,x)}{\partial x} \right| = 0;$$

$$N_5) \quad \sup_{t\in[0,T],x\in D} \left| \frac{\partial^2 N^\epsilon(t,x)}{\partial x^2} \right| \le \L.$$

Suppose also that

$$|B^\epsilon(t,x)| + Q^\epsilon(t,x) \le \alpha^\epsilon(t) + h^\epsilon(t,x),$$
$$\lim_{\epsilon\to 0}\left(||h^\epsilon||_2 + \int_0^T \alpha^\epsilon(t)dt \right) = 0.$$

Then $x^\epsilon \implies x$.

It is known [7] that the limit coefficients in conditions (V), (N) are uniquely determined.

Proof of the theorem. Denote $\eta^\epsilon(t) = f^\epsilon(\xi^\epsilon(t))$. To use the result of the lemma, we apply the Itô formula for the process $\xi^\epsilon(t)$ and for the function $f^\epsilon(x)$. We have

$$\eta^\epsilon(t) = f^\epsilon(x) + \int_0^t (\gamma^\epsilon(\eta^\epsilon(s)) + B^\epsilon(s,\eta^\epsilon(s))ds$$
$$(4) \qquad + \int_0^t (q^\epsilon(\eta^\epsilon(s)) + Q^\epsilon(s,\eta^\epsilon(s))^{\frac{1}{2}} dw$$

In (4),

$$\gamma^\epsilon(x) = F^\epsilon(\phi^\epsilon(x)) r^\epsilon(\phi^\epsilon(x)),$$
$$B^\epsilon(t,x) = F^\epsilon(\phi^\epsilon(x))(g^\epsilon(t,\phi^\epsilon(x)) - r^\epsilon(\phi^\epsilon(x))) + \frac{1}{2}(F^\epsilon)'(\phi^\epsilon(x)) A^\epsilon(t,\phi^\epsilon(x)),$$
$$q^\epsilon(x) = (F^\epsilon)^2(\phi^\epsilon(x) a^\epsilon(\phi^\epsilon(x)),$$
$$Q^\epsilon(t,x) = (F^\epsilon)^2(\phi^\epsilon(x)) A^\epsilon(t,\phi^\epsilon(x)).$$

Now we verify that the functions $\gamma^\epsilon, q^\epsilon$ satisfy conditions (V), (N) in the lemma and conditions V_4, V_5, N_4, N_5 are valid for the functions $V^\epsilon(x)$ and $N^\epsilon(x)$. From condition i) of the theorem, we have

$$\int_0^x \frac{1}{q^\epsilon(y)}dy = \int_0^{\phi^\epsilon(x)} \frac{1}{F^\epsilon(y)a^\epsilon(y)}dy \underset{\epsilon\to 0}{\longrightarrow} \int_0^{\phi(x)} a(y)dy =$$

$$= \int_0^x a(\phi(y))D\phi(y)dy. \tag{5}$$

Denote

$$N^\epsilon(x) = \int_0^x \int_0^y \left[\frac{1}{q^\epsilon(z)a(\phi(z))D\phi(z)} - 1\right]dzdy \tag{6}$$

From (6), we get

$$q^\epsilon(x) + q^\epsilon(x)\frac{d^2N^\epsilon(x)}{dx^2} = \frac{1}{a(\phi(x))D\phi(x)}.$$

From (6) and (5), we have

$$\leq \sup_{x\in D}\left(|N^\epsilon(x)| + \left|\frac{dN^\epsilon(x)}{dx}\right|\right) = 0.$$

It is obvious that

$$\sup_{x\in D}\left|\frac{d^2N^\epsilon(x)}{dx^2}\right| \leq Ł.$$

So, conditions (N_1) - (N_5) from the lemma are valid, and the limit coefficient equals

$$q(x) = \frac{1}{a(\phi(x))D\phi(x)}.$$

Reasoning similarly and using condition ii) in the theorem, we conclude that the function

$$V^\epsilon(x) = 2\int_0^x \int_0^y \left[\frac{r(\phi(z))}{q^\epsilon(z)a(\phi(z))} - \frac{\gamma^\epsilon(z)}{q^\epsilon(z)}\right]dzdy$$

satisfies conditions (V_1) - (V_5) from the lemma with the limit coefficient

$$\gamma(x) = \frac{r(\phi(x))}{a(\phi(x))}.$$

For the functions $B^\epsilon(t,x)$ and $Q^\epsilon(t,x)$ from condition (I_3), we have

$$|B^\epsilon(t,x)| + Q^\epsilon(t,x) \leq Ł(\alpha^\epsilon(t) + h^\epsilon(t,x)).$$

Thus, $\eta^\epsilon(t) \implies \eta(t)$, where

$$\eta(t) = f(x) + \int_0^t \gamma(\eta(s))ds + \int_0^t \sqrt{q(\eta(s))}dw(s). \tag{7}$$

As follows from condition (I_2), the limit $\leq f_\epsilon(x) = f(x)$ and $\leq \phi_\epsilon(x) = \phi(x)$ uniformly on the compact sets. From Theorem 1.5.5 [1], we conclude that $\xi^\epsilon(t) \implies \xi(t) = \phi(\eta(t))$. Observing that $D\phi(f(x)) = 1$ for $x \neq 0$, we can rewrite Eq. (7) as

$$\eta(t) = f(x) + \int_0^t \frac{r(\xi(s))}{a(\xi(s))}ds + \int_0^t \frac{1}{\sqrt{a(\xi(s))}}dw(s). \tag{8}$$

Using formula (3) for the function $\phi(x)$ and for the process $\eta(t)$ from (8), we get the statement of the theorem by Lemma 1 [8]. The theorem is proved.

Consider the model example. Introduce the functions

$$\alpha^\epsilon(t) = \frac{\epsilon^3|2t-1|}{[t(t-1)+\epsilon^2]^2},\ \ h^\epsilon(x) = \frac{\epsilon^{\frac{1}{8}}}{(2\pi\epsilon)^{\frac{1}{4}}}\exp\left\{-\frac{x^2}{4\epsilon}\right\},\ \ \tau^\epsilon(t,x) = \alpha^\epsilon(t) + h^\epsilon(x),$$

and study the solutions of the stochastic equations

$$\xi^\epsilon(t) = x + \frac{1}{\epsilon}\int_0^t b\left(\frac{\xi^\epsilon(s)}{\epsilon}\right)ds + \int_0^t \left[g\left(\frac{\xi^\epsilon(s)}{\epsilon}\right) + \tau^\epsilon(s, \xi^\epsilon(s))\right]ds + \tag{9}$$
$$+ \int_0^t \sigma\left(\frac{\xi^\epsilon(s)}{\epsilon}\right)dw(s).$$

It is obvious that, for $t = 0$ or $t = 1$ or $x = 0$, the function $\tau^\epsilon(t, x)$ tends to infinity as $\epsilon \to 0$, but condition I_3 is valid.

Suppose that

$$\left|\int_0^x \frac{b(y)}{\sigma^2(y)}dy\right| < const, \quad \int_{-\infty}^0 \frac{b(y)}{\sigma^2(y)}dy = B_1, \quad \int_0^\infty \frac{b(y)}{\sigma^2(y)}dy = B_2$$

and that the limits

$$\lim_{|x|\to\infty} \frac{1}{x}\int_0^x \frac{g(y)}{\sigma^2(y)}dy = A_1,$$
$$\lim_{|x|\to\infty} \frac{1}{x}\int_0^x \frac{dy}{\sigma^2(y)} = A_2 > 0$$

exist. In this case, the conditions of the theorem are valid, and

$$\beta_1 = \exp(2B_1), \beta_2 = \exp(-2B_2), \beta = th(B_1 + B_2), a(x) = \frac{1}{A_2}, r(x) = \frac{A_1}{A_2}, N_\phi(x) = 0.$$

Then the limit process for Eq. (9) is

$$\xi(t) = x + th(B_1 + B_2)L^\xi(t, 0) + \frac{A_1}{A_2}t + \frac{1}{\sqrt{A_2}}w(t)$$

Bibliography

1. P.Billingsley, *Convergence of probability measures*, Wiley, New York, 1968.
2. H.J.Engelbert, W.Schmidt., *Strong Markov continuous local martingales and solutions of one dimensional stochastic differential equations.I, II, III*, Math.Nachr., **143,144,151** (1989, 1989, 1991), 167-184, 241-281, 149- 197.
3. J.-F.Le Gall, *One-dimensional stochastic equations involving local times of the unknown processes*, Lecture Notes in Mathematics **1095** (1983), 51-82.
4. G. L. Kulinich, *On necessary and sufficient conditions for the convergence of solutions of one dimensional diffusion stochastic equations with a non-regular dependence of coefficients on a parameter*, Theory Prob. Appl. **27** (1982), no. 4, 795–802.
5. O.A.Ladyzhenskaya, V.A.Solonnikov, N.N. Uraltzeva, *Linear and quasilinear equations of parabolic type*, AMS, Providence, RI., 1968.
6. R.Sh.Liptser, A.N.Shiryaev, *Theory of martingales*, Kluwer, Dordrecht, 1989.
7. S.Makhno, *Convergence of diffusion processes, I, II*, Ukr. Math. J. **44** (1992), 284 - 289, 1389 - 1395.
8. S.Makhno, *Limit theorem for one dimensional stochastic equation*, Theory Probab. Appl. **48** (2002), 156 - 161.
9. S.Makhno, *Preservation of the convergence of solutions of stochastic equations under the perturbation of their coefficients*, Ukr. Math. Bull. **1** (2004), 251 - 264.
10. N.I.Portenko, *Generalized Diffusion Processes*, AMS, Providence, RI., 1990.
11. Rosenkrants W., *Limit theorems for solutions to a class of stochastic differential equations*, Indiana Univ. Math. J. **24** (1975), 613 - 624.

Institute for Applied Mathematics and Mechanics, R.Luxemburg str. 74, Donetsk 83114, Ukraine

E-mail: makhno@iamm.ac.donetsk.ua