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EXIT FROM AN INTERVAL BY A DIFFERENCE OF TWO RENEWAL PROCESSES

Integral transforms of the joint distribution of the first exit time from an interval, the value of the overshoot through a boundary, and the value of a linear component at the epoch of the exit are determined for the difference of two renewal processes.

1. Main definitions

Let η, ξ, $\varkappa$, $\delta \in (0,\infty)$ be positive random variables. By $F(x) = \mathbf{P}\,[\,\eta \le x\,]$, $G(y) = \mathbf{P}\,[\,\xi \le y\,]$ we denote the distribution functions of the random variables η, ξ. Introduce the sequences $\{\eta, \eta_n\}$, $\{\xi, \xi_n\}$, $n \in \mathbb{N}$, of independent identically distributed variables and, for $x, y \ge 0$, define the random sequences

$$\begin{aligned} &\eta_0(x) = 0, \quad \eta_1(x) = \eta_x, \quad \eta_{n+1}(x) = \eta_x + \eta_1 + \cdots + \eta_n, \qquad n \in \mathbb{N}, \\ &\xi_0(y) = 0, \quad \xi_1(y) = \xi_y, \quad \xi_{n+1}(y) = \xi_y + \xi_1 + \cdots + \xi_n, \qquad n \in \mathbb{N}, \end{aligned} \tag{1}$$

where η_x, ξ_y are the random variables given by their distribution functions

$$\mathbf{P}\,[\,\eta_x \le u\,] = \frac{F(x+u) - F(x)}{1 - F(x)}, \qquad \mathbf{P}\,[\,\xi_y \le v\,] = \frac{G(y+v) - G(y)}{1 - G(y)}, \qquad u, v \ge 0.$$

For all $x, y \ge 0$, we define the renewal processes generated by sequences (1):

$$\alpha_y(t) = \max\{\, n \in \mathbb{N} \cup 0 : \ \xi_n(y) \le t\,\}, \qquad \beta_x(t) = \max\{\, n \in \mathbb{N} \cup 0 : \ \eta_n(x) \le t\,\}.$$

Introduce the monotone independent random walks generated by $\varkappa$, δ:

$$\varkappa_0 = 0, \quad \varkappa_n = \varkappa_1' + \cdots + \varkappa_n', \qquad \delta_0 = 0, \quad \delta_n = \delta_1' + \cdots + \delta_n', \quad n \in \mathbb{N}.$$

Here, $\{\varkappa, \varkappa_n'\}$, $\{\delta, \delta_n'\}$, $n \in \mathbb{N}$, are the sequences of independent identically distributed random variables. Define the right-continuous step process

$$\{D_{xy}(t)\}_{\{t\ge 0\}} = \varkappa_{\alpha_y(t)} - \delta_{\beta_x(t)} \in \mathbb{R}, \qquad x, y \ge 0; \qquad D_{xy}(0) = 0. \tag{2}$$

This process takes positive jumps at the epochs $\xi_n(y)$, $n \in \mathbb{N}$, of the value $\varkappa_n'$. At the epochs $\eta_n(x)$ $n \in \mathbb{N}$, there occur negative jumps of the value δ_n'. Let $\{\,D_{xy}(t)\,\}_{\{t\ge 0\}}$ be the difference of two renewal processes. Note that this process is not Markovian (apart from the case where η, ξ are exponentially distributed). For all $t \ge 0$, we introduce the right-continuous linear components

$$\eta_x^+(t) = \begin{cases} t + x, & 0 \le t < \eta_x, \\ t - \eta_{\beta_x(t)}(x), & t \ge \eta_x \end{cases} \in \mathbb{R}_+, \qquad x \ge 0, \tag{3}$$

$$\xi_y^+(t) = \begin{cases} t + y, & 0 \le t < \xi_y, \\ t - \xi_{\alpha_y(t)}(y), & t \ge \xi_y \end{cases} \in \mathbb{R}_+, \qquad y \ge 0. \tag{4}$$

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The process $\{\eta_x^+(t)\}_{\{t\geq 0\}}$ increases linearly on the intervals $[\eta_n(x),\, \eta_{n+1}(x))$, $n \in \mathbb{N}\cup 0$, and vanishes to zero at the epochs $\eta_n(x)$, $n \in \mathbb{N}$. The value of the process at the time $t_0 \geq \eta_x$ is the time elapsed since the last negative jump of the process $\{D_{xy}(t)\}_{\{t\geq 0\}}$ till t_0. The process $\{\xi_x^+(t)\}_{\{t\geq 0\}}$ increases linearly on the interval $[\xi_n(x),\, \xi_{n+1}(x))$, $n \in \mathbb{N}\cup 0$, and vanishes to zero at the epochs $\xi_n(x)$, $n \in \mathbb{N}$. The value of the process at the time $t_0 \geq \xi_x$ is the time elapsed since the last positive jump of the process $\{D_{xy}(t)\}_{\{t\geq 0\}}$ till t_0.

Define a right-continuous Markov process

$$(5) \qquad Y_{xy}^t = \big\{D_{xy}(t),\, \eta_x^+(t),\, \xi_y^+(t)\big\}_{\{t\geq 0\}} \in \mathbb{R}\times\mathbb{R}_+^2, \qquad Y_{xy}^0 = \{0,x,y\}, \quad x,y \geq 0,$$

accompanying (governing) the process $\{D_{xy}(t)\}_{\{t\geq 0\}}$. The so defined process is a Markov process which is homogeneous in the first component [1]. If, at the instant t_0, $Y_{xy}^{t_0} = \{k,u,v\}$, then the future behaviour of the process $\{Y_{xy}^t\}_{\{t\geq t_0\}}$ does not depend on the value k of the first component, and the first negative jump of the process $\{D_{xy}(t)\}_{\{t\geq t_0\}}$ of the value δ occurs after the random time η_u has elapsed, and the process $\{D_{xy}(t)\}_{\{t\geq t_0\}}$ will take the first positive jump of the value $\varkappa$ after the time ξ_v has elapsed.

For $k \in \mathbb{R}_+$, we define

$$\tau_{xy}^k = \inf\{\, t:\ D_{xy}(t) > k\,\}, \qquad T_{xy}^k = D_{xy}(\tau_{xy}^k) - k, \qquad \eta_{xy}^k = \eta_x^+(\tau_{xy}^k)$$

which are the first passage time of the upper level k by the process $\{D_{xy}(t)\}_{\{t\geq 0\}}$, the value of the overshoot through the upper level k, and the value of the linear component $\eta_x^+(\cdot)$ at the passage instant. Note that the overleap of the upper level can take place only at the epochs $\xi_n(y)$, $n \in \mathbb{N}$, and, in view of (4), $\xi_y^+(\tau_{xy}^k) = 0$.

Similarly, for $k \in \mathbb{R}_+$, we introduce the random variables

$$\tau_k^{xy} = \inf\{\, t:\ D_{xy}(t) < -k\,\}, \qquad T_k^{xy} = -D_{xy}(\tau_k^{xy}) - k, \qquad \xi_k^{xy} = \xi_y^+(\tau_k^{xy})$$

which are the first passage time of the lower level $-k$ by the process $\{D_{xy}(t)\}_{\{t\geq 0\}}$, the value of the overshoot, and the value of the linear component $\xi_y^+(\cdot)$ at the passage instant. Observe that the overleap of the lower level occurs at the epochs $\eta_n(x)$, $n \in \mathbb{N}$, and, in view of (3), $\eta_x^+(\tau_k^{xy}) = 0$. Note, that the random variables τ_{xy}^k, τ_k^{xy} are the Markov times of the process $\{Y_{xy}^t\}_{\{t\geq 0\}}$, because they take values from countable sets $\{\,\xi_n(y),\ n \in \mathbb{N}\}$, $\{\,\eta_n(x),\ n\in\mathbb{N}\}$. The integral transforms of the joint distributions

$$(6) \qquad \big\{\tau_{xy}^k,\, T_{xy}^k,\, \eta_{xy}^k\big\}, \qquad \big\{\tau_k^{xy},\, T_k^{xy},\, \xi_k^{xy}\big\}, \qquad k,x,y \in \mathbb{R}_+,$$

have been obtained in [2] in terms of the solutions of integral equations. For particular types of the process $\{Y_{xy}^t\}_{\{t\geq 0\}}$, the integral transforms of distribution (6) have been studied in [3]–[6] and also in [8], [9]. We assume that the integral transforms of the joint distributions of the boundary functionals (6) are known, and we will solve the two-boundary problem for the process $\{D_{xy}(t)\}_{\{t\geq 0\}}$ in terms of these integral transforms.

2 Exit of the process $\{D_{xy}(t)\}_{\{t\geq 0\}}$ from the interval

Let us fix $B \geq 0$, $k \in [0,B]$, $r = B-k$, $Y_{xy}^0 = \{0,x,y\}$, $x,y \geq 0$, and introduce the random variable

$$\chi = \inf\{\, t:\ D_{xy}(t) \notin [-r,k]\,\}$$

as the first exit time from the interval $[-r,k]$ by the process $\{D_{xy}(t)\}_{\{t\geq 0\}}$. This random variable is the Markov time of the process $\{Y_{xy}^t\}_{\{t\geq 0\}}$, since it takes values from a countable set $\{\,\xi_n(y),\ n\in\mathbb{N}\}\cup\{\,\eta_n(x),\ n\in\mathbb{N}\}$. Note that the exit can take place either through the upper boundary k, or through the lower boundary $-r$. We introduce the events $A^k = \{\, D_{xy}(\chi) > k\,\}$ when the process exits the interval by crossing the upper boundary and $A_r = \{\, D_{xy}(\chi) < -r \,\}$ when the process exits the interval by crossing the lower boundary. Define the random variable,

$$X = (D_{xy}(\chi) - k)\,\mathbf{I}_{A^k} + (-D_{xy}(\chi) - r)\,\mathbf{I}_{A_r}, \qquad L = \eta_x^+(\chi)\,\mathbf{I}_{A^k} + \xi_y^+(\chi)\,\mathbf{I}_{A_r}$$

which are the value of the overshoot through the interval $[-r, k]$ by the process

$$\{D_{xy}(t)\}_{\{t\geq 0\}}$$

at the epoch of the first exit and the value of the linear component at the exit epoch, where $\mathbf{I}_A = \mathbf{I}_A(\omega)$ is the indicator function of the set A. Denote

$$\begin{aligned}
&f_{xy}^k(du, dl, s) = \mathbf{E}\left[\exp\{-s\tau_{xy}^k\};\ T_{xy}^k \in du,\ \eta_{xy}^k \in dl\right], \qquad k, x, y \geq 0,\\
&f_k^{xy}(du, dl, s) = \mathbf{E}\left[\exp\{-s\tau_k^{xy}\};\ T_k^{xy} \in du,\ \xi_k^{xy} \in dl\right], \qquad k, x, y \geq 0,\\
&V_s^+(du, dl) = \mathbf{E}\left[\, e^{-s\chi};\ X \in du, L \in dl, A^k\,\right],\\
&V_s^-(du, dl) = \mathbf{E}\left[\, e^{-s\chi};\ X \in du, L \in dl, A_r\right].
\end{aligned}$$

Theorem 1. *Let $B \geq 0$, $k \in [0, B]$, $r = B - k$, $Y_{xy}^0 = \{0, x, y\}$, $x, y \geq 0$. Then the Laplace transforms $V_s^{\pm}(du, dl)$ of the joint distribution of $\{\,\chi,\ X,\ L\,\}$, being the first exit time from the interval $[-r, k]$ by the process $\{D_{xy}(t)\}_{\{t\geq 0\}}$, the value of overshoot through the boundary, and the value of the linear component at the exit epoch, satisfy the equalities*

$$\begin{aligned}
&V_s^+(du, dl) = F_{xy}^+(k, du, dl, s) + \iint_{\mathbb{R}_+^2} F_{xy}^+(k, du_1, dl_1, s)\,\mathbf{K}_{u_1 l_1}^+(du, dl, s),\\
&V_s^-(du, dl) = F_{xy}^-(k, du, dl, s) + \iint_{\mathbb{R}_+^2} F_{xy}^-(r, du_1, dl_1, s)\,\mathbf{K}_{u_1 l_1}^-(du, dl, s),
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
F_{xy}^+(k, du, dl, s) &= f_{xy}^k(du, dl, s) - \iint_{\mathbb{R}_+^2} f_r^{xy}(du_1, dl_1, s)\, f_{0 l_1}^{u_1+B}(du, dl, s),\\
F_{xy}^-(r, du, dl, s) &= f_r^{xy}(du, dl, s) - \iint_{\mathbb{R}_+^2} f_{xy}^k(du_1, dl_1, s)\, f_{u_1+B}^{l_1 0}(du, dl, s);
\end{aligned}$$

$\mathbf{K}_{u_1 l_1}^{\pm}(du, dl, s) = \sum\limits_{n=1}^{\infty} K_{u_1 l_1}^{\pm}(du, dl, s)^{*(n)}$ *are the series of iterations* $K_{u_1 l_1}^{\pm}(du, dl, s)^{*(n)}$;

$$\begin{aligned}
&K_{u_1 l_1}^{\pm}(du, dl, s)^{*(1)} = K_{u_1 l_1}^{\pm}(du, dl, s),\\
&K_{u_1 l_1}^{\pm}(du, dl, s)^{*(n+1)} = \iint_{\mathbb{R}_+^2} K_{u_1 l_1}^{\pm}(du_2, dl_2, s)\, K_{u_2 l_2}^{\pm}(du, dl, s)^{*(n)}, \quad n \in \mathbb{N},
\end{aligned} \tag{8}$$

are the successive iterations of the kernels $K_{u_1 l_1}^{\pm}(du, dl, s)$ which are given by the formulae

$$\begin{aligned}
K_{u_1 l_1}^+(du, dl, s) &= \iint_{\mathbb{R}_+^2} f_{u_1+B}^{l_1 0}(du_2, dl_2, s)\, f_{0 l_2}^{u_2+B}(du, dl, s),\\
K_{u_1 l_1}^-(du, dl, s) &= \iint_{\mathbb{R}_+^2} f_{0 l_1}^{u_1+B}(du_2, dl_2, s)\, f_{u_2+B}^{l_2 0}(du, dl, s).
\end{aligned} \tag{9}$$

Proof. The integral transforms of the joint distribution of the first exit time and the value of the overshoot through the boundaries have been determined for the processes with independent increments and random walks in [7]. These integral transforms were obtained in terms of the distributions of one-boundary functionals of the process and the random walk. We used the idea (utilizing the one-boundary functional of the process) and the method (solving a system of linear integral equations) for this problem in the case where the process is a difference of two renewal processes. Following this framework, we derive a system of integral equations to determine the function $V_s^{\pm}(du, dl)$:

$$
\begin{aligned}
& f_{xy}^{k}(du, dl, s) = V_s^{+}(du, dl) + \iint_{\mathbb{R}_+^2} V_s^{-}(du_1, dl_1)\, f_{0l_1}^{u_1+B}(du, dl, s), \\
(10) \qquad & f_r^{xy}(du, dl, s) = V_s^{-}(du, dl) + \iint_{\mathbb{R}_+^2} V_s^{+}(du_1, dl_1)\, f_{u_1+B}^{l_1 0}(du, dl, s).
\end{aligned}
$$

To derive these equations, we applied the total probability law and also used the fact that the process $\{Y_{xy}^t\}_{\{t\geq 0\}}$ is homogeneous in the first component and τ_{xy}^k, τ_k^{xy}, and χ are Markov times. To get the first equation of (10), observe that the first crossing through the upper boundary k by the process $\{D_{xy}(t)\}_{\{t\geq 0\}}$ (the expression on the left-hand side of the equation) occurs on a sample path which does not intersect the lower level $-r$ (the first term on the right-hand side) or on the path which does intersect the lower level $-r$ and then crosses the upper boundary k (the second term on the right-hand side). The second equation is derived analogously. This system of integral equations is similar to a system of linear equations with two variables. Substituting the expression for $V_s^{-}(du, dl)$ from the second equation into the first one yields

$$
\begin{aligned}
V_s^{+}(du, dl) = f_{xy}^{k}(du, dl, s) &- \iint_{\mathbb{R}_+^2} f_r^{xy}(du_1, dl_1, s)\, f_{0l_1}^{u_1+B}(du, dl, s) + \\
&+ \iint_{u_1, l_1 \in \mathbb{R}_+^2} \iint_{u_2, l_2 \in \mathbb{R}_+^2} V_s^{+}(du_2, dl_2)\, f_{u_2+B}^{l_2 0}(du_1, dl_1, s)\, f_{0l_1}^{u_1+B}(du, dl, s).
\end{aligned}
$$

Changing the order of integration in the third term of the right-hand side of this equation, we obtain a linear integral equation for the function $V_s^{+}(du, dl)$:

$$
(11) \qquad V_s^{+}(du, dl) = F_{xy}^{+}(k, du, dl, s) + \iint_{\mathbb{R}_+^2} V_s^{+}(du_1, dl_1)\, K_{u_1 l_1}^{+}(du, dl, s).
$$

Here,

$$
K_{u_1 l_1}^{+}(du, dl, s) = \iint_{\mathbb{R}_+^2} f_{u_1+B}^{l_1 0}(du_2, dl_2, s)\, f_{0l_2}^{u_2+B}(du, dl, s)
$$

is the kernel of this equation. Observe that the random variable τ_k^{x0} takes values from the set $\{\eta_n(x),\ n \in \mathbb{N}\}$. Therefore, for $s > 0$,

$$
f_k^{x0}(du, dl, s) \leq \mathbf{E} \exp\{-s\tau_k^{x0}\} \leq f_x(s) \leq f^*(s),
$$

where $f_x(s) = \mathbf{E}\exp\{-s\eta_x\}$ and $f^*(s) = \sup_{x\geq 0} f_x(s)$. The analogous estimation is valid for the function $f_{0y}^{k}(du, dl, s)$:

$$
f_{0y}^{k}(du, dl, s) \leq \mathbf{E}\exp\{-s\tau_{0y}^{k}\} \leq g_y(s) \leq g^*(s),
$$

where $g_y(s) = \mathbf{E}\exp\{-s\xi_y\}$, $g^*(s) = \sup_{y\geq 0} g_y(s)$. Then, for $s > s_0 > 0$, the kernel $K_{u_1 l_1}^{+}(du, dl, s)$ satisfies the estimation

$$
K_{u_1 l_1}^{+}(du, dl, s) \leq f^*(s)\, g^*(s) \leq \lambda = f^*(s_0)\, g^*(s_0) < 1, \qquad s_0 > 0.
$$

Utilizing formula (8) defining the successive iterations of this kernel by the method of mathematical induction, we get, for $s > s_0 > 0$,

$$
K_{u_1 l_1}^{+}(du, dl, s)^{*(n+1)} < \lambda^{n+1}, \qquad n \in \mathbb{N}.
$$

Then the series of successive iterations converges uniformly with respect to $u_1, l_1, u, l \in \mathbb{R}_+$, $s > s_0$, and

$$
\mathbf{K}_{u_1 l_1}^{+}(du, dl, s) < \lambda(1-\lambda)^{-1}.
$$

Therefore, we can apply the method of successive iterations [10] to solve the linear integral equation (11). This yields the first equality of the theorem. The second equality of the theorem can be proved analogously.

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