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PARAMETER ESTIMATORS OF NONLINEAR QUANTILE REGRESSION

We have obtained the asymptotic normality of parameter estimators of a nonlinear quantile regression with nonsymmetric random noise.

INTRODUCTION

Here, we examine the asymptotic normality of Koenker and Basset estimators [1] or the generalized least moduli estimators (GLME) of nonlinear regression model parameters that generalize least moduli estimators for non-symmetric observation errors.

The consistency property of GLME has been considered in [2].

1. ASSUMPTIONS AND THE MAIN RESULT

Suppose that an observation X_j is a r.v. with values in $(\mathbb{R}^1, \mathcal{B}^1)$ (\mathbb{R}^1 is a real line, \mathcal{B}^1 - σ -algebra of its Borel subsets) and distribution P_j . We also assume that the unknown distribution P_j belongs to a certain parametric family $\{P_{i\theta}, \theta \in \Theta\}$. We call the triple $\mathcal{E}_j = \{\mathbb{R}^1, \mathcal{B}^1, P_{j\theta}, \theta \in \Theta\}$ a statistical experiment generated by the observation X_j .

We say that a statistical experiment $\mathcal{E}^n = \{\mathbb{R}^n, \mathcal{B}^n, P_\theta^n, \theta \in \Theta\}$ is the product of the statistical experiments $\mathcal{E}_i, i = 1, \dots, n$, if $P_\theta^n = P_{1\theta} \times \dots \times P_{n\theta}$ (\mathbb{R}^n - n -dimensional Euclidean space and \mathcal{B}^n - σ -algebra of its Borel subsets). We say that the experiment \mathcal{E}^n is generated by n independent observations $X = (X_1, \dots, X_n)$.

Let the observations have the form

$$(1.1) \quad X_j = g(j, \theta) + \varepsilon_j, \quad j = 1, \dots, n,$$

where $g(j, \theta)$ is a non-random sequence of functions defined on Θ^c , Θ^c is the closure of an open convex set $\Theta \subset \mathbb{R}^q$ in \mathbb{R}^q , and

A1. ε_j are independent identically distributed random variables (r.v.) with zero mean, distribution function \mathcal{P} , and

$$(1.2) \quad \mathcal{P}(0) = \beta, \quad \beta \in (0, 1).$$

It is not supposed that the functions $g(j, \theta)$ are the linear forms of coordinates of the vector θ .

Definition. *GLME of the parameter $\theta \in \Theta$ obtained by the observations $X_j, j = 1, \dots, n$ of the form (1.1) is said to be any random vector $\hat{\theta}_n = \hat{\theta}_n(X_j, j = 1, \dots, n) \in \Theta^c$ having the property*

$$(1.3) \quad S_\beta(\hat{\theta}_n) = \inf_{\tau \in \Theta^c} S_\beta(\tau), \quad S_\beta(\tau) = \sum \rho_\beta(X_j - g(j, \tau)),$$

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where $\sum = \sum_{j=1}^n$ and

$$(1.4) \quad \rho_\beta(x) = \begin{cases} \beta x, & x \geq 0 \\ (\beta - 1)x, & x < 0 \end{cases}, \quad \beta \in (0, 1).$$

Since $P_\theta^n\{X_j < g(j, \theta)\} = P_\theta^n\{\varepsilon_j < 0\} = F(0) = \beta$, the observation model (1.1) can be interpreted as a nonlinear quantile regression [1]. Indeed, $\hat{\theta}_n$ estimates the β -quantile $g(j, \theta)$ of observations X_j , $j = 1, \dots, n$.

Let us impose some restrictions on r.v. ε_j :

A2. $\mu_s = E|\varepsilon_j|^s < \infty$ for some natural s .

A3. R.v. ε_j has a bounded density $p(x) = \mathcal{P}'(x)$ with the property

$$|p(x) - p(0)| \leq H|x|, \quad p(0) > 0,$$

where $H < \infty$ is a certain constant.

Example. A r.v. $\xi = \chi_{2m}^2 - 2m$, where χ_{2m}^2 has chi-squared distribution with even degrees of freedom, satisfies conditions **A1-A3**.

Denote, by $\mathcal{C}^q \subset \mathcal{B}^q$, the class of all convex Borel subsets of \mathbb{R}^q and, by $T \subset \Theta$, some compact.

Let us introduce the notation

$$g_i(j, \tau) = \frac{\partial}{\partial \tau^i} g(j, \tau), \quad g_{il}(j, \tau) = \frac{\partial^2}{\partial \tau^i \partial \tau^l} g(j, \tau), \\ d_{in}^2(\theta) = \sum g_i^2(j, \theta), \quad d_{il,n}^2(\tau) = \sum g_{il}^2(j, \tau), \quad \tau \in \Theta^c, \quad i, l = 1, \dots, q.$$

Here, $d_n^2(\theta)$ is a diagonal matrix with elements $d_{in}^2(\theta)$, $i = 1, \dots, q$ on the diagonal.

Consider the change of variables $u = n^{-1/2}d_n(\theta)(\tau - \theta)$, i.e.

$$g(j, \tau) = g(j, \theta + n^{1/2}d_n^{-1}(\theta)u) = f(j, u),$$

assuming that θ is a true value of the parameter. Under this change of variables, the set Θ turns to the set $\tilde{U}_n(\theta) = n^{-1/2}U_n(\theta)$, where $U_n(\theta) = d_n(\theta)(\Theta - \theta)$, and GLME $\hat{\theta}_n$ turns to a normed random vector $\hat{u}_n = n^{-1/2}d_n(\theta)(\hat{\theta}_n - \theta)$.

We will denote positive constants by the letter k . Suppose that

B1. Functions $g(j, \theta)$, $j \geq 1$ are continuous on Θ^c together with all the first partial derivatives, and $g_i(j, \theta)$, $i = 1, \dots, q$, $j \geq 1$, are continuously differentiable in Θ . Moreover, for any $R \geq 0$,

$$(1.5) \quad (i) \quad \sup_{\theta \in T} \sup_{u \in v(R) \cap \tilde{U}_n^c(\theta)} \max_{1 \leq j \leq n} \frac{|f_i(j, u)|}{d_{in}(\theta)} \leq k^i(R)n^{-1/2}, \quad i = 1, \dots, q,$$

$$(1.6) \quad (ii) \quad \sup_{\theta \in T} \sup_{u \in v(R) \cap \tilde{U}_n^c(\theta)} \frac{d_{il,n}(\theta + n^{1/2}d_n^{-1/2}(\theta)u)}{d_{in}(\theta)d_{ln}(\theta)} \leq k^{il}(R)n^{-1/2}, \quad i, l = 1, \dots, q.$$

It follows from (1.5) that

$$(1.7) \quad \sup_{\theta \in T} \sup_{u_1, u_2 \in v^c(R) \cap \tilde{U}_n^c(\theta)} n^{-1} \frac{\Phi_n(u_1, u_2)}{|u_1 - u_2|^2} \leq k(R),$$

where $\Phi_n(u_1, u_2) = \sum (f(j, u_1) - f(j, u_2))^2$.

Similarly, relation (1.6) yields the inequality

$$(1.8) \quad \sup_{\theta \in T} \sup_{u_1, u_2 \in v^c(R) \cap \tilde{U}_n^c(\theta)} \frac{\Phi_n^{(i)}(u_1, u_2)}{d_{in}^2(\theta)|u_1 - u_2|^2} \leq \tilde{k}^{(i)}(R),$$

with $\Phi_n^{(i)}(u_1, u_2) = \sum ((f_i(j, u_1) - f_i(j, u_2))^2, \quad i = 1, \dots, q.$

Suppose that GLME is consistent, namely:

C. For any $r > 0$

$$\sup_{\theta \in T} P_{\theta}^n \{ |n^{-1/2} d_n(\theta)(\hat{\theta}_n - \theta)| \geq r \} = \begin{cases} O(n^{-s+1}), & s \geq 2, \\ o(1), & s = 1. \end{cases}$$

The sufficient conditions for **C** to be fulfilled are stated in [2].

Let us denote

$$I(\theta) = \left(d_{in}^{-1}(\theta) d_{in}^{-1}(\theta) \sum_{i,l=1}^q g_i(j, \theta) g_l(j, \theta) \right)_{i,l=1}^q, \quad \theta \in \Theta.$$

The matrix $I(\theta)$ is symmetric and non-negative definite. Let $\lambda_{\min}(I(\theta))$ be the smallest eigenvalue of $I(\theta)$. Assume that

B2. For $n > n_0$, $\inf_{\theta \in T} \lambda_{\min}(I(\theta)) \geq \lambda_0 > 0$.

Let l be an arbitrary direction in \mathbb{R}^q , and $\tau \in \Theta$. Then

$$\frac{\partial}{\partial l} S_{\beta}(\tau) = \sum \langle \nabla g(j, \tau), l \rangle (\chi \{ X_j * g(j, \tau) - \beta \}),$$

where " $*$ " denotes " \leq " if $\langle \nabla g(j, \tau), l \rangle \geq 0$ and " $<$ " if $\langle \nabla g(j, \tau), l \rangle < 0$. Let r_0 be a distance between T and $\mathbb{R}^q \setminus \Theta$. If an event $\{ |\hat{\theta}_n - \theta| < r \}$ occurs for $\theta \in T$ and $r < r_0$, then, for any direction l ,

$$\frac{\partial}{\partial l} S_{\beta}(\hat{\theta}_n) \geq 0.$$

This remark will be used in the proof of the main result.

Theorem. *If conditions **A1** - **A3**, **B1**, **B2**, and **C** are fulfilled, then*

$$(1.9) \quad \sup_{\theta \in T} \sup_{C \in \mathcal{C}^q} \left| P_{\theta}^n \left\{ \frac{p(0)}{\sqrt{\beta(1-\beta)}} I^{1/2}(\theta) d_n(\theta)(\hat{\theta}_n - \theta) \in C \right\} - \Phi(C) \right| \xrightarrow[n \rightarrow \infty]{} 0,$$

where $\Phi(C) = \int_C \frac{1}{(2\pi)^{q/2}} e^{-\frac{\|x\|^2}{2}} dx$.

In other words, the normal distribution $N\left(0, \frac{\beta(1-\beta)}{p^2(0)} I^{-1}(\theta)\right)$ is the accompanying law for the distribution of the normed estimator $d_n(\theta)(\hat{\theta}_n - \theta)$.

2. AUXILIARY ASSERTIONS

We carry out the proof by the scheme of the theorem on asymptotic normality of the least moduli estimators [3], by using the method of partitioning a parametric set [4,5].

Let l_1, \dots, l_q be the positive directions of the coordinate axes. Let us consider the vectors $S_{\beta}^{\pm}(\tau)$ with coordinates

$$S_{i\beta}^{\pm}(\tau) = d_{in}^{-1}(\theta) \left(\frac{\partial}{\partial(\pm l_i)} \right) S_{\beta}(\tau), \quad i = 1, \dots, q,$$

and the vectors $E_{\theta}^n S_{\beta}^{\pm}(\theta)$ with coordinates

$$E_{\theta}^n S_{i\beta}^{\pm}(\tau) = \pm d_{in}^{-1}(\theta) \sum g_i(j, \tau) [\mathcal{P}(g(j, \tau) - g(j, \theta)) - \beta], \quad i = 1, \dots, q.$$

Clearly,

$$E_{\theta}^n S_{\beta}^{\pm}(\theta) = 0,$$

due to assumption **A1**. Let us denote $S_\beta^{\pm}(u) = S_\beta^{\pm}(\theta + n^{1/2}d_n^{-1}(\theta)u)$ and

$$z_n^\pm(\theta, u) = \frac{\left| S_\beta^{\pm}(u) - S_\beta^{\pm}(0) - E_\theta^n S_\beta^{\pm}(u) \right|}{1 + \left| E_\theta^n S_\beta^{\pm}(u) \right|}.$$

Lemma 1. *Under the conditions of the theorem, for any $\epsilon > 0$ and sufficiently small $r > 0$,*

$$(2.1) \quad \sup_{\theta \in T} P_\theta^n \left\{ \sup_{u \in v^c(r) \cap \bar{U}_n^c(\theta)} z_n^\pm(\theta, u) > \epsilon \right\} \xrightarrow{n \rightarrow \infty} 0.$$

Proof. We will prove the statement for $z_n^+(\theta, u)$. Assume, for simplicity, that $r = 1$ and the inner supremum in (2.1) is defined in a cube

$$C_0 = \left\{ u : |u|_0 = \max_{1 \leq i \leq q} |u_i| \leq 1 \right\} \supset v(1).$$

Let us cover the cube C_0 with $N_0 = O(\ln n)$ cubes $C_{(1)}, \dots, C_{(N_0)}$ in the following way. For the number $t \in (0, 1)$, we consider a concentric system of sets

$$\begin{aligned} C^{(m)} &= \{u : |u|_0 \in [(1-t)^{m+1}, (1-t)^m]\}, \quad m = 0, \dots, m_0 - 1, \\ C^{(m_0)} &= \{u : |u|_0 \leq (1-t)^{m_0}\}. \end{aligned}$$

We cover each of the sets $C^{(m)}$ by identical cubes with sides

$$a_m = (1-t)^m - (1-t)^{m+1} = t(1-t)^m$$

and enumerate these cubes. They form the required covering

$$C_{(1)}, \dots, C_{(N_0-1)}, C_{(N_0)} \stackrel{\text{def}}{=} C^{(m_0)}.$$

Let us choose $m_0 = m_0(n)$ from the condition $(1-t)^{\tilde{m}_0} = n^{-\gamma}$, $m_0 = [\tilde{m}_0]$, $\gamma \in (\frac{1}{2}, 1)$.

We denote, by $|\cdot|_0$, the distance from $C_{(j)}$ to 0 which is equal to

$$r(j) = (1-t)n^{-\gamma m/\tilde{m}_0},$$

and, by $|\cdot|$, the diameter of $C_{(j)}$ which is equal to

$$a(j) = tn^{-\gamma m/\tilde{m}_0}$$

for some $m = m(j)$, $j = 1, \dots, N_0 - 1$. Moreover, if the cube $C_{(j)}$ is an element of the covering of the sets $C^{(m)}$, then

$$a(j) = a_m, \quad r(j) = t(1-t)^{m+1} + \dots + t(1-t)^{m_0-1} + (1-t)^{m_0}.$$

The number of cubes $C_{(j)}$ covering each set $C^{(m)}$ can be made not depending on m and, consequently, on n . In order to verify this, let us consider any octant in \mathbb{R}^q . The volume occurring in its part of the set $C^{(m)}$ is $(1-t)^{mq} - (1-t)^{(m+1)q}$, and the volume of the sets $C_{(j)}$ is equal to $a^q(j) = t^q(1-t)^{mq}$. In this way, the maximum number of cubes $C_{(j)}$ that can be "placed" in the part of $C^{(m)}$ that belongs to the given octant is equal to

$$\frac{(1-t)^{mq} - (1-t)^{(m+1)q}}{t^q(1-t)^{mq}} = \frac{1 - (1-t)^q}{t^q}$$

cubes. Since $m_0 = O(\ln n)$, $N_0 = O(\ln n)$ as well. Let us fix $\theta \in T$. Then

$$(2.2) \quad P_\theta^n \left\{ \sup_{u \in C_0} z_n^+(\theta, u) > \epsilon \right\} \leq \sum_{j=1}^{N_0} P_\theta^n \left\{ \sup_{u \in C_{(j)}} z_n^+(\theta, u) > \epsilon \right\}.$$

Let us estimate each term in (2.2). The general element of the derivative matrix $D_n(u)$ of the mapping

$$u \longrightarrow E_\theta^n S_\beta^{*+}(u)$$

has the form

$$\begin{aligned} D_n^{il}(u) &= \frac{\partial}{\partial u_l} E_\theta^n S_{i\beta}^{*+}(u) \\ &= n^{1/2} d_{in}^{-1}(\theta) d_{ln}^{-1}(\theta) \sum f_{il}(j, u) [\mathcal{P}(g(j, \tau) - g(j, \theta)) - \beta] \\ &\quad + n^{1/2} d_{in}^{-1}(\theta) d_{ln}^{-1}(\theta) \sum f_i(j, u) f_l(j, u) p(g(j, \tau) - g(j, \theta)) \\ &= {}_1 D_n^{il}(u) + {}_2 D_n^{il}(u). \end{aligned}$$

Taking into account (1.6), (1.7), and the inequality

$$\sup_{x \in \mathbb{R}^1} p(x) = p_0 < \infty,$$

we obtain, for $|u| < r$,

$$(2.3) \quad \begin{aligned} n^{-1/2} |{}_1 D_n^{il}(u)| &\leq n^{1/2} d_{in}^{-1}(\theta) d_{ln}^{-1}(\theta) d_{il,n}(\theta + n^{1/2} d_n^{-1}(\theta) u) \times \\ &\times \left(n^{-1} \sum (\mathcal{P}(f(j, u) - f(j, 0)) - \mathcal{P}(0))^2 \right)^{1/2} \leq k^{(il)}(r) k^{1/2}(r) p_0 |u|. \end{aligned}$$

On the other hand,

$$(2.4) \quad \begin{aligned} &\left| n^{-1/2} {}_2 D_n^{il}(u) - p(0) I_{il}(\theta) \right| \leq \\ &\leq p_0 \left[d_{in}^{-1}(\theta) d_{in}(\theta + n^{1/2} d_n^{-1}(\theta) u) d_{ln}^{-1}(\theta) \left(\Phi_n^{(l)}(u, 0) \right)^{1/2} + d_{in}^{-1}(\theta) \left(\Phi_n^{(i)}(u, 0) \right)^{1/2} \right] \\ &+ d_{in}^{-1}(\theta) d_{ln}^{-1}(\theta) \left| \sum g_i(j, \theta) g_l(j, \theta) (p(f(j, u) - f(j, 0)) - p(0)) \right|. \end{aligned}$$

It follows from (1.5) and (1.8) that the terms in square brackets are bounded by the quantity

$$p_0 \left((\tilde{k}^{(i)})^{1/2} + k^{(i)}(r) (\tilde{k}^{(l)})^{1/2} \right) |u|.$$

For another term on the right-hand side of (2.4), we can find, by using condition **A3** and (1.5), the upper bound

$$(2.5) \quad \begin{aligned} &n^{1/2} d_{in}^{-1}(\theta) \max_{1 \leq j \leq n} |g_i(j, \theta)| \left(n^{-1} \sum (p(f(j, u) - f(j, 0)) - p(0))^2 \right)^{1/2} \\ &\leq k^{(i)}(r) H k^{1/2}(r) |u|. \end{aligned}$$

Since the matrix $n^{-1/2} D_n(0) = p(0) I(\theta)$ is positive definite by condition **B2**, it follows from the above-presented considerations that, for sufficiently small u (for simplicity we assume that $u \in C_0$) and some $k_0 > 0$,

$$(2.6) \quad \inf_{\theta \in T} \left| E_\theta^n S_\beta^+(\theta + n^{1/2} d_n^{-1}(\theta) u) \right| \geq k_0 n^{1/2} |u|_0.$$

Let $l \neq N_0$, and let $v \in C_{(l)}$ be an arbitrary point. Then, in view of (2.6), we can write

$$\begin{aligned} \sup_{u \in C_{(l)}} z_n^+(\theta, u) &\leq \left(\sup_{u \in C_{(l)}} M_n^{(l)}(\theta, u, v) + L_n^{(l)}(\theta, v) \right) (1 + k_0 n^{1/2} r(l))^{-1}, \\ M_n^{(l)}(\theta, u, v) &= \sum_{\lambda=1}^4 M_{\lambda n}^{(l)}(\theta, u, v) \quad (\text{mod } P_\theta^n) \end{aligned}$$

$$\begin{aligned}
M_{1n}^{(l)}(\theta, u, v) &= \left| d_n^{-1}(\theta) \sum \nabla f(j, u) (\chi\{X_j * f(j, u)\} - \chi\{X_j < f(j, v)\}) \right| \\
M_{2n}^{(l)}(\theta, u, v) &= \left| d_n^{-1}(\theta) \sum (\nabla f(j, u) - \nabla f(j, v)) (\chi\{X_j < f(j, v)\} - \beta) \right| \\
M_{3n}^{(l)}(\theta, u, v) &= \left| d_n^{-1}(\theta) \sum \nabla f(j, u) (\mathcal{P}(f(j, u) - f(j, 0)) - \mathcal{P}(f(j, v) - f(j, 0))) \right| \\
M_{4n}^{(l)}(\theta, u, v) &= \left| d_n^{-1}(\theta) \sum (\nabla f(j, u) - \nabla f(j, v)) (\mathcal{P}(f(j, v) - f(j, 0)) - \beta) \right| \\
L_n^{(l)}(\theta, v) &= \left| d_n^{-1}(\theta) \sum (\nabla f(j, v) (\chi\{X_j < f(j, v)\} - \beta) - \nabla f(j, 0) (\chi\{\varepsilon_j * 0\} - \beta) \right. \\
&\quad \left. - \nabla f(j, v) (\mathcal{P}(f(j, v) - f(j, 0)) - \beta) \right| \quad (\text{mod } P_\theta^n).
\end{aligned}$$

By (1.8) and for $u, v \in C_{(l)}$, we obtain

$$(2.7) \quad n^{-1/2} M_{2n}^{(l)}(\theta, u, v) \leq \beta' \left(\sum_{i=1}^q d_{in}^{-2}(\theta) \Phi_n^{(i)}(u, v) \right)^{1/2} \leq k_1 a(l).$$

Furthermore, in accordance with (1.5), (1.7), and **A3**, we get

$$(2.8) \quad n^{-1/2} M_{3n}^{(l)}(\theta, u, v) \leq p_0 n^{-1/2} \Phi_{2n}^{1/2}(u, v) \left(\sum_{i=1}^q \frac{d_{in}^2(\theta + n^{1/2} d_n^{-1}(\theta) u)}{d_{in}^2(\theta)} \right)^{1/2} \leq k_2 a(l).$$

Analogously,

$$(2.9) \quad n^{-1/2} M_{4n}^{(l)}(\theta, u, v) \leq p_0 n^{-1/2} \Phi_{2n}^{1/2}(v, 0) \left(\sum_{i=1}^q d_{in}^{-2}(\theta) \Phi_n^{(i)}(u, v) \right)^{1/2} \leq k_3 a(l).$$

Let us estimate $M_{1n}^{(l)}(\theta, u, v)$. For any $u, v \in C_{(l)}$,

$$\begin{aligned}
&|\chi\{X_j * f(j, u)\} - \chi\{X_j < f(j, v)\}| \\
&\leq \chi \left\{ \inf_{u \in C_{(l)}} f(j, u) - f(j, 0) \leq \varepsilon_j \leq \sup_{u \in C_{(l)}} f(j, u) - f(j, 0) \right\} = \chi_j \quad (\text{mod } P_\theta^n).
\end{aligned}$$

Consequently, by (1.5),

$$(2.10) \quad \begin{aligned} n^{-1/2} M_{1n}^{(l)}(\theta, u, v) &\leq n^{-1/2} \left(\sum_{i=1}^q \left(d_{in}^{-1}(\theta) \max_{1 \leq j \leq n} |f_i(j, u)| \right)^2 \right)^{1/2} \sum \chi_j \\ &\leq k_4 n^{-1} \sum \chi_j. \end{aligned}$$

Using the formula for finite increments, we find

$$(2.11) \quad \begin{aligned} n^{-1} \sum E_\theta^n \chi_j &= n^{-1} \sum \left(\mathcal{P} \left(\sup_{u \in C_{(l)}} f(j, u) - f(j, 0) \right) - \mathcal{P} \left(\inf_{u \in C_{(l)}} f(j, u) - f(j, 0) \right) \right) \\ &\leq p_0 n^{-1} \sum \sup_{u_1, u_2 \in C_{(l)}} |f(j, u_1) - f(j, u_2)| \\ &\leq p_0 q^{1/2} \left(\sum_{i=1}^q \left(n^{1/2} d_{in}^{-1}(\theta) \sup_{u \in C_{(l)}} \max_{1 \leq j \leq n} |f_i(j, u)| \right)^2 \right)^{1/2} a(l) \leq k_5 a(l). \end{aligned}$$

Estimates (2.7)-(2.11) show that there exist constants k_6 and k_7 such that

$$P_\theta^n \left\{ \sup_{u \in C_{(l)}} M_n^{(k)}(\theta, u, v) (1 + k_0 n^{1/2} r(l))^{-1} > \frac{\epsilon}{2} \right\}$$

$$(2.12) \quad \leq P_\theta^n \left\{ k_6 n^{-1} \sum (\chi_j - E_\theta^n \chi_j) > \frac{\epsilon}{2} r(l) - k_7 a(l) \right\}.$$

Note that $\frac{\epsilon}{2} r(l) - k_7 a(l) = \left(\frac{\epsilon}{2} (1-t) - k_7 t \right) n^{-\gamma m / \tilde{m}_0} > 0$, if t is chosen sufficiently small. Therefore, probability (2.12) can be estimated, with the help of the Chebyshev inequality and (2.11), by the quantity

$$(2.13) \quad \frac{4k_6^2}{(\epsilon(1-t) - 2k_7 t)^2} n^{-2+2\gamma m / \tilde{m}_0} \sum E_\theta^n \chi_j \leq k_8 n^{-1+\gamma m / \tilde{m}_0}.$$

Using the notation

$$\begin{aligned} L_{1i}(j) &= (f_i(j, v) - f_i(j, 0))(\chi\{X_j < f(j, v)\} - \beta), \\ L_{2i}(j) &= f_i(j, 0)(\chi\{X_j < f(j, v)\} - \chi\{\varepsilon_j * 0\}), \quad i = 1, \dots, q, \end{aligned}$$

we obtain

$$(2.14) \quad \begin{aligned} P_1 &= P_\theta^n \left\{ L_n^{(k)}(\theta, v) (1 + k_0 n^{1/2} r(l))^{-1} > \frac{\epsilon}{2} \right\} \\ &\leq \frac{4}{n(k_0 \epsilon)^2 r^2(l)} \sum_{i=1}^q d_{in}^{-2}(\theta) \sum_{\lambda=1}^2 E_\theta^n \left(\sum (L_{\lambda i}(j) - E_\theta^n L_{\lambda i}(j)) \right)^2, \end{aligned}$$

$$(2.15) \quad D_\theta^n \left(\sum L_{1i}(j) \right) \leq \Phi_{2n}^{(i)}(v, 0),$$

$$(2.16) \quad \begin{aligned} D_\theta^n \left(\sum L_{2i}(j) \right) &\leq \sum f_i^2(j, 0) |\mathcal{P}(f(j, v) - f(j, 0)) - \mathcal{P}(0)| \\ &\leq p_0 \max_{1 \leq j \leq n} |g_i(j, \theta)| d_{in}(\theta) \Phi_{2n}^{1/2}(v, 0). \end{aligned}$$

It follows from relations (2.14)-(2.16) and the conditions of the theorem that

$$(2.17) \quad \begin{aligned} P_1 &\leq \frac{4n^{-1}}{(k_0 \epsilon)^2} \left[\frac{(r(l) + a(l))^2}{r^2(l)} \sum_{i=1}^q \tilde{k}^{(i)}(1) + \frac{r(l) + a(l)}{r^2(l)} p_0 k^{1/2}(1) \sum_{i=1}^q k^{(i)}(1) \right] \\ &\leq k_9 n^{-1} \left[(1-t)^{-2} + (1-t)^{-2} n^{\gamma m / \tilde{m}_0} \right] = O \left(n^{-1+\gamma m / \tilde{m}_0} \right). \end{aligned}$$

Inequalities (2.13) and (2.17) show that, for $l = 1, \dots, N_0 - 1$ and some $m = m(l) < m_0$,

$$(2.18) \quad \sup_{\theta \in T} P_\theta^n \left\{ \sup_{u \in C_{(l)}} z_n^+(\theta, u) > \epsilon \right\} = O \left(n^{-1+\gamma m / \tilde{m}_0} \right).$$

Let us consider the case $l = N_0$. Clearly,

$$(2.19) \quad \begin{aligned} &P_\theta^n \left\{ \sup_{u \in C_{(N_0)}} z_n^+(\theta, u) > \epsilon \right\} \leq \\ &\leq P_\theta^n \left\{ \sup_{|u|_0 < n^{-\gamma m / \tilde{m}_0}} \left| S_\beta^{*+}(u) - S_\beta^{*+}(0) - E_\theta^n S_\beta^{*+}(u) \right| > \epsilon \right\}. \end{aligned}$$

Let us rewrite the expression standing under the sign of supremum in (2.19) in the form of $\nu_1(\theta, u) + \nu_2(\theta, u) + \nu_3(\theta, u)$, where

$$\begin{aligned} \nu_1(\theta, u) &= d_n^{-1}(\theta) \sum (\nabla f(j, u) - \nabla f(j, 0))(\chi\{X_j * f(j, u)\} - \beta), \\ \nu_2(\theta, u) &= d_n^{-1}(\theta) \sum \nabla f(j, 0)(\chi\{X_j * f(j, u)\} - \chi\{\varepsilon_j * 0\}), \\ \nu_3(\theta, u) &= d_n^{-1}(\theta) \sum \nabla f(j, u)(\mathcal{P}(f(j, u) - f(j, 0)) - \beta). \end{aligned}$$

It is easy to show that, for $|u|_0 < n^{-\gamma m/\bar{m}_0}$,

$$(2.20) \quad |\nu_1(\theta, u)| \leq \beta' n^{\frac{1}{2}} \left(\sum_{i=1}^q d_{in}^{-2}(\theta) \Phi_{2n}^{(i)}(u, 0) \right)^{1/2} \leq k_1 n^{\frac{1}{2} - \frac{\gamma m}{\bar{m}_0}},$$

$$(2.21) \quad |\nu_3(\theta, u)| \leq p_0 \Phi_{2n}^{\frac{1}{2}}(u, 0) \left(\sum_{i=1}^q \frac{d_{in}^2(\theta + n^{1/2} d_n^{-1}(\theta) u)}{d_{in}^2(\theta)} \right)^{1/2} \leq k_2 n^{\frac{1}{2} - \frac{\gamma m}{\bar{m}_0}},$$

where k_1 and k_2 are the same as in (2.7) and (2.8), correspondingly.

If $\gamma > \frac{1}{2}$, then the exponents in (2.20) and (2.21) are negative for $n > n_0$. That is, for $\epsilon' < \epsilon$, it remains to estimate the probability

$$(2.22) \quad \begin{aligned} & P_\theta^n \left\{ \sup_{|u|_0 < n^{-\gamma m/\bar{m}_0}} |\nu_2(\theta, u)| > \epsilon' \right\} \\ & \leq P_\theta^n \left\{ \left(\sum_{i=1}^q \left(d_{in}^{-1}(\theta) \max_{1 \leq j \leq n} |g_i(j, \theta)| \right)^2 \right)^{1/2} \sum \tilde{\chi}_j > \epsilon' \right\}, \\ & \leq P_\theta^n \left\{ k_4 n^{-1/2} \sum \tilde{\chi}_j > \epsilon' \right\}, \end{aligned}$$

$$\tilde{\chi}_j = \chi \left\{ \inf_{|u|_0 \leq n^{-\gamma m/\bar{m}_0}} f(j, u) - f(j, 0) \leq \varepsilon_j \leq \sup_{|u|_0 \leq n^{-\gamma m/\bar{m}_0}} f(j, u) - f(j, 0) \right\}.$$

From the conditions of the theorem,

$$\sum E_\theta^n \tilde{\chi}_j \leq k_5 n^{-\gamma m/\bar{m}_0}, \quad j = 1, \dots, n.$$

Hence, instead of (2.22), it is sufficient to estimate, for any $\epsilon'' > 0$, the probability

$$P_\theta^n \left\{ n^{-1/2} \sum (\tilde{\chi}_j - E_\theta^n \tilde{\chi}_j) > \epsilon'' \right\} \leq (\epsilon'')^2 k_5 n^{-\gamma m/\bar{m}_0}.$$

Taking into account the fact that all the bounds are uniform in $\theta \in T$, we obtain that the lemma is proved for $z_n^+(\theta, u)$. The case of $z_n^-(\theta, u)$ is investigated similarly. \square

Let us set

$$E_\theta^n S_\beta^\pm(\hat{\theta}_n) = (E_\theta^n S_\beta^\pm(\tau))_{\tau=\hat{\theta}_n}.$$

Lemma 2. *Under the conditions of the theorem, for any $\epsilon > 0$,*

$$(2.23) \quad \sup_{\theta \in T} P_\theta^n \left\{ |S_\beta^\pm(\theta) + E_\theta^n S_\beta^\pm(\hat{\theta}_n)| > \epsilon \right\} \xrightarrow{n \rightarrow \infty} 0.$$

Proof. Let us introduce the events

$$\begin{aligned} A_i^\pm(\theta) &= \{S_{i\beta}^\pm(\theta) + E_\theta^n S_{i\beta}^\pm(\hat{\theta}_n) - S_{i\beta}^\pm(\hat{\theta}_n) \geq -\epsilon(1 + |E_\theta^n S_{i\beta}^\pm(\hat{\theta}_n)|)\}, \\ & \quad i = 1, \dots, q. \end{aligned}$$

It follows from (1.11) and the previous lemma that

$$(2.24) \quad \inf_{\theta \in T} P_\theta^n \{A_i^\pm(\theta)\} \xrightarrow{n \rightarrow \infty} 1, \quad i = 1, \dots, q.$$

For the events $\{|\hat{\theta}_n - \theta| < r\}$, $r < r_0$, $S_\beta^\pm(\hat{\theta}_n) \geq 0$. Therefore, relation (2.24) is true for the events

$$B_i^\pm(\theta) = \{S_{i\beta}^\pm(\theta) + E_\theta^n S_{i\beta}^\pm(\hat{\theta}_n) \geq -\epsilon(1 + |E_\theta^n S_{i\beta}^\pm(\hat{\theta}_n)|)\} \supset A_i^\pm(\theta).$$

On the other hand,

$$S_{i\beta}^+(\theta) + S_{i\beta}^-(\theta) = \sum |g_i(j, \theta)| \chi\{\varepsilon_j = 0\} = 0 \pmod{P_\theta^n},$$

and the events $B_i^-(\theta)$ are equally like to the events

$$C_i^+(\theta) = \{S_{i\beta}^+(\theta) + E_\theta^n S_{i\beta}^+(\hat{\theta}_n) \leq \epsilon(1 + |E_\theta^n S_\beta^+(\hat{\theta}_n)|)\}.$$

Furthermore, for $\epsilon < q^{-1}$, the events $D_i^+(\theta) = B_i^+(\theta) \cap C_i^+(\theta)$, $i = 1, \dots, q$,

$$(2.25) \quad D_i^+(\theta) = \left\{ \left| S_{i\beta}^+(\theta) + E_\theta^n S_{i\beta}^+(\hat{\theta}_n) \right| \leq \epsilon(1 + |E_\theta^n S_\beta^+(\hat{\theta}_n)|) \right\},$$

$$\begin{aligned} \bigcap_{i=1}^q D_i^+(\theta) &\subseteq \left\{ \left| S_\beta^+(\theta) + E_\theta^n S_\beta^+(\hat{\theta}_n) \right| \leq q\epsilon(1 + |E_\theta^n S_\beta^+(\hat{\theta}_n)|) \right\} \\ &\subseteq \left\{ \left| E_\theta^n S_\beta^+(\hat{\theta}_n) \right| \leq (1 - q\epsilon)^{-1}(q\epsilon + |S_\beta^+(\theta)|) \right\} = X^+(\theta), \end{aligned}$$

i.e.,

$$(2.26) \quad \inf_{\theta \in T} P_\theta^n \{X^+(\theta)\} \xrightarrow{n \rightarrow \infty} 1.$$

Let us note that

$$(2.27) \quad P_\theta^n \{|E_\theta^n S_\beta^+(\hat{\theta}_n)| > M\} \leq P_\theta^n \{\overline{X^+(\theta)}\} + P_\theta^n \{|S_\beta^+(\theta)| > M(1 - q\epsilon) - q\epsilon\},$$

where $\overline{X^+(\theta)}$ is a complement of the event $X^+(\theta)$. Let us denote

$$\begin{aligned} \eta_j &= \chi\{\varepsilon_j < 0\} - \beta, \quad j \geq 1, \\ I_{in}(\theta) &= \{1, \dots, n\} \cap \{j : g_i(j, \theta) > 0\}. \end{aligned}$$

Then P_θ^n - a.s.

$$S_\beta^+(\theta) - d_{in}^{-1}(\theta) \sum g_i(j, \theta) \eta_j = d_{in}^{-1}(\theta) \sum_{j \in I_{in}(\theta)} g_i(j, \theta) \chi\{\varepsilon_j = 0\} = 0.$$

Therefore, by the Chebyshev inequality,

$$P_\theta^n \{|S_\beta^+(\theta)| > M(1 - q\epsilon) - q\epsilon\} \leq q(M(1 - q\epsilon) - q\epsilon)^{-2} \xrightarrow{M \rightarrow \infty} 0,$$

i.e., the vector $S_\beta^+(\theta)$ is bounded in probability. It follows from (2.26) and (2.27) that the vector $E_\theta^n S_\beta^+(\hat{\theta}_n)$ is also bounded in probability uniformly in $\theta \in T$.

According to (2.25),

$$\sup_{\theta \in T} P_\theta^n \left\{ \left| S_\beta^+(\theta) + E_\theta^n S_\beta^+(\hat{\theta}_n) \right| > \epsilon \left(1 + |E_\theta^n S_\beta^+(\hat{\theta}_n)| \right) \right\} \xrightarrow{n \rightarrow \infty} 0.$$

Therefore, (2.23) holds. We remark that the boundedness in probability of the r.v. $E_\theta^n S_\beta^+(\hat{\theta}_n)$ can also be obtained immediately from condition **C**, the explicit form of $E_\theta^n S_\beta^+(\hat{\theta}_n)$, and from the conditions of the theorem. \square

Lemma 3. *Under the conditions of the theorem, for any $\epsilon > 0$,*

$$(2.28) \quad P_\theta^n \left\{ \left| E_\theta^n S_\beta^+(\hat{\theta}_n) - p(0)I(\theta)d_n(\theta)(\hat{\theta}_n - \theta) \right| > \epsilon \right\} \xrightarrow{n \rightarrow \infty} 0.$$

Proof. If the quantity $n^{-1/2}|d_n(\theta)(\hat{\theta}_n - \theta)|$ is small, then it follows from inequality (2.6) and the boundedness of the r.v. $E_\theta^n S_\beta^+(\hat{\theta}_n)$ in probability that the norm of the vector $d_n(\theta)(\hat{\theta}_n - \theta)$ is bounded in probability. The statement of Lemma 3 follows from condition **C** and inequalities (2.3)-(2.5). \square

3. PROOF OF THE THEOREM

Relations (2.23) and (2.28) show that, for any $\epsilon > 0$,

$$(3.1) \quad P_\theta^n \left\{ |(p(0))^{-1} \Lambda(\theta) S_\beta^+(\theta) + d_n(\theta)(\hat{\theta}_n - \theta)| > \epsilon \right\} \xrightarrow{n \rightarrow \infty} 0.$$

As was noted above,

$$S_\beta^+(\theta) = d_n^{-1}(\theta) \sum \nabla g(j, \theta) \eta_j \quad (\text{mod } P_\theta^n).$$

Let us apply Corollary 17.2 in ([5], p. 165) to the random vectors

$$\xi_{jn} = n^{1/2} d_n^{-1}(\theta) \nabla g(j, \theta) \eta_j, \quad j = 1, \dots, n.$$

It follows from (1.5) that

$$n^{-1} \sum E_\theta^n |\xi_{jn}|^3 \leq q^{1/2} \sum_{i=1}^q n^{-1} \sum d_{in}^{-3}(\theta) |g_i(j, \theta)|^3 n^{3/2} \leq k_{10} < \infty$$

uniformly in $\theta \in T$. Then

$$(3.2) \quad \sup_{\theta \in T} \sup_{C \in \mathcal{C}^q} \left| P_\theta^n \left\{ I^{-1/2}(\theta) S_\beta^+(\theta) \in C \right\} - \Phi(C) \right| = O(n^{-1/2}).$$

Let us find the correlation matrix of $S_\beta^+(\theta)$. Clearly, $ES_\beta^+(\theta) = 0$. Then, taking into account **A1**, we get

$$E_\theta^n S_{i\beta}^+(\theta) S_{l\beta}^+(\theta) = d_{in}^{-1}(\theta) d_{ln}^{-1}(\theta) \sum g_i(j, \theta) g_l(j, \theta) E \eta_j^2, \quad i, l = 1, \dots, q.$$

It follows from the form of η_j that $E \eta_j^2 = \beta(1 - \beta)$. Then

$$(3.3) \quad E_\theta^n S_\beta^+(\theta) (S_\beta^+(\theta))^T = \beta(1 - \beta) I(\theta).$$

Relations (3.1)-(3.3) yield that, for any $\epsilon > 0$ and $C \in \mathcal{C}^q$,

$$(3.4) \quad -\Delta_n + \Phi(C_{-\epsilon}) \leq P_\theta^n \left\{ \frac{p(0)}{\sqrt{\beta(1 - \beta)}} I^{1/2}(\theta) d_n(\theta) (\hat{\theta}_n - \theta) \in C \right\} \leq \Delta_n + \Phi(C_\epsilon),$$

where $C_{-\epsilon}$ and C_ϵ are the exterior and interior sets parallel to C , and $\Delta_n \xrightarrow{n \rightarrow \infty} 0$ uniformly in $\theta \in T$ and $C \in \mathcal{C}^q$. The statement of the theorem follows from (3.4) and the theorem from Section 3 in [6] which state that, for any $\epsilon > 0$,

$$\sup_{C \in \mathcal{C}^q} |\Phi(C_{\pm\epsilon}) - \Phi(C)| \leq k\epsilon,$$

where k is a constant that does not depend on ϵ .

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