Theory of Stochastic Processes
Vol. 11 (27), no. 3–4, 2005, pp. 29–41

UDC 519.21

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ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF A DIFFERENTIAL EQUATION WITH INTERACTION GOVERNED BY GENERALIZED FUNCTION IN ABSTRACT WIENER SPACE

We consider the following differential equation with interaction governed by a generalized function $\varkappa_0$:

$$\frac{dx(u,t)}{dt} = a(x(u,t), \varkappa_t), \quad x(u,0) = u, \quad \varkappa_t = \varkappa_0 \circ x(\cdot, t)^{-1}.$$

The conditions that guarantee the existence and uniqueness of a solution when mapping a belongs to some Sobolev space are obtained.

1. Introduction

Let (X, H, μ) be an abstract Wiener space, i.e. X is a real separable Banach space and μ is a Gaussian measure on X with the Cameron–Martin space H (cf. [1]). Consider the evolution of a material system in X in the case where the behaviour of each particle depends not only on the position of this particle but also on some characteristic of the whole system represented by a generalized function. Examples of such characteristics are mass distributions at some surfaces, their derivatives, ets.

Let $W_p^k = W_p^k(X, H, \mu), k \in \mathbb{N}, p \geq 1$, be the Sobolev space (cf. [1], the precise definition will be given later), and let $W_q^{-k} = (W_p^k)^*$, where $\frac{1}{p} + \frac{1}{q} = 1$, be the space of generalized functions equipped with $*$-weak topology. Denote, by $x(u,t)$, the position of the particle starting from u at time t. Assume that the characteristic of the material system at time t is $\varkappa_t \in W_q^{-k}$, and the evolution of the system is described by the differential equation with interaction

$$(1) \qquad \begin{cases} \frac{dx(u,t)}{dt} = a(x(u,t), \varkappa_t) \\ x(u,0) = u, \ \varkappa_t = \varkappa_0 \circ x(\cdot, t)^{-1}, \ t \geq 0. \end{cases}$$

where $a : X \times W_q^{-k} \to H$ is a measurable transformation. Here, the generalized function $\varkappa_t = \varkappa_0 \circ x(\cdot, t)^{-1}$ is said to be the image of the generalized function $\varkappa_0$ under transformation $x(\cdot, t)$ if, for every test function $f \in W_p^k$, we have $f \circ x(\cdot, t) \in W_p^k$ and $\langle f, \varkappa_t \rangle = \langle f \circ x(\cdot, t), \varkappa_0 \rangle$. Note that if $\varkappa_0$ is a measure on X, then the definition of $\varkappa_t$ coincides with the standard definition of the image of a measure.

2000 *AMS Mathematics Subject Classification.* Primary 60H07,28C20.

Key words and phrases. Differential equation with interaction, Sobolev space, generalized function.

This research has been partially supported by the Ministry of Education and Science of Ukraine, project N GP/F8/0086.

Definition 1. Measurable mapping $x : X \times \mathbb{R} \to X$ is said to be a solution of Eq. (1) if

1) for every $t \geq 0$, the generalized function $\varkappa_t = \varkappa_0 \circ x(\cdot, t)^{-1}$ belongs to W_q^{-k}.

2) for μ-almost all u,

$$x(u,t) = u + \int_0^t a(x(u,s), \varkappa_s) ds \text{ holds for all } t \geq 0;$$

3) for every $t \geq 0$, the measure $\mu \circ x(\cdot, t)^{-1}$ is absolutely continuous with respect to μ;

Remark. Condition 3) in Definition 1 provides that the solution does not depend on the particular choice of a modification of a.

In this article, we obtain some sufficient conditions for the existence and uniqueness of solution of (1). To formulate them, we need to recall some standard constructions and notations from the Malliavin calculus (cf. [1]).

For any separable Hilbert space E, we denote, by $\mathcal{F}C^\infty(X, E)$, a set of smooth cylindical functions, i.e. functions of the form

$$f(u) = \sum_{l=1}^m \varphi_l(\langle y_1, u\rangle, \dots, \langle y_n, u\rangle) e_l,$$

where $y_1, \dots, y_n \in X^*$, $\varphi_1, \dots, \varphi_m \in C_b^\infty(\mathbb{R}^n)$ and $e_1, \dots, e_m \in E$. The derivative ∇ along H is defined, for $f \in \mathcal{F}C^\infty(X, E)$, by

$$\nabla f(u) = \sum_{l=1}^m \sum_{i=1}^n \frac{\partial \varphi_l}{\partial x_i}(\langle y_1, u\rangle, \dots, \langle y_n, u\rangle) j^* y_i \otimes e_l \in \mathcal{F}C^\infty(X, E_1),$$

where $E_1 = \mathcal{H}(H, E)$ is the space of Hilbert–Schmidt operators from H to E equipped with the Hilbert–Schmidt norm. Define higher order derivatives on $\mathcal{F}C^\infty(X, E)$ iteratively by setting $E_0 = E$, $\nabla^0 = 1\!\!1_{\mathcal{F}C^\infty(X,E)}$ and, for $k \in \mathbb{N}$,

$$E_k = \mathcal{H}(H, E_{k-1}),\ \nabla^k = \nabla \circ \nabla^{k-1} : \mathcal{F}C^\infty(X, E) \to \mathcal{F}C^\infty(X, E_k).$$

Note that E_k can be identified with the space of k-linear Hilbert–Schmidt operators on H with range in E.

For any $k \in \mathbb{N}$ and $p \in [1, +\infty)$, the operator ∇^k is closable under the norm $\|f\|_{p,k} = \sum_{i=0}^k \|\nabla^i f\|_{L_p(X, E_i, \mu)}$.

The completion of $\mathcal{F}C^\infty(X, E)$ under this norm is a Sobolev space $W_p^k(X, E, \mu) \subset L_p(X, E, \mu)$. The extensions $D^k : W_p^k(X, E, \mu) \to L_p(X, E, \mu)$ of derivatives ∇^k to W_p^k are called stochastic derivatives. By $\delta : \mathcal{D}(\delta) \subset L_q(X, H, \mu) \to L_q(X, \mathbb{R}, \mu)$, $\frac{1}{p} + \frac{1}{q} = 1$, we denote the divergence operator, i.e. the operator adjoint to D. Denote, by $\|\cdot\|_{\mathcal{H}}$, the Hilbert–Schmidt norm in each of E_k, $k \geq 1$, and, by $\|\cdot\|_{\mathrm{op}}$, the operator norm in $\mathcal{L}(H)$.

Now we can formulate the results.

Theorem 1. *Let $a : X \times W_q^{-k} \to H$ be such that*

1) $\exists p_0 \geq 1\ \forall \varkappa \in W_q^{-k} \quad a(\cdot, \varkappa) \in W_{p_0}^k(X, H, \mu)$;

2) $c_0 = \sup\limits_{\substack{u \in X \\ \varkappa \in W_q^{-k}}} \|a(u, \varkappa)\|_H < \infty, \quad \forall 1 \leq l \leq k \quad c_l = \sup\limits_{\substack{u \in X \\ \varkappa \in W_q^{-k}}} \|D^l a(u, \varkappa)\|_{\mathcal{H}} < \infty;$

3) $\forall c > 0 \quad \theta(c) = \sup\limits_{\varkappa \in W_q^{-k}} \int_X \exp(c|\delta a(u, \varkappa)|) \mu(du) < \infty;$

4) if $\{\varkappa, \varkappa_n, n \geq 1\} \subset W_q^{-k}$ *and* $\varkappa_n \to \varkappa,\ n \to \infty$, *$*$-weakly in* W_q^{-k}, *then* $a(u, \varkappa_n) \to a(u, \varkappa),\ n \to \infty$, *in measure* μ.

Suppose that

$$\exists \varepsilon > 0 \quad \varkappa_0 \in W_{q+\varepsilon}^{-k}.$$

Then Eq. (1) has a solution on $[0,+\infty)$.

Theorem 2. *Let* $a, \varkappa_0$ *satisfy the conditions of Theorem 1 and, moreover,* $\exists L>0\ \exists q_1<q\ \forall u\in X\ \forall h\in H\ \forall \varkappa_1,\varkappa_2\in W_q^{-k}$

$$\|a(u,\varkappa_1)-a(u+h,\varkappa_2)\|_H \le L(\|h\|_H+\|\varkappa_1-\varkappa_2\|_{q_1,-k-1}),$$

$\forall 1\le l\le k$

$$\|D^l a(u,\varkappa_1)-D^l a(u+h,\varkappa_2)\|_{\mathcal{H}} \le L(\|h\|_H+\|\varkappa_1-\varkappa_2\|_{q_1,-k-1}),$$

where $\|\varkappa\|_{q,-k}=\sup\limits_{\substack{f\in W_p^k\\ \|f\|_{p,k}\le 1}}|\langle f,\varkappa\rangle|$. *Then Eq. (1) has a unique solution on* $[0,+\infty)$.

Remark. If the transformation a in (1) depends only on the first argument, i.e. $a(u,\mu)=a_0(u)$, then Eq. (1) turns to be an ordinary differential equation

$$\text{(1')}\qquad \begin{cases} \frac{dx(u,t)}{dt}=a_0(x(u,t)),\\ x(u,0)=u. \end{cases}$$

It is well known that Eq. $(1')$ has a unique solution if the transformation a is Lipschitzian. The sufficient conditions for the existence and uniqueness of solution of $(1')$ were studied in [2–4] in the case where the transformation a belongs to some Sobolev space, instead of being Lipschitzian. In particular, it was proved in [3] that if $a_0\in W_p^1(X,H,\mu_0)$ and $\exp(|\delta a_0|)\in L_c(X,H,\mu_0)$, $\exp(\|Da_0\|_{\mathrm{op}})\in L_c(X,H,\mu_0)$ for some $c>0$, then Eq. $(1')$ has a unique solution.

2. The space of generalized functions W_{q-}^{-k}

We shall prove Theorem 1 in a slightly different form involving other spaces of generalized functions. Note that if $\widetilde{p}>p$, $\frac1p+\frac1q=1$ and $\frac1{\widetilde p}+\frac1{\widetilde q}=1$, then $W_{\widetilde p}^k\subset W_p^k$, $W_q^{-k}\subset W_{\widetilde q}^{-k}$. Denote $W_{p+}^k=\bigcup_{\widetilde p>p}W_{\widetilde p}^k$, $W_{q-}^{-k}=\bigcap_{\widetilde q<q}W_{\widetilde q}^{-k}$.

The elements of W_{q-}^{-k} are linear functionals on W_{p+}^k. Define the topology τ on W_{q-}^{-k} as $\tau=C(W_{p+}^k,W_{q-}^{-k})$. Then $\varkappa_n\xrightarrow{\tau}\varkappa$, $n\to\infty$, in W_{q-}^{-k} means by definition that, for every $\widetilde p>p$, the sequence $\varkappa_n$, $n\ge1$ converges to $\varkappa$ $*$-weakly in $W_{\widetilde p}^{-k}$, i.e., for every $\widetilde p>p$ and for every test function $f\in W_{\widetilde p}^k$, we have $\langle f,\varkappa_n\rangle\to\langle f,\varkappa\rangle$, $n\to\infty$.

We now can formulate the result in terms of the spaces W_{q-}^{-k}.

Theorem 1$'$. *Let* $a: X\times W_{q-}^{-k}\to H$ *be such that*

1) $\exists p_0\ge1\ \forall\varkappa\in W_{q-}^{-k}\quad a(\cdot,\varkappa)\in W_{p_0}^k(X,H,\mu)$;

2) $c_0=\sup\limits_{\substack{u\in X\\ \varkappa\in W_{q-}^{-k}}}\|a(u,\varkappa)\|_H<\infty$, $\forall 1\le l\le k\ \ c_l=\sup\limits_{\substack{u\in X\\ \varkappa\in W_{q-}^{-k}}}\|D^la(u,\varkappa)\|_{\mathcal H}<\infty$;

3) $\forall c>0\ \ \theta(c)=\sup\limits_{\varkappa\in W_{q-}^{-k}}\int_X\exp(c|\delta a(u,\varkappa)|)\mu(du)<\infty$;

4) if $\{\varkappa,\varkappa_n,n\ge1\}\subset W_{q-}^{-k}$ *and* $\varkappa_n\xrightarrow{\tau}\varkappa$, $n\to\infty$, *in* W_{q-}^{-k} *then* $a(u,\varkappa_n)\to a(u,\varkappa)$, $n\to\infty$, *in measure* μ.

Then, for every $\varkappa_0\in W_{q-}^{-k}$, *Eq. (1) has a solution on* $[0,+\infty)$ *such that* $\varkappa_t\in W_{q-}^{-k}$, $t\ge0$.

Remark. Condition 1) in Definition 1 is caused by the fact that the transformation a is defined on $X\times W_q^{-k}$. Hence, it is reasonable to replace it by $\varkappa_t\in W_{q-}^{-k}$, $t\ge0$ in the setup of Theorem $1'$.

Theorem 1 follows from Theorem 1′ immediately since $W^{-k}_{q+\varepsilon-} \subset W^{-k}_q$ and $\varkappa_0 \in W^{-k}_{q+\varepsilon} \subset W^{-k}_{q+\varepsilon-}$ imply $\varkappa_t \in W^{-k}_{q+\varepsilon-} \subset W^{-k}_q,\ t \geq 0$.

Theorem 1′ shows that the solution of Eq. (1) preserves the space W^{-k}_{q-}, i.e. if the initial value $\varkappa_0$ belongs to W^{-k}_{q-}, then the images $\varkappa_t,\ t \geq 0$, remain the elements of the same space. The following example shows that the solution of Eq. (1) does not preserve the space W^{-k}_q, hence the condition $\varkappa_0 \in W^{-k}_{q+\varepsilon}$ in Theorem 1 cannot be replaced by $\varkappa_0 \in W^{-k}_q$. Therefore, the spaces W^{-k}_{q-} are more natural when dealing with Eq. (1) than usual spaces W^{-k}_q.

Example 1. Let $X = \mathbb{R}$, $\mu(du) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$, and let a generalized function $\varkappa_0$ be defined by $\langle f, \varkappa_0 \rangle = \int_{\mathbb{R}} f(u) 1\!\!1_{\{u \geq 0\}} e^{u^2/2q - \sqrt{u}} \mu(du)$, $f \in W^1_p$. Since

$$\int_{\mathbb{R}} (1\!\!1_{\{u\geq 0\}} e^{u^2/2q-\sqrt{u}})^q \mu(du) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{u^2/2-\sqrt{u}} \cdot e^{-u^2/2} du = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-q\sqrt{u}} du < \infty,$$

we get $\varkappa_0 \in L_q(d\mu) \subset W^{-1}_q(d\mu)$.

Let $a(u, \varkappa) \equiv 1$. Then Eq. (1) turns out to be an ordinary differential equation which has a unique solution $x(u,t) = u + t$. We have

$$\langle f, \varkappa_t \rangle = \int_{\mathbb{R}} f(u+t) 1\!\!1_{\{u\geq 0\}} e^{u^2/2q-\sqrt{u}} \mu(du) = \frac{1}{\sqrt{2\pi}} \int_t^\infty f(v) e^{(v-t)^2/2q - \sqrt{v-t}} \cdot$$

$$\cdot e^{-(v-t)^2/2} dv = \int_{\mathbb{R}} f(v) 1\!\!1_{\{v \geq t\}} e^{v^2/2q+(q-1)vt/q-(q-1)t^2/2-\sqrt{v-t}} \mu(dv).$$

Hence, $\varkappa_t$ must be a regular generalized function, but

$$\int_{\mathbb{R}} \left(1\!\!1_{\{v\geq t\}} e^{v^2/2q+(q-1)vt/q-(q-1)t^2/2-\sqrt{v-t}} \right)^q \mu(dv) =$$

$$= \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{(q-1)vt-(q-1)t^2/2-q\sqrt{v-t}} dv = \infty$$

and $\varkappa_t \notin W^{-1}_q, t > 0$. On the other hand, for every $\widetilde{q} < q$, we have $\varkappa_t \in L_{\widetilde{q}}(d\mu) \subset W^{-1}_{\widetilde{q}}(d\mu)$. Therefore, $\varkappa_t \in W^{-1}_{q-}$.

3. The proof of Theorem 1′

Note that it is sufficient to obtain the existence of a solution on $[0,1]$. Really, since $\varkappa_1 \in W^{-k}_{q-}$, one can determine the solution on $[k, k+1], k \geq 0$, in succession solving (1) on $[0,1]$ with the initial value $\varkappa_k$ instead of $\varkappa_0$.

Let $g : [0,1] \to W^{-k}_{q-}$ be some measurable mapping. By [3, Theorem 5.3.1], the equation

$$x^g(u,t) = u + \int_0^t a(x^g(u,s), g(s)) ds$$

has a unique (up to μ-equivalence) solution $x^g(u,t), t \in [0,1]$ and, moreover, for every $t \in [0,1]$, the measure $\mu \circ x^g(\cdot,t)^{-1}$ is absolutely continuous with respect to μ. In the next section, we will verify that, for every $t \in [0,1]$ and $f \in W^k_{p+}$, the function $f \circ x^g(\cdot,t)$ belongs to W^k_{p+}, and there exists the image of the generalized function $\varkappa_0 \circ x^g(\cdot,t)^{-1}$, i.e. an element of W^{-k}_{q-} such that, for every $f \in W^k_{p+}$, we have $\langle f, \varkappa_0 \circ x^g(\cdot,t)^{-1} \rangle = \langle f \circ x^g(\cdot,t), \varkappa_0 \rangle$ (see Proposition 1).

Define

$$F(g)(t) = \varkappa_0 \circ x^g(\cdot,t)^{-1},\ t \in [0,1]. \tag{2}$$

Then if the function g satisfies $F(g)(t) = \varkappa_0 \circ x^g(\cdot,t)^{-1} = g(t),\ t \in [0,1]$, then $x^g(u,t)$ solves (1).

Thus, the solutions of (1) correspond to fixed points of the transformation F. To prove the existence of a fixed point, we apply the Schauder theorem.

Theorem 3 [5, Theorem 2(3.XVI)]**.** *Let Y_0 be a closed convex subset of a linear normed space Y, and let $F : Y_0 \to Y_0$ be a continuous transformation such that $F(Y_0)$ is relatively compact. Then F has a fixed point.*

Set $Y = C([0,1], W_{q-}^{-k})$, where W_{q-}^{-k} is equipped with a metric λ defined as follows. Fix a sequence $1 < q_1 < \ldots < q_n < q_{n+1} < \ldots < q$ such that $q_n \to q,\ n \to \infty$, and, for every $n \geq 1$, find the family of functions $\{f_{nm}, m \geq 1\}$ dense in $W_{p_n}^k$, where p_n is determined by the condition $\frac{1}{p_n} + \frac{1}{q_n} = 1$. For every $\varkappa_1, \varkappa_2 \in W_{q_n}^{-k}$, we set

$$\lambda_n(\varkappa_1,\varkappa_2) = \sum_{m\geq 1} \frac{1}{2^m}(1 \wedge |\langle f_{nm},\varkappa_1\rangle - \langle f_{nm},\varkappa_2\rangle|).$$

Then λ_n is a metric in $W_{q_n}^{-k}$, because $\varkappa_1 \neq \varkappa_2$ implies $\lambda_n(\varkappa_1,\varkappa_2) \neq 0$. Therefore, $\lambda(\varkappa_1,\varkappa_2) = \sum_{n\geq 1} \frac{1}{2^n}\lambda_n(\varkappa_1,\varkappa_2),\ \varkappa_1,\varkappa_2 \in W_{q-}^{-k}$ is a metric in W_{q-}^{-k}. Without loss of generality, we may assume that each of $f_{nm}, n, m \geq 1$, belongs to $\mathcal{F}C^\infty(X,\mathbb{R})$ and has bounded derivatives of any orders.

Remark. Let us compare the convergence in the metric λ with convergence in the topology τ in W_{q-}^{-k}. The sequence of elements of $W_{q_n}^{-k}$ converges $*$-weakly if and only if it is bounded in the norm of $W_{q_n}^{-k}$ and converges in the metric λ_n. Hence, the sequence of elements of W_{q-}^{-k} converges in the topology τ if and only if it is bounded in the norms of $W_{q_n}^{-k}, n \geq 1$, and converges in the metric λ.

In the next section, we will verify that, for every $g \in Y$, the function $F(g)$ defined by (2) belongs to Y, and the set $F(Y)$ is relatively compact. Also we will find a closed convex set Y_0 such that $F(Y) \subset Y_0 \subset Y$ and $F : Y_0 \to Y_0$ is continuous. Then the conditions of the Schauder theorem are valid, and the transformation F has a fixed point, which proves Theorem $1'$. □

4. Properties of the transformation F

Proposition 1. *For every $g \in Y, t \in [0,1]$, the generalized function $F(g)(t)$ belongs to W_{q-}^{-k}.*

Proof. Fix $g \in Y, t \in [0,1]$. First, we prove that there exist constants $\widetilde{c}_l, 1 \leq l \leq k$, which depend on a but do not depend on $g \in Y$ and $t \in [0,1]$ such that

$$\text{(3)} \qquad \operatorname*{esssup}_{u\in X} \|Dx^g(u,t) - 1\!\!1_H\|_{\mathcal{H}} \leq \widetilde{c}_1 \text{ and } \operatorname*{esssup}_{u\in X} \|D^l x^g(u,t)\|_{\mathcal{H}} \leq \widetilde{c}_l,\ 2 \leq l \leq k.$$

Since, for μ-almost all u, the derivative $Dx^g(u,t)$ satisfies the equation

$$Dx^g(u,t) = 1\!\!1_H + \int_0^t Da(x^g(u,s), g(s))Dx^g(u,s)ds \text{ for every } t \geq 0$$

(cf. [4, Lemma 5.17]), we have

$$\|Dx^g(u,t)\|_{\text{op}} \leq 1 + \int_0^t \|Da(x^g(u,s), g(s))\|_{\mathcal{H}} \|Dx^g(u,s)\|_{\text{op}} ds \leq$$

$$\leq 1 + c_1 \int_0^t \|Dx^g(u,s)\|_{\text{op}} ds$$

for μ-almost all u. By the Gronwall inequality, this implies $\|Dx^g(u,t)\|_{\mathrm{op}} \le e^{c_1 t} \le e^{c_1} = \widetilde{c}_{\mathrm{op}}$ for μ-a.a $u \in X, t \in [0,1]$. Hence, for μ-almost all u and for every $t \in [0,1]$,

$$\|Dx^g(u,t) - 1\!\!1_H\|_{\mathcal{H}} \le \int_0^t \|Da(x^g(u,s), g(s))\|_{\mathcal{H}} \|Dx^g(u,s)\|_{\mathrm{op}} ds \le c_1 \widetilde{c}_{\mathrm{op}} t \le c_1 \widetilde{c}_{\mathrm{op}} = \widetilde{c}_1.$$

Now

$$D^2 x^g(u,t) = \int_0^t Da(x^g(u,s), g(s)) D^2 x^g(u,s) ds + \int_0^t D^2 a(x^g(u,s), g(s))(Dx^g(u,s))^2 ds$$

implies

$$\|D^2 x^g(u,t)\|_{\mathcal{H}} \le c_1 \int_0^t \|D^2 x^g(u,s)\|_{\mathcal{H}} ds + c_2 \widetilde{c}_{\mathrm{op}}^2 \text{ for } \mu\text{-a.a } u, t \in [0,1],$$

and, by the Gronwall inequality,

$$\exists \widetilde{c}_2 > 0 \ \forall t \in [0,1] \ \forall g \in Y \quad \operatorname*{esssup}_{u \in X} \|D^2 x^g(u,t)\|_{\mathcal{H}} \le \widetilde{c}_2.$$

Similar calculations prove (3) in succession for every $l \le k$.

Fix any $n \ge 1$. Let us verify that $F(g)(t) \in W_{q_n}^{-k}$, i.e. $F(g)(t)$ is a linear continuous functional on $W_{p_n}^k$, where $p_1 > \ldots > p_n > p_{n+1} > \ldots$ are taken from the definition of the metric λ. We prove that, for every $f \in W_{p_n}^k$, the function $f \circ x^g(\cdot,t)$ belongs to $W_{p_{n+1}}^k$ and

$$\|f \circ x^g(\cdot,t)\|_{p_{n+1},k} \le \widetilde{c} \|f\|_{p_n,k}, \ f \in W_{p_n}^k. \tag{4}$$

To simplify notations, we denote here and thereafter, by $\widetilde{c}$, any constants which depend on a, p_n, and p_{n+1} and do not depend on f, g, and t.

Denote, by L_t^g, the density of the measure $\mu \circ x^g(\cdot,t)^{-1}$ with respect to μ. Then, for every $f \in L_{p_n}(X, \mathbb{R}, \mu)$ we have, by the Hölder inequality,

$$\int_X |f \circ x^g(u,t)|^{p_{n+1}} \mu(du) = \int_X |f(u)|^{p_{n+1}} L_t^g(u) \mu(du) \le$$

$$\le \left(\int_X |f(u)|^{p_n} \mu(du) \right)^{\frac{p_{n+1}}{p_n}} \cdot \left(\int_X (L_t^g(u))^{\frac{p_n}{p_n - p_{n+1}}} \mu(du) \right)^{\frac{p_n - p_{n+1}}{p_n}}.$$

By [3, Theorem 5.12] for every $c > 1$ and $0 \le t \le 1$, we have

$$\int_X (L_t^g(u))^c \mu(du) \le \left(1 + \frac{c-1}{c} \sup_{\varkappa \in W_{q-}^{-k}} \int_X \exp(c|\delta a(u,\varkappa)|) \mu(du) \right) e^{1/c},$$

hence

$$\forall f \in L_{p_n}(X, \mathbb{R}, \mu) \quad f \circ x^g(\cdot,t) \in L_{p_{n+1}}(X, \mathbb{R}, \mu) \text{ and } \|f \circ x^g(\cdot,t)\|_{L_{p_{n+1}}} \le \widetilde{c} \|f\|_{L_{p_n}}.$$

For every $f \in W_{p_n}^1$ similarly to [4, Corollary 5.6], we have

$$\int_X \|D(f \circ x^g(u,t))\|_{\mathcal{H}}^{p_{n+1}} \mu(du) = \int_X \|Df \circ x^g(u,t) Dx^g(u,t)\|_{\mathcal{H}}^{p_{n+1}} \mu(du) \le$$

$$\le \int_X \|Df \circ x^g(u,t)\|_{\mathcal{H}}^{p_{n+1}} \cdot \widetilde{c}_{\mathrm{op}}^{p_{n+1}} \mu(du) = \widetilde{c}_{\mathrm{op}}^{p_{n+1}} \int_X \|Df(u)\|_{\mathcal{H}}^{p_{n+1}} L_t^g(u) \mu(du) \le$$

$$\le \widetilde{c} \left(\int_X \|Df(u)\|_{\mathcal{H}}^{p_n} \mu(du) \right)^{\frac{p_{n+1}}{p_n}},$$

hence

$$\forall f \in W^1_{p_n} \quad f \circ x^g(\cdot,t) \in W^1_{p_{n+1}} \text{ and } \|f \circ x^g(\cdot,t)\|_{p_{n+1},1} \le \widetilde{c}\|f\|_{p_n,1}.$$

Similarly for every $f \in W^k_{p_n}$ and for every $2 \le l \le k$,

$$\int_X \|D^l(f \circ x^g(u,t))\|^{p_{n+1}}_{\mathcal{H}} \mu(du) \le \widetilde{c} \sum_{i=1}^{l} \int_X \|D^i f \circ x^g(u,t)\|^{p_{n+1}}_{\mathcal{H}} \mu(du) \le$$

$$\le \widetilde{c} \sum_{i=1}^{l} \left(\int_X \|D^i f(u)\|^{p_n}_{\mathcal{H}} \mu(du) \right)^{\frac{p_{n+1}}{p_n}}.$$

Therefore,

$$\forall f \in W^k_{p_n} \quad f \circ x^g(\cdot,t) \in W^k_{p_{n+1}} \text{ and } \|f \circ x^g(\cdot,t)\|_{p_{n+1},k} \le \widetilde{c}\|f\|_{p_n,k},$$

and (4) is proved. Hence, for every $f \in W^k_{p_n}$, we can define $\langle f, F(g)(t)\rangle = \langle f \circ x^g(\cdot,t), \varkappa_0\rangle$ because of $f \circ x^g(\cdot,t) \in W^k_{p_{n+1}}$ and $\varkappa_0 \in W^{-k}_{q-} \subset W^{-k}_{q_{n+1}}$. Moreover,

$$|\langle f, F(g)(t)\rangle| = |\langle f \circ x^g(\cdot,t), \varkappa_0\rangle| \le \|\varkappa_0\|_{q_{n+1},-k} \cdot \|f \circ x^g(\cdot,t)\|_{p_{n+1},k} \le$$

$$\le \widetilde{c}\|\varkappa_0\|_{q_{n+1},-k}\|f\|_{p_n,k} = R_n\|f\|_{p_n,k}.$$

Thus, we have $F(g)(t) \in \cap_{n\ge1} W^{-k}_{q_n} = W^{-k}_{q-}$. Proposition 1 is proved.

Proposition 2. *For every $g \in Y = C([0,1], W^{-k}_{q-})$, the function $F(g)$ belongs to Y and, moreover, the family of functions $\{F(g), g \in Y\}$ is equicontinuous.*

Proof. By Proposition 1 for every $g \in Y$, the function $F(g)$ maps $[0,1]$ to W^{-k}_{q-}. Hence, the first assertion of Proposition 2 follows from the second one.

We have to check that

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall g \in Y \ \forall t_1, t_2 \in [0,1] \quad |t_1 - t_2| < \delta \Rightarrow \lambda(F(g)(t_1), F(g)(t_2)) < \varepsilon.$$

Since

$$\lambda(F(g)(t_1), F(g)(t_2)) \le$$

$$\le \sum_{n=1}^{N} \frac{1}{2^n} \left(1 \wedge \sum_{m=1}^{N} \frac{1}{2^m} (1 \wedge |\langle f_{nm}, F(g)(t_1)\rangle - \langle f_{nm}, F(g)(t_2)\rangle|) \right) + \frac{1}{2^{N-1}},$$

it is sufficient to prove that

$\forall n, m \in \mathbb{N} \ \forall \varepsilon > 0 \ \exists \delta = \delta_{nm} > 0 \ \forall g \in Y \ \forall t_1, t_2 \in [0,1]$

$$|t_1 - t_2| < \delta \Rightarrow |\langle f_{nm}, F(g)(t_1)\rangle - \langle f_{nm}, F(g)(t_2)\rangle| < \varepsilon. \tag{5}$$

Fix $n, m \in \mathbb{N}$ and $f = f_{nm}$. By definition of the metric λ, we have $f \in \mathcal{F}C^\infty(X, \mathbb{R}) \subset W^k_{p_{n-1}}$. Then

$$\forall t \in [0,1] \quad f \circ x^g(\cdot,t) \in W^k_{p_n}$$

and

$$\forall t_1, t_2 \in [0,1] \quad |\langle f, F(g)(t_1)\rangle - \langle f, F(g)(t_2)\rangle| = |\langle f \circ x^g(\cdot,t_1) - f \circ x^g(\cdot,t_2), \varkappa_0\rangle| \le$$

$$\le \|f \circ x^g(\cdot,t_1) - f \circ x^g(\cdot,t_2)\|_{p_n,k} \cdot \|\varkappa_0\|_{q_n,-k}.$$

Since f has bounded derivatives of any orders, we have

$$\int_X |f \circ x^g(\cdot,t_1) - f \circ x^g(\cdot,t_2)|^{p_n} \mu(du) \le$$

$$\le \widetilde{c}_f \int_X \|x^g(\cdot,t_1) - x^g(\cdot,t_2)\|^{p_n}_H \mu(du).$$

Here and thereafter, we denote, by $\widetilde{c}_f$, any constants which depend on f but do not depend on $g \in Y$ and $t_1, t_2 \in [0,1]$. Also we have

$$\int_X \|D(f \circ x^g(u,t_1)) - D(f \circ x^g(u,t_2))\|^{p_n}_{\mathcal{H}} \mu(du) \le$$

$$\le 2^{p_n - 1} \int_X \|Df \circ x^g(u,t_1) - D(f \circ x^g(u,t_2)\|^{p_n}_{\mathcal{H}} \cdot$$

$$\cdot \|Dx^g(u,t_1)\|^{p_n}_{\text{op}} \mu(du) + 2^{p_n-1} \int_X \|Df \circ x^g(u,t_1)\|^{p_n}_{\mathcal{H}} \cdot \|Dx^g(u,t_1) - Dx^g(u,t_2)\|^{p_n}_{\mathcal{H}} \mu(du) \le$$

$$\le \widetilde{c}_f \left(\int_X \|x^g(u,t_1) - x^g(u,t_2)\|^{p_n}_H \mu(du) + \int_X \|Dx^g(u,t_1) - Dx^g(u,t_2)\|^{p_n}_{\mathcal{H}} \mu(du) \right)$$

and similarly

$$\forall l \le k \quad \int_X \|D^l(f \circ x^g(u,t_1)) - D^l(f \circ x^g(u,t_2))\|^{p_n}_{\mathcal{H}} \mu(du) \le$$

$$\le \widetilde{c}_f \left(\int_X \|x^g(u,t_1) - x^g(u,t_2)\|^{p_n}_H \mu(du) + \sum_{i=1}^{l} \int_X \|D^i x^g(u,t_1) - D^i x^g(u,t_2)\|^{p_n}_{\mathcal{H}} \mu(du) \right).$$

It remains to check that

(6) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall g \in Y\ \forall t_1, t_2 \in [0,1] \quad |t_1 - t_2| < \delta \Rightarrow \|x^g(\cdot,t_1) - x^g(\cdot,t_2)\|_{p_n,k} < \varepsilon.$

Let $0 \le t_1 < t_2 \le 1$. Then $x^g(u,t_2) = x^g(u,t_1) + \int_{t_1}^{t_2} a(x^g(u,s), g(s))ds$ implies, for μ-almost all u, $\operatorname{esssup}_{u \in X} \|x^g(u,t_2) - x^g(u,t_1)\|_H \le (t_2 - t_1)c_0$,

$$Dx^g(u,t_2) = Dx^g(u,t_1) + \int_{t_1}^{t_2} Da(x^g(u,s), g(s)) Dx^g(u,s)ds$$

implies, for μ-almost all u,

$$\operatorname*{esssup}_{u \in X} \|Dx^g(u,t_2) - Dx^g(u,t_1)\| \le (t_2 - t_1)c_1 \widetilde{c}_{\text{op}},$$

and similarly

$$\forall l \le k \quad \operatorname*{esssup}_{u \in X} \|D^l x^g(u,t_2) - D^l x^g(u,t_1)\|_{\mathcal{H}} \le |t_2 - t_1| \widetilde{\widetilde{c}}_l,$$

where $\widetilde{\widetilde{c}}_l$ is a function of the constants $c_0, \dots, c_l$ from condition 2) of Theorem 1′ and of the constants $\widetilde{c}_{\text{op}}, \widetilde{c}_1, \dots, \widetilde{c}_l$ defined in the proof of Proposition 1.

Therefore, $\|x^g(\cdot,t_1) - x^g(\cdot,t_2)\|_{p_n,k} \le \widetilde{\widetilde{c}}|t_1 - t_2|$, where $\widetilde{\widetilde{c}}$ does not depend on $g \in Y$ and $t_1, t_2 \in [0,1]$. Thus, (6) holds true, and the equicontinuity of the family of functions $F(Y)$ is proved.

Proposition 3. *The set $F(Y)$ is relatively compact.*

Proof. Let us verify the conditions of the Arzela–Ascoli theorem. Since, by Proposition 2, the family of functions $F(Y)$ is equicontinuous, it remains to check that, for every $t \in [0,1]$, there exists a compact set $K_t \subset W^{-k}_{q-}$ such that

$$\forall g \in Y \quad F(g)(t) \in K_t.$$

It was obtained at the end of the proof of Proposition 1 that

(7) $\forall n \ge 1\ \exists R_n > 0\ \forall f \in W^k_{p_n}\ \forall g \in Y\ \forall t \in [0,1] \quad |\langle f, F(g)(t)\rangle| \le R_n \|f\|_{p_n,k}.$

Set $K(n) = \{\varkappa \in W_{q_n}^{-k} \big| \|\varkappa\|_{q_n,-k} \le R_n\}$. Then, by the Banach–Alaoglu theorem, $K(n)$ is a $*$-weak compact in $W_{q_n}^{-k}$. Therefore, $K(n)$ is a compact in $W_{q_n}^{-k}$ with the metric λ_n. Let $K = \cap_{n\ge 1} K(n)$. Then (7) implies that, for every $t \in [0,1]$, we have $\{F(g)(t), g \in Y\} \subset K$. Moreover, K is a compact in W_{q-}^{-k}. Really, fix any sequence $\{\varkappa_m, m \ge 1\} \subset K$. For every $n \ge 1$, there exist a subsequence which converges in $W_{q_n}^{-k}$ since $K \subset K(n)$ and $K(n)$ is a compact in $W_{q_n}^{-k}$. Then, by applying the diagonal method, we can find a subsequence which converges in each of $W_{q_n}^{-k}$. Hence, this subsequence converges in W_{q-}^{-k}. Thus, K is a compact in W_{q-}^{-k}. The Arzela–Ascoli theorem implies that $F(Y)$ is relatively compact.

Set $Y_0 = C([0,1], K)$, where K is a compact constructed in the proof of Proposition 3. Then $F(Y) \subset Y_0 \subset Y$, and Y_0 is a closed convex subset of Y. It is evident that F maps Y_0 to Y_0.

Proposition 4. *The transformation F is continuous on Y_0.*

Proof. Let $g_n \to g_0,\ n \to \infty$, in Y_0. We have to check that $F(g_n) \to F(g_0),\ n \to \infty$, in Y_0. Since $F(Y_0)$ is relatively compact, there exist a subsequence $\{n_i,\ i \ge 1\}$ such that $F(g_{n_i})$ converges in Y_0 as $i \to \infty$. Thus, it is sufficient to verify that if $g_n \to g_0$ and $F(g_n) \to \widetilde{g}$ in $Y_0,\ n \to \infty$, then $\widetilde{g} = F(g_0)$. We prove that, for every f from the definition of the metric λ and for every $t \in [0,1]$, $\langle f, F(g_0)(t)\rangle = \langle f, \widetilde{g}(t)\rangle$. This will imply $\widetilde{g} = F(g_0)$. Fix any f from the definition of the metric λ for $t \in [0,1]$.

Since $\langle f, F(g_n)(t)\rangle \to \langle f, \widetilde{g}(t)\rangle,\ n \to \infty$, it is sufficient to check that $\langle f, F(g_n)(t)\rangle \to \langle f, F(g_0)(t)\rangle,\ n \to \infty$, or, equivalently $\langle f \circ x^{g_n}(\cdot,t) - f \circ x^{g_0}(\cdot,t), \varkappa_0\rangle \to 0,\ n \to \infty$. Fix any $m \in \mathbb{N}$. We will verify that

$$\forall \varkappa \in W_{q_m}^{-k} \quad \langle f \circ x^{g_n}(\cdot,t) - f \circ x^{g_0}(\cdot,t), \varkappa\rangle \to 0,\ n \to \infty. \tag{8}$$

Since $f \in \mathcal{F}C^\infty(X,\mathbb{R}) \subset W_{p_{m-1}}^k$, we get $f \circ x^{g_n}(\cdot,t) \in W_{p_m}^k$ and $\|f \circ x^{g_n}(\cdot,t)\|_{p_m,k} \le \widetilde{c}\|f\|_{p_{m-1},k}, n \ge 0$, where $\widetilde{c}$ is a constant which depends on p_{m-1}, p_m but does not depend on n. It is sufficient to check (8) for $\varkappa$ from a dense subset of $W_{q_m}^{-k}$, for example for regular generalized functions $\varkappa$ of the form

$$\langle f, \varkappa\rangle = \int_X f(u)\rho(u)\mu(du),\ \text{where } \rho \in L_{q_m}(X,\mathbb{R},\mu). \tag{9}$$

Fix any $\varkappa$ defined by (9). We have

$$|\langle f \circ x^{g_n}(\cdot,t) - f \circ x^{g_0}(\cdot,t), \varkappa\rangle| = \left|\int_X (f \circ x^{g_n}(\cdot,t) - f \circ x^{g_0}(\cdot,t))\rho(u)\mu(du)\right| \le$$

$$\le \widetilde{c}_f \int_X \|x^{g_n}(u,t) - x^{g_0}(u,t)\|_H |\rho(u)|\mu(du) \le$$

$$\le \widetilde{c}_f \left(\int_X \|x^{g_n}(u,t) - x^{g_0}(u,t)\|_H^{p_m}\mu(du)\right)^{1/p_m} \left(\int_X |\rho(u)|^{q_m}\mu(du)\right)^{1/q_m}, \tag{10}$$

where $\widetilde{c}_f$ is a constant which depends only on f.

Similar to the proof of Theorem 5.21 in [3], it can be checked that

$$\int_X \|x^{g_n}(u,t) - x^{g_0}(u,t)\|_H^{p_m}\mu(du) \le$$

$$\le \overline{c}\left(\int_X \int_0^1 \|a(u,g_n(s)) - a(u,g_0(s))\|_H^{p_{m-1}}\mu(du)ds\right)^{\frac{p_m}{p_{m-1}}},$$

where $\overline{c}$ is a constant which depends only on c_1 and $\theta(c)$ defined in conditions 2), 3) of Theorem 1′.

Note that $g_n(t) \in K, n \geq 1$. Hence, the sequence $\{g_n(t),\ n \geq 1\}$ is bounded in the norms $W_{q_m}^{-k},\ m \geq 1$. Moreover, by the remark after the definition of the metric λ, the convergence $g_n(t) \to g_0(t),\ n \to \infty$, in W_{q-}^{-k} with the metric λ implies the convergence $g_n(t) \xrightarrow{\tau} g_0(t),\ n \to \infty$. By condition 4) of Theorem 1′, this implies

$$\forall t \in [0,1] \quad a(u, g_n(t)) \xrightarrow{\ \mu\ } a(u, g_0(t)),\ n \to \infty,$$

and, by the Lebesgue dominated convergence theorem,

$$\int_X \int_0^1 \|a(u, g_n(s)) - a(u, g_0(s))\|_H^{p_m - 1} \mu(du) ds \to 0,\ n \to \infty.$$

Therefore, (10) proves (8) for any regular generalized function $\varkappa \in W_{q_m}^{-k}$. Since regular generalized functions are dense in $W_{q_m}^{-k}$, (8) is proved for every $\varkappa \in W_{q_m}^{-k}$. In particular, (8) holds for $\varkappa = \varkappa_0$. Then $F(g_0)(t) = \widetilde{g}(t), t \in [0,1]$, and the continuity of F is proved.

5. The proof of Theorem 2

Assume that Eq. (1) has solutions $x(u,t)$ and $y(u,t)$. Then

$$x(u,t) = u + \int_0^t a(x(u,s), \varkappa_s^x) ds,\ t \geq 0,$$
$$y(u,t) = u + \int_0^t a(y(u,s), \varkappa_s^y) ds,\ t \geq 0,$$

where $\varkappa_s^x = \varkappa_0 \circ x(\cdot, s)^{-1} \in W_q^{-k}$, $\varkappa_s^y = \varkappa_0 \circ y(\cdot, s)^{-1} \in W_q^{-k},\ s \geq 0$. We will find $t_0 > 0$ which depends only on $c_0, \ldots, c_k, L$ and $\|\varkappa_0\|_{q,-k}$ from the conditions of Theorems 1 and 2 such that $x(u,s) = y(u,s),\ s \leq t_0$ for μ-almost all u. This implies the uniqueness of the solution.

Set

$$\Delta_l x(t) = \operatorname*{esssup}_{u \in X} \sup_{s \leq t} \|D^l x(u,s) - D^l y(u,s)\|_{\mathcal{H}},\ 0 \leq l \leq k,$$
$$\Delta \varkappa(t) = \sup_{s \leq t} \|\varkappa_s^x - \varkappa_s^y\|_{q_1, -k-1},$$

where $q_1 < q$ is defined in the formulation of Theorem 2.

Note that $\Delta\varkappa(t)$ is correctly defined since

$$\varkappa_s^x - \varkappa_s^y \in W_q^{-k} \subset W_{q_1}^{-k-1},\ q_1 < q.$$

We have $\|x(u,t) - y(u,t)\|_H \leq L \left(\int_0^t \|x(u,t) - y(u,t)\|_H ds + \int_0^t \|\varkappa_s^x - \varkappa_s^y\|_{q_1,-k-1} ds \right)$, thus

$$\Delta_0 x(t) \leq Lt\Delta_0 x(t) + Lt\Delta\varkappa(t). \tag{11}$$

Let us estimate $\Delta\varkappa(t)$. We have

$$\|\varkappa_s^x - \varkappa_s^y\|_{q_1,-k-1} = \sup_{\|f\|_{p_1,k+1} \leq 1} |\langle f \circ x(\cdot,s) - f \circ y(\cdot,s), \varkappa_0 \rangle| \leq$$

$$\leq \|\varkappa\|_{q,-k} \sup_{\|f\|_{p_1,k+1} \leq 1} \|f \circ x(\cdot,s) - f \circ y(\cdot,s)\|_{p,k},$$

where $\frac{1}{p_1} + \frac{1}{q_1} = 1$, since, for every $f \in W_{p_1}^{k+1} \subset W_{p_1}^k$, we have $f \circ x(\cdot,s) \in W_p^k,\ f \circ y(\cdot,s) \in W_p^k$.

Set

$$\Delta_l \varkappa(t) = $$
$$= \sup_{s\le t} \sup_{\|f\|_{p_1,k+1}\le 1} \left(\int_X \|D^l(f\circ x(u,s)) - D^l(f\circ y(u,s))\|^p_{\mathcal{H}} \mu(du)\right)^{1/p}, \ 0\le l\le k.$$

Then

$$\Delta\varkappa(t) \le \|\varkappa_0\|_{q,-k} \sum_{l=0}^{k} \Delta_l \varkappa(t). \tag{12}$$

By (3), there exists a constant $\widetilde{c}$ which depends only on a and is such that

$$\operatorname*{esssup}_{u\in X} \sup_{t\le 1} \left(\|x(u,t)-u\|_H + \|Dx(u,t) - \mathbb{I}_H\|_{\mathcal{H}} + \sum_{l=2}^{k} \|D^l x(u,t)\|_{\mathcal{H}} \right) \le \widetilde{c},$$

$$\operatorname*{esssup}_{u\in X} \sup_{t\le 1} \left(\|y(u,t)-u\|_H + \|Dy(u,t) - \mathbb{I}_H\|_{\mathcal{H}} + \sum_{l=2}^{k} \|D^l y(u,t)\|_{\mathcal{H}} \right) \le \widetilde{c}.$$

Similarly to the proof of Proposition 5.2.1 in [3], it can be checked that

$$\int_X |f\circ x(u,s) - f\circ y(u,s)|^p \mu(du) \le$$
$$\le \widetilde{c} \left(\int_X \|Df(u)\|^{p_1}_{\mathcal{H}} \mu(du)\right)^{p/p_1} \left(\int_0^s \int_X \|a(u,\varkappa^x_r) - a(u,\varkappa^y_r)\|^{p_2}_H \mu(du)dr\right)^{p/p_2} \le$$
$$\le \widetilde{c}\Delta\varkappa(t) t^{p/p_2}, s\le t,$$

where $\widetilde{c}$ is a constant which depends only on a and $p_2 > p$. Thus, $\Delta_0\varkappa(t) \le \widetilde{c}\Delta\varkappa(t)t^{1/p_2}$. Let us estimate $\Delta_1\varkappa(t)$. By the chain rule, we have

$$\int_X \|D(f\circ x(u,s)) - D(f\circ y(u,s))\|^p_{\mathcal{H}}\mu(du) \le$$
$$\le 2^{p-1} \int_X \|Df\circ x(u,s) - Df\circ y(u,s)\|^p_{\mathcal{H}} \mu(du)\cdot$$
$$\cdot \left(\operatorname*{esssup}_{u\in X} \sup_{s\le t} \|Dx(u,s)\|_{\text{op}}\right)^p + 2^{p-1} \left(\int_X \|Df\circ y(u,s)\|^{\widetilde{p}}_{\mathcal{H}} \mu(du)\right)^{p/\widetilde{p}} \cdot \tag{13}$$
$$\cdot \left(\int_X \|Dx(u,s) - Dy(u,s)\|^{\widetilde{q}}_{\mathcal{H}} \mu(du)\right)^{p/\widetilde{q}},$$

where $\frac{1}{\widetilde{p}} + \frac{1}{\widetilde{q}} = \frac{1}{p}$ and $p < \widetilde{p} < p_1$.

Denote, by L^y_s, the density of the measure $\mu\circ y(\cdot,s)^{-1}$ with respect to μ. Then

$$\int_X \|Df\circ y(u,s)\|^{\widetilde{p}}_{\mathcal{H}} \mu(du) = \int_X \|Df(u)\|^{\widetilde{p}}_{\mathcal{H}} L^y_s(u)\mu(du) \le$$
$$\le \left(\int_X \|Df(u)\|^{p_1}_{\mathcal{H}} \mu(du)\right)^{\widetilde{p}/p_1} \left(\int_X (L^y_s(u))^{\widetilde{q}_1} \mu(du)\right)^{\widetilde{p}/q_1},$$

where $\frac{1}{p_1} + \frac{1}{\widetilde{q}_1} = \frac{1}{\widetilde{p}}$.

Note that $y(u,t)$ satisfie the equation

$$y(u,t) = u + \int_0^t \overline{a}^y_s(y(u,s))ds$$

with $\overline{a}_s^y = a(\cdot, \varkappa_s^y)$. Then, by [3, Theorem 5.1.2], there exists a constant $\widetilde{c} > 0$ which depends only on a such that

$$\int_X (L_s^y(u))^{\widetilde{q}_1} \mu(du) \le \widetilde{c},\ 0 \le s \le t.$$

Also we have

$$\|Dx(u,t) - Dy(u,t)\|_{\mathcal{H}} \le \int_0^t \|Da(x(u,s), \varkappa_s^x)\|_{\mathcal{H}} \cdot \|Dx(u,s) - Dy(u,s)\|_{\mathcal{H}} ds +$$
$$+ \int_0^t \|Da(x(u,s), \varkappa_s^x) - Da(y(u,s), \varkappa_s^y)\|_{\mathcal{H}} \cdot \|Dy(u,s)\|_{\text{op}} ds \le$$
$$\le \widetilde{c} \left(\int_0^t (\|Dx(u,s) - Dy(u,s)\|_{\mathcal{H}} + \|x(u,s) - y(u,s)\|_H + \|\varkappa_s^x - \varkappa_s^y\|_{q_1,-k-1})\, ds \right)$$

for μ-almost all u, where $\widetilde{c}$ depends only on a and L.

Similarly to the proof of Proposition 5.2.1 in [3], we get

$$\int_X \|Df \circ x(u,s) - Df \circ y(u,s)\|_{\mathcal{H}}^p \mu(du) \le \widetilde{c} \left(\int_X \|D^2 f(u)\|_{\mathcal{H}}^{p_1} \right)^{p/p_1} \cdot$$
$$\cdot \left(\int_0^s \|Da(u, \varkappa_r^x) - Da(u, \varkappa_r^y)\|_{\mathcal{H}}^{p_2} \mu(du) dr \right)^{p/p_2} \le \widetilde{c} \Delta\varkappa(t) t^{p/p_2}, s \le t,$$

where $\widetilde{c}$ is a constant which depends only on a. Hence, (13) implies

$$\Delta_1 \varkappa(t) \le \widetilde{c}(\Delta\varkappa(t) t^{1/p_2} + (\Delta_1 x(t) + \Delta_0 x(t) + \Delta\varkappa(t))t).$$

Similarly,

$$\Delta_l \varkappa(t) \le \widetilde{c} \sup_{s \le t} \sup_{\|f\|_{p_1,k+1} \le 1} \left(\sum_{i \le l} \int_X \|D^i f \circ x(u,s) - D^i f \circ y(u,s)\|_{\mathcal{H}}^p \mu(du) \right)^{1/p} +$$
$$+ \widetilde{c} \sum_{i \le l} \Delta_i x(t),$$
$$\int_X \|D^i f \circ x(u,s) - D^i f \circ y(u,s)\|_{\mathcal{H}}^p \mu(du) \le \widetilde{c} t^{p/p_2} \Delta\varkappa(t), s \le t$$

and

$$\Delta_i x(t) \le \widetilde{c} t \left(\sum_{j \le i} \Delta_j x(t) + \Delta\varkappa(t) \right). \tag{14}$$

Hence,

$$\Delta_l \varkappa(t) \le \widetilde{c} \left(\Delta\varkappa(t) t^{1/p_2} + \left(\sum_{i \le l} \Delta_i x(t) + \Delta\varkappa(t))t \right) \right). \tag{15}$$

By (11), (12), (14), and (15), we have

$$\Delta\varkappa(t) + \sum_{i \le k} \Delta_i x(t) \le c t^{1/p_2} \left(\Delta\varkappa(t) + \sum_{i \le k} \Delta_i x(t) \right), t < 1,$$

where c depends only on a, $\|\varkappa_0\|_{q,-k}$, and L. Thus, for $0 < t_0 < 1$ such that $ct_0^{1/p_2} < 1$, we have $\Delta_0 x(t_0) = 0$. That is, for μ-almost all u for every $s \le t_0$, we have $x(u,s) = y(u,s)$. The uniqueness is proved. □

Consider an example which shows that one actually needs Lipschitzian conditions with the $\|\cdot\|_{q_1,-k-1}$-norm in Theorem 2, although generalized functions belong to W_q^{-k}.

Example 2. Let $X=\mathbb{R}$, $\mu(du)=\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du$, and the generalized function $\varkappa_0$ is the delta-function δ_0, i.e. $\langle f,\varkappa_0\rangle=\langle f,\delta_0\rangle=f(0)$. Fix $1<p<2$. By the Sobolev embedding theorem, $W_p^1(\mathbb{R},\mu)\subset C(\mathbb{R})$. Hence, $\delta_0\in W_q^{-1}$, where $\frac{1}{p}+\frac{1}{q}=1$. Set $f(x)=\sqrt{|x|},\ x\in\mathbb{R}$. Then $f\in W_p^1$ for $1<p<2$ and $\langle f,\varkappa\rangle,\varkappa\in W_q^{-1}$, is correctly defined. Let $\varphi\in C_0^\infty(\mathbb{R})$ be such that $|x|\le\frac{1}{2}$ implies $\varphi(x)=x$, $|x|\ge 1$ implies $\varphi(x)=0$, $\max_{x\in\mathbb{R}}|\varphi(x)|\le 1$, and $\max_{x\in\mathbb{R}}|\varphi'(x)|\le 2$. Let $a(u,\varkappa)=a(\varkappa)=\varphi(\langle f,\varkappa\rangle),\varkappa\in W_q^{-1}$. Then Eq. (1) has the form

$$\begin{cases}\frac{dx(u,t)}{dt}=\varphi(\langle f,\varkappa_t\rangle)=\varphi(\langle f\circ x(\cdot,t),\varkappa_0\rangle)=\varphi(\sqrt{|x(0,t)|}),\\ x(u,0)=u.\end{cases}$$

Consider $x_1(0,t)=0$, $x_2(0,t)=\frac{1}{4}t^2, t\in[0,1]$. Then $x_{1,2}(u,t)=u+x_{1,2}(0,t)$ are two distinct solutions of (1) on $[0,1]$. It is straightforward to verify that the conditions of Theorem 1 are valid. Moreover, we have

$$\forall\varkappa_1,\varkappa_2\in W_q^{-1}\quad |a(\varkappa_1)-a(\varkappa_2)|=|\varphi(\langle f,\varkappa_1\rangle)-\varphi(\langle f,\varkappa_2\rangle)|\le$$
$$\le 2|\langle f,\varkappa_1\rangle-\langle f,\varkappa_2\rangle|\le 2\|f\|_{p,1}\|\varkappa_1-\varkappa_2\|_{q,-1},$$

but the solution of (1) is not unique.

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