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TARAS O. ANDROSHCHUK AND ALEXEY M. KULIK

LIMIT THEOREMS FOR OSCILLATORY FUNCTIONALS OF A MARKOV PROCESS

We study the limit behavior of a family of functionals from a given Markov process which are called *oscillatory functionals*. The typical oscillatory functional is homogeneneous and non-negative but neither additive nor continuous. We claim that the discontinuity and non-additivity of functionals from a given family vanish in the limit and, in this framework, prove a generalization of the theorem by E.B. Dynkin on the convergence of a family of W-functionals.

Introduction

The well-known theorem by E.B. Dynkin (see [1], Ch. 6.3) provides a characterization of the L_2-convergence of W-functionals from a given Markov process in terms of their characteristics. In this paper, we consider the families of functionals satisfying some weaker version of the conditions of the Dynkin theorem. Namely, elements of the family may fail to be continuous and additive (and therefore to be W-functionals), but their discontinuity and non-additivity vanish in the limit in an appropriate sense. The limit results for some families of functionals of such a type have been known for a long time. We mention three such examples: the normalized numbers of downcrossings of the interval by a diffusion (see [2], §2.4, 6.5a), the normalized numbers of intersections of a level by the diffusion (see [3], §6), and the entropy-like metric characteristics of the set of zeros of the diffusion (see [2], §2.5, 6.5b). Our aim is to give a unified approach to the limit theorems for functionals of such a type (we informally call them oscillatory functionals), similar to the one introduced by E.B. Dynkin in [1].

1. Main theorems

In a sequel, we suppose a locally compact metric space $(\mathcal{X}, \rho)$ with a homogeneous Markov process $(\{X_t, t \geq 0\}, \{\mathcal{M}_t, t \geq 0\}, \{P_x, x \in \mathcal{X}\})$ to be fixed. Here and below, the E.B.Dynkin's notation (see [1], Ch. 3.1) is used. In order to shorten the exposition, we consider only the case where the lifetime of the process $\zeta = +\infty$. The process is claimed to be a Feller one and to have right continuous trajectories. Therefore (see ([1], Ch. 3.2), the flow $\{\mathcal{M}_t, t \geq 0\}$ can be supposed to be right continuous and complete w.r.t. every probability $P_x, x \in \mathcal{X}$. We also suppose (this does not restrict generality, but makes notation more simple) that, for every t, $\mathcal{M}_t$ is the smallest σ-algebra, complete w.r.t. every probability $P_x, x \in \mathcal{X}$ and containing all sets of the type

$$\{\omega : X_{s_1}(\omega) \in \Delta_1, \dots, X_{s_n}(\omega) \in \Delta_n\}, \quad s_1, \dots, s_n \leq t, \quad \Delta_1, \dots, \Delta_n \in \mathcal{B}(\mathcal{X}).$$

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Let us recall some notions. The family $\{\phi_t^s, 0 \le s \le t < +\infty\}$ of random variables is called a functional of the Markov process $\{X_t\}$ if, for every $\Gamma \in \mathcal{B}(\Re)$ $\{\omega : \phi_t^s(\omega) \in \Gamma\} \in \mathcal{M} \equiv \bigvee_t \mathcal{M}_t$. A functional $\{\phi_t^s\}$ is called homogeneous if, for every $s \le t, h > 0$, $\theta_h \phi_t^s = \phi_{t+h}^{s+h}$ almost surely. Here, θ_h denotes the operator of the time shift ([1], Ch. 3.1).

Definition 1. *The family $\{\phi^\varepsilon \equiv \{\phi_t^{s,\varepsilon}\}, \varepsilon > 0\}$ is called a QW-family (quasi-W-family) of functionals if, for every ε $\{\phi_t^{s,\varepsilon}\}$ is a homogeneous non-negative functional, non-decreasing and càdlàg w.r.t. to t, and two following conditions hold true:*

VD. *("vanishing discontinuity"): almost surely for all $0 \le s \le t$,*

$$\phi_t^{s,\varepsilon} - \phi_{t-0}^{s,\varepsilon} \le \varepsilon;$$

VN. *("vanishing non-additivity"): almost surely for all $0 \le s \le t \le u$,*

$$|\phi_u^{s,\varepsilon} - \phi_t^{s,\varepsilon} - \phi_u^{t,\varepsilon}| \le \varepsilon;$$

For a given QW-family $\{\phi^\varepsilon\}$, we denote the corresponding family of characteristics by $\{f^\varepsilon\}$:

$$f_t^\varepsilon(x) = E_x \phi_t^{0,\varepsilon} = E_x \phi_{t+s}^{s,\varepsilon}, \quad x \in \mathcal{X}, s, t \ge 0.$$

Here and below, E_x denotes the expectation w.r.t. P_x. One version of the main result of the paper is given in the following theorem analogous to Theorem 6.4 [1].

Theorem 1. *1. Let QW-family $\{\phi^\varepsilon\}$ be such that*

$$\|f_t^\varepsilon\| \equiv \sup_{x \in \mathcal{X}} f_t^\varepsilon(x) < \infty, \quad \varepsilon > 0,\ t \ge 0, \tag{1}$$

and there exists a function $f = \{f_t(x), t \ge 0, x \in \mathcal{X}\}$ such that

$$\lim_{\varepsilon \to 0} \sup_{0 \le u \le t} \|f_u^\varepsilon - f_u\| = 0, \quad t \ge 0. \tag{2}$$

Then f is a characteristic of some non-negative homogenous additive functional ϕ, and

$$\phi_t^s = \underset{\varepsilon \to 0}{l.i.m.}\ \phi_t^{s,\varepsilon}, \quad 0 \le s \le t < +\infty.$$

2. Let the following additional condition hold true:

US. *("upper semiadditivity"): almost surely for all $0 \le s \le t \le u$,*

$$\phi_u^{s,\varepsilon} - \phi_t^{s,\varepsilon} - \phi_u^{t,\varepsilon} \ge 0.$$

Then ϕ is a V-functional.

The method of proof is analogous to that of Theorem 6.4 [1] and is based on two auxiliary lemmas corresponding to Lemma 6.4 and Lemma 6.5 [1].

Lemma 1. *Let conditions* **VD**, **VN** *and (1) hold true, and let $\varepsilon > 0$ be fixed. Then*

$$E_x[\phi_t^{0,\varepsilon}]^2 \le 2\|f_t^\varepsilon\|^2 + 3\varepsilon\|f_t^\varepsilon\|, \quad t \ge 0, x \in \mathcal{X}.$$

Proof. For a fixed t, let us take a partition $S = \{0 = s_0 < s_1 < \cdots < s_M = t\}$ and decompose $\phi_t^{0,\varepsilon}$ into a sum

$$\phi_t^{0,\varepsilon} = \sum_{j=0}^{M-1} \Phi_j^S, \quad \Phi_j^S \equiv \phi_{s_{j+1}}^{0,\varepsilon} - \phi_{s_j}^{0,\varepsilon}.$$

One has

$$[\phi_t^{0,\varepsilon}]^2 = \sum_{j=0}^{M-1} [\Phi_j^S]^2 + 2\sum_{i<j} \Phi_i^S \Phi_j^S = \Sigma_1^S + 2\Sigma_2^S.$$

Since $\phi_t^{0,\varepsilon}$ is non-decreasing as a function of t, we have $\Phi_j^S \geq 0$. Therefore, if we have two partitions $S \subset \tilde{S}$, then $\Sigma_1^S \geq \Sigma_1^{\tilde{S}}$, and thus $\Sigma_2^S \leq \Sigma_2^{\tilde{S}}$. Now let us take a sequence of partitions S_n such that $S_1 \subset S_2 \subset \dots$ and the diameter $|S_n|$ of S_n tends to 0 as $n \to +\infty$. Then, with probability 1,

$$\Sigma_1^{S_n} \to \mathrm{var}_2(\phi_\cdot^{0,\varepsilon})_t = \sum_{s \leq t} [\phi_s^{0,\varepsilon} - \phi_{s-}^{0,\varepsilon}]^2 =: \zeta_t,$$

and $2\Sigma_2^{S_n} \uparrow [\phi_t^{0,\varepsilon}]^2 - \zeta_t, n \to +\infty$. Denote $\Delta_i^{S_n} = \phi_t^{0,\varepsilon} - \phi_{s_{i+1}^n}^{0,\varepsilon} - \phi_t^{s_{i+1}^n,\varepsilon}$. Due to condition **VN**, $|\Delta_i^{S_n}| \leq \varepsilon$. Then

$$E_x \Sigma_2^{S_n} = E_x \sum_{i=0}^{M_n-1} \Phi_i^{S_n} [\phi_t^{s_{i+1}^n,\varepsilon} + \Delta_i^{S_n}] \leq \varepsilon E_x \sum_{i=1}^{M_n-1} \Phi_i^{S_n} + E_x \sum_{i=0}^{M_n-1} \Phi_i^{S_n} \theta_{s_{i+1}^n} \phi_t^{0,\varepsilon}. \quad (3)$$

The first summand on the right-hand side of (3) is equal to $\varepsilon f_t^\varepsilon(x)$. The second summand is estimated by (see [1], (6.25) and Theorem 3.1)

$$E_x \sum_{i=0}^{M_n-1} \left(\Phi_i^{S_n} E_{X_{s_{i+1}^n}} \phi_t^{0,\varepsilon} \right) = E_x \sum_{i=0}^{M_n-1} \Phi_i^{S_n} f_t^{0,\varepsilon}(X_{s_{i+1}^n}) \leq \|f_t^\varepsilon\| E_x \phi_t^{0,\varepsilon} \leq \|f_t^\varepsilon\|^2. \quad (4)$$

Thus, due to the theorem on monotone convergence,

$$E_x[\phi_t^{0,\varepsilon}]^2 - \zeta_t \leq 2\|f_t^\varepsilon\|(\|f_t^\varepsilon\| + \varepsilon). \quad (5)$$

On the other hand, condition **VD** implies that $\zeta_t \leq \varepsilon \phi_t^{0,\varepsilon}$ and therefore $E_x \zeta_t \leq \varepsilon f_t^\varepsilon(x) \leq \varepsilon \|f_t^\varepsilon\|$, which together with (5) gives the needed statement. The lemma is proved.

Lemma 2. *Under the conditions of Lemma 1,*

$$E_x \left(\phi_t^{s,\varepsilon} - \phi_t^{s,\tilde{\varepsilon}} \right)^2 \leq \left[2 \sup_{u \leq \ t-s} \|f_u^\varepsilon - f_u^{\tilde{\varepsilon}}\| + 5 \max(\varepsilon, \tilde{\varepsilon}) \right] \times \left[f_t^{s,\varepsilon}(x) + f_t^{s,\tilde{\varepsilon}}(x) \right],$$

$0 \leq s \leq t$, $\varepsilon, \tilde{\varepsilon} > 0$, *where* $f_t^{s,\varepsilon}(x) \equiv E_x \phi_t^{s,\varepsilon}$.

Proof. We consider only the case where $s = 0$, the general case is analogous. In order to simplify the notation, we write $\phi_t^s = \phi_t^{s,\varepsilon}, \tilde{\phi}_t^s = \phi_t^{s,\tilde{\varepsilon}}, \delta = \max(\varepsilon, \tilde{\varepsilon})$. For a fixed t and a fixed partition S of $[0,t]$, we denote $\Phi_j = \phi_{s_{j+1}}^0 - \phi_{s_j}^0, \tilde{\Phi}_j = \tilde{\phi}_{s_{j+1}}^0 - \tilde{\phi}_{s_j}^0$. We have

$$(\phi_t^0 - \tilde{\phi}_t^0)^2 = \Sigma_3^S + 2\Sigma_4^S, \qquad \Sigma_3^S = \sum_{j=0}^{M-1} (\Phi_j - \tilde{\Phi}_j)^2,$$

$$\Sigma_4^S = \sum_{i=0}^{M-1} (\Phi_i - \tilde{\Phi}_i) \Big[(\phi_t^0 - \phi_{s_{i+1}}^0) - (\tilde{\phi}_t^0 - \tilde{\phi}_{s_{i+1}}^0) \Big].$$

Denoting also $\Delta_i = \phi_t^0 - \phi_{s_{i+1}}^0 - \phi_t^{s_{i+1}}$, $\tilde{\Delta}_i = \tilde{\phi}_t^0 - \tilde{\phi}_{s_{i+1}}^0 - \tilde{\phi}_t^{s_{i+1}}$, we have that $\Delta_i, \tilde{\Delta}_i \in [-\delta, \delta]$. The expectation of Σ_4^S can be expressed and estimated analogously to (3),(4):

$$E_x \Sigma_4^S = E_x \sum_{i=0}^{M-1} (\Phi_i - \tilde{\Phi}_i)(\Delta_i - \tilde{\Delta}_i) + E_x \sum_{i=1}^{M-1} (\Phi_i - \tilde{\Phi}_i)[f_{t-s_{i+1}}(X_{s_{i+1}}) - \tilde{f}_{t-s_{i+1}}(X_{s_{i+1}})], \quad (6)$$

where $\tilde{f}$ denotes the characteristic of $\tilde{\phi}$. The first summand in (6) is estimated by

$$E_x \sum_{i=0}^{M-1} (\Phi_i + \tilde{\Phi}_i)(|\Delta_i| + \tilde{|}\Delta_i|) \leq 2\delta E_x(\phi_t^0 + \tilde{\phi}_t^0) \leq 2\delta[f_t(x) + \tilde{f}_t(x)].$$

The second summand is estimated by

$$E_x\sum_{i=0}^{M-1}(\Phi_i+\tilde{\Phi}_i)\Big|f_{t-s_{i+1}}(X_{s_{i+1}})-\tilde{f}_{t-s_{i+1}}(X_{s_{i+1}})\Big|\le\sup_{u\le t}\|f_u-\tilde{f}_u\|\times[f_t(x)+\tilde{f}_t(x)].$$

Therefore,

$$E_x(\phi_t^0-\tilde{\phi}_t^0)^2\le\left(4\delta+2\sup_{u\le t}\|f_u-\tilde{f}_u\|\right)\times[f_t(x)+\tilde{f}_t(x)]+\limsup_{|S|\to0}E_x\Sigma_3^S. \qquad (7)$$

We have $\Sigma_3^S\le2(\zeta_t^S+\zeta_t^S)$, where

$$\zeta_t^S=\sum_{j=0}^{M-1}\Phi_j^2,\quad\tilde{\zeta}_t^S=\sum_{j=1}^{M-1}\tilde{\Phi}_j^2.$$

For every S, the variables $\zeta_t^S,\tilde{\zeta}_t^S$ are majorized by the variables $[\phi_t^0]^2,[\phi_t^0]^2$ which are integrable due to Lemma 1. On the other hand,

$$\zeta_t^S\to\zeta_t=\sum_{s\le t}[\phi_s^0-\phi_{s-}^0]^2,\quad\tilde{\zeta}_t^S\to\tilde{\zeta}_t=\sum_{s\le t}[\tilde{\phi}_s^0-\tilde{\phi}_{s-}^0]^2,\quad|S|\to0,$$

$E_x\zeta_t\le\varepsilon f_t(x),E_x\tilde{\zeta}_t\le\tilde{\varepsilon}\tilde{f}_t(x)$, thus

$$\limsup_{|S|\to0}E_x\Sigma_3^S\le\delta[f_t(x)+\tilde{f}_t(x)].$$

This together with (7) gives the needed estimate. The lemma is proved.

Proof of the theorem. It follows immediately from the statement of Lemma 2 that, for every s,t,

$$\lim_{\varepsilon,\ \tilde{\varepsilon}\to0}\sup_{x\in\mathcal{X}}E_x\left(\phi_t^{s,\varepsilon}-\phi_t^{s,\tilde{\varepsilon}}\right)^2=0$$

Therefore (see Lemma 6.3 [1]), there exists the limit ϕ in mean square of functionals ϕ^ε as $\varepsilon\to0$.

By virtue of general results about the convergence of functionals ([1], Ch. 6.2)), the functional ϕ is a non-negative and homogeneous functional of the process X. (note that we have changed slightly the terminology from [1] Ch.6; our functionals are *almost* non-negative and *almost* homogenous in the terminology of E.B. Dynkin). Taking limits on the both parts of condition **VN**, we obtain the additivity of the limit functional ϕ. The function f from condition (2) of the Theorem is obviously the characteristic of the functional ϕ. Statement 1 is proved.

To prove Statement 2, it is sufficient to prove (see Theorem 6.2 [1]) that, for an arbitrary starting distribution μ of the process X, the following convergence takes place:

$$\lim_{|S|\to0}E_\mu\sum_{i=0}^{M-1}(\phi_{s_{i+1}}^{s_i})^2=0. \qquad (8)$$

Obviously,

$$E_x(\phi_{s_{i+1}}^{s_i})^2\le2E_x[\phi_{s_{i+1}}^{s_i}-\phi_{s_{i+1}}^{s_i,\varepsilon}]^2+2E_x[\phi_{s_{i+1}}^{s_i,\varepsilon}]^2.$$

From the definition of the functional ϕ and the statement of Lemma 2, we have

$$E_x[\phi_{s_{i+1}}^{s_i}-\phi_{s_{i+1}}^{s_i,\varepsilon}]^2=\lim_{\tilde{\varepsilon}\to0}E_x[\phi_{s_{i+1}}^{s_i,\tilde{\varepsilon}}-\phi_{s_{i+1}}^{s_i,\varepsilon}]^2\le$$

$$\le\lim_{\tilde{\varepsilon}\to0}[2\sup_{u\le s_{i+1}-s_i}||f_u^{\tilde{\varepsilon}}-f_u^\varepsilon||+5\max(\varepsilon,\tilde{\varepsilon})]\times[f_{s_{i+1}}^{s_i,\tilde{\varepsilon}}(x)+f_{s_{i+1}}^{s_i,\varepsilon}(x)]=$$

$$=[2\sup_{u\le s_{i+1}-s_i}||f_u-f_u^\varepsilon||+5\varepsilon]\times[f_{s_{i+1}}^{s_i}(x)+f_{s_{i+1}}^{s_i,\varepsilon}(x)].$$

Thus

$$E_x(\phi^{s_i}_{s_{i+1}})^2 \le 2\left[2\sup_{u\le s_{i+1}-s_i}||f_u - f^\varepsilon_u|| + 5\varepsilon\right]\times\left[f^{s_i}_{s_{i+1}}(x) + f^{s_i,\varepsilon}_{s_{i+1}}(x)\right] + 2E_x\left[\phi^{s_i,\varepsilon}_{s_{i+1}}\right]^2.$$

Taking the sum over $i=\overline{0,M-1}$ and noting that the inequality $f^{s,\varepsilon}_t + f^{t,\varepsilon}_u \le f^{s,\varepsilon}_u$ holds under condition **US** for $s<t<u$, we have

$$E_x\sum_{i=0}^{M-1}(\phi^{s_i}_{s_{i+1}})^2 \le \left[4\sup_{u\le t-s}||f_u - f^\varepsilon_u|| + 10\varepsilon\right]\times[f^s_t(x) + f^{s,\varepsilon}_t(x)] + 2E_x\sum_{i=0}^{M-1}\left[\phi^{s_i,\varepsilon}_{s_{i+1}}\right]^2$$

. This implies that, for an arbitrary probability distribution μ,

$$E_\mu\sum_{i=0}^{M-1}(\phi^{s_i}_{s_{i+1}})^2 \le \left[4\sup_{u\le t-s}||f_u - f^\varepsilon_u|| + 10\varepsilon\right]\times\|f^s_t + f^{s,\varepsilon}_t\| + 2E_\mu\sum_{i=0}^{M-1}\left[\phi^{s_i,\varepsilon}_{s_{i+1}}\right]^2.$$

We have that $\sum_{i=0}^{M-1}\left[\phi^{s_i,\varepsilon}_{s_{i+1}}\right]^2 \le \Sigma^S_3$ (see notations in the proof of Lemma 1). Due to the estimate of the second moment of Σ^S_1 given in this proof,

$$\limsup_{|S|\to 0} E_\mu\sum_{i=0}^{M-1}(\phi^{s_i}_{s_{i+1}})^2 \le \left[4\sup_{u\le t-s}||f_u - f^\varepsilon_u|| + 10\varepsilon\right]\times\|f^s_t + f^{s,\varepsilon}_t\| + 2\varepsilon\|f^\varepsilon_t\|.$$

Now taking $\varepsilon\to 0$, we obtain (8), which completes the proof of the theorem.

Let us formulate another version of Theorem 1 under slightly different suppositions on the family of functionals. Let us suppose functionals ϕ^ε to be homogeneous not at an arbitrary moment of the time but at the points of some partitions $S^\varepsilon = \{0 = s^\varepsilon_0 < s^\varepsilon_1 < \dots\}$ of $\Re^+$:

$$\phi^{s^\varepsilon_i,\varepsilon}_{s^\varepsilon_{i+1}} = \theta_{s^\varepsilon_i}\phi^{0,\varepsilon}_{s^\varepsilon_{i+1}-s^\varepsilon_i}. \tag{9}$$

Let also $|S^\varepsilon|\to 0$ as $\varepsilon\to 0$. Such a supposition enlarges the class of families which can be treated in our approach, but now we cannot use the construction of Lemma 1 in order to estimate $E\Sigma^S_1$ (and $E\Sigma^S_3$ in a sequel). Therefore, we impose the following additional condition: there exists a non-random $r(\varepsilon)$ such that $\lim_{\varepsilon\to 0} r(\varepsilon) = 0$ and

$$\phi^{s^\varepsilon_i,\varepsilon}_{s^\varepsilon_{i+1}} \le r(\varepsilon), \quad i\ge 0. \tag{10}$$

Theorem 2. *Let the family $\{\phi^\varepsilon\}$ satisfy conditions (1),(2),(9),(10),* **VN**,**VD**, *and let the function f be continuous w.r.t. $(x,t)\in\mathcal{X}\times\Re^+$.*

Then f is a characteristic of some non-negative homogenous additive functional ϕ, and

$$\phi^s_t = \underset{\varepsilon\to 0}{l.i.m.}\ \phi^{s,\varepsilon}_t, \quad 0\le s\le t<+\infty.$$

Under additional condition **US**, *ϕ is a V-functional.*

Proof. We will prove the theorem under the additional supposition that $\mathcal{X}$ is compact (this, due to standard localization arguments, does not restrict generality). Then, for every $\delta>0$,

$$\varpi(t,\delta) \equiv \sup_{u\le t}\sup_x P_x(\rho(X_u,x)>\delta)\to 0, \quad t\to 0+.$$

By $W^\gamma(\cdot)$ $W(\cdot)$, we denote, respectively, the moduli of continuity of the family $\{f^\varepsilon,\varepsilon\le\gamma\}$ and of f:

$$W^\gamma(\delta) = \sup_{\varepsilon\le\gamma,\rho(x,y)\le\delta}|f^\varepsilon(x) - f^\varepsilon(y)|, \quad W(\delta) = \sup_{\rho(x,y)\le\delta}|f(x) - f(y)|.$$

One can see that $W^\gamma(\delta)\to W(\delta), \gamma\to 0$ for every $\delta>0$.

The main point of the proof is an estimate analogous to one given in Lemma 2. In order to provide such an estimate, let us introduce some notations. Suppose $\varepsilon, \tilde{\varepsilon} > 0$ to be fixed. For $u > 0$, we denote

$$r(u) = \max\{i : s_i^\varepsilon \le u\},\ \tilde{r}(u) = \max\{i : s_i^{\tilde{\varepsilon}} \le u\}, \quad \nu(u) = s^\varepsilon_{r(u)},\ \tilde{\nu}(u) = s^{\tilde{\varepsilon}}_{\tilde{r}(u)}.$$

Denote also $s_i = s_i^\varepsilon, \tilde{s}_i = s_i^{\tilde{\varepsilon}}, q(i) = \tilde{r}(s_i), \tilde{q}(i) = r(\tilde{s}_i)$. Note that if either $j > q(i)$ or $i > \tilde{q}(j)$, then the segments (s_i, s_{i+1}) and $(\tilde{s}_j, \tilde{s}_{j+1})$ do not intersect. Let us estimate $E_x(\phi_t^0 - \tilde{\phi}_t^0)^2$ (see notations in the proof of Lemma 2). We suppose that, for a given t, $\nu(t) = \tilde{\nu}(t) = t$ (this does not restrict generality since $\phi_t^0 - \phi^0_{\nu(t)}, \tilde{\phi}_t^0 - \tilde{\phi}^0_{\tilde{\nu}(t)} \in [0, \max(\varepsilon + r(\varepsilon), \tilde{\varepsilon} + r(\tilde{\varepsilon}))])$. We put $\Phi_j = \phi^0_{s^\varepsilon_{j+1}} - \phi^0_{s^\varepsilon_j}, \tilde{\Phi}_j = \tilde{\phi}^0_{\tilde{s}_{j+1}} - \tilde{\phi}^0_{\tilde{s}_j}$, $M = r(t), \tilde{M} = \tilde{r}(t)$.

One has

$$(\phi_t^0 - \tilde{\phi}_t^0)^2 = \left(\sum_{i=0}^{M-1} \Phi_i\right)^2 + \left(\sum_{j=0}^{\tilde{M}-1} \tilde{\Phi}_j\right)^2 - 2\sum_{i=0}^{M-1}\sum_{j=0}^{\tilde{M}-1} \Phi_i\tilde{\Phi}_j = \Sigma_5 + 2\Sigma_6,$$

where

$$\Sigma_5 = \sum_{i=0}^{M-1} \Phi_i^2 + \sum_{j=0}^{\tilde{M}-1} \tilde{\Phi}_j^2 - 2\sum_{i,j: j\le q(i), i\le \tilde{q}(j)} \Phi_i\tilde{\Phi}_j \le \sum_{i=0}^{M-1} \Phi_i^2 + \sum_{j=0}^{\tilde{M}-1} \tilde{\Phi}_j^2,$$

$$\Sigma_6 = \left[\sum_{i<l} \Phi_i\Phi_l - \sum_{q(i)<j} \Phi_i\tilde{\Phi}_j\right] + \left[\sum_{j<k} \tilde{\Phi}_j\tilde{\Phi}_k - \sum_{\tilde{q}(j)<i} \Phi_i\tilde{\Phi}_j\right] =$$

$$= \sum_{i=0}^{M-1} \Phi_i\left[(\phi_t^0 - \phi^0_{s_{i+1}}) - (\tilde{\phi}_t^0 - \tilde{\phi}^0_{\tilde{s}_{q(i)+1}})\right] + \sum_{j=0}^{\tilde{M}-1} \tilde{\Phi}_i\left[(\tilde{\phi}_t^0 - \tilde{\phi}^0_{\tilde{s}_{j+1}}) - (\phi_t^0 - \phi^0_{s_{\tilde{q}(j)+1}})\right]. \tag{11}$$

Due to conditions **VN** and (10),

$$E_x\Sigma_5 \le (r(\varepsilon)+\varepsilon)E_x\sum_{i=0}^{M-1}\Phi_i + (r(\tilde{\varepsilon})+\tilde{\varepsilon})E_x\sum_{j=0}^{\tilde{M}-1}\tilde{\Phi}_j \le \max(r(\varepsilon)+\varepsilon, r(\tilde{\varepsilon})+\tilde{\varepsilon})[f_t^0(x) + \tilde{f}_t^0(x)].$$

Next, analogously to (3),(4) we have

$$E_x\sum_{i=0}^{M-1}\Phi_i\left[(\phi_t^0 - \phi^0_{s_{i+1}}) - (\tilde{\phi}_t^0 - \tilde{\phi}^0_{\tilde{s}_{q(i)+1}})\right] \le 2\max(\varepsilon, \tilde{\varepsilon})[f_t(x) + \tilde{f}_t(x)] +$$

$$+ E_x\sum_{i=0}^{M-1}\Phi_i|f_{t-s_{i+1}}(X_{s_{i+1}}) - \tilde{f}_{t-\tilde{s}_{q(i)+1}}(X_{\tilde{s}_{q(i)+1}})|. \tag{12}$$

Note that $|s_{i+1} - \tilde{s}_{q(i)+1}|, |\tilde{s}_{j+1} - s_{\tilde{q}(j)+1}| \le \theta \equiv |S^\varepsilon| + |S^{\tilde{\varepsilon}}|$. Thus, the second summand in (12) can be estimated by

$$f_t(x)\Big[\sup_{u\le t}\|f_u - \tilde{f}_u\| + W^\gamma(\theta+\delta) + F\cdot\varpi(\theta,\delta)\Big],$$

where $F = \sup_\varepsilon \sup_{u\le t}\|f_t^\varepsilon\|, \gamma = \max(\varepsilon, \tilde{\varepsilon})$, and $\delta > 0$ is arbitrary. Writing the same inequalities for the second summand on the right-hand side of (11), we obtain the following estimate analogous to one given in Lemma 2:

$$E_x(\phi_t^0 - \tilde{\phi}_t^0)^2 \le [f_t(x) + \tilde{f}_t(x)]\times$$

$$\times\Big[\sup_{u\le t}\|f_u - \tilde{f}_u\| + 5\max(\varepsilon, \tilde{\varepsilon}) + \max(r(\varepsilon), r(\varepsilon)) + 2W^\gamma(\theta+\delta) + 2F\cdot\varpi(\theta,\delta)\Big].$$

The same estimate, of course, can be written also for the L_2-distance between ϕ_t^s and $\tilde{\phi}_t^s$. Therefore, for every s, t,

$$\limsup_{\varepsilon,\ \tilde{\varepsilon}\to 0} \sup_{x\in\mathcal{X}} E_x \left(\phi_t^{s,\varepsilon} - \phi_t^{s,\tilde{\varepsilon}}\right)^2 \leq 4F \cdot W(2\delta).$$

After taking $\delta \to 0$, we obtain that there exists a mean square limit ϕ of the family $\{\phi^\varepsilon\}$. The proof of the fact that ϕ is a V-functional is analogous to the same proof in Theorem 1. The theorem is proved.

Remark. If the function f in Theorems 1,2 is already known to be a W-function, i.e. a characteristic of some W-functional $\tilde{\phi}$, then one can show the mean square convergence of the family $\{\phi^\varepsilon\}$ to the functional $\tilde{\phi}$ without use of the additional condition **US**. This is an important improvement since condition **US** is a quite significant restriction, while, in typical examples, f is known and is a W-function. However, we omit the proof of this statement since it can be given in a completely analogous way to the proofs of the first parts of Theorems 1,2.

2. Examples

In this section, we illustrate the general statements obtained before by two examples of oscillatory functionals.

2.1. Number of passings through a template. Let $\mathcal{X} = \Re$, and let $a_1, \dots, a_M \in \Re$ be fixed ($a_i \neq a_{i+1}, i = 1, \dots, M-1, a_1 \neq a_M$). The set $\{\tau_i\}$ of random times $\tau_1, \dots, \tau_M$ such that $\tau_1 < \cdots < \tau_M$ and $X_{\tau_i} = a_M$ is called a *passing of the process $X_\cdot$ through the template* $\{a_i\}$. Two such sets $\{\tau_i\}$ and $\{\tilde{\tau}_i\}$ are called adjusted if either $\tau_M \leq \tilde{\tau}_1$ or $\tilde{\tau}_M \leq \tau_1$. For every $s < t$, we denote, by $N_{s,t}^{\{a_i\}}$, the maximal number of adjusted passings of the process $X_\cdot$ through the template $\{a_i\}$ happened on the time interval $[s, t]$.

Now we take a family of templates $\{\varepsilon a_i\}, \varepsilon > 0$ and define the functionals ϕ^ε as the properly normalized numbers of passings of the given process $X_\cdot$ through these templates:

$$\phi_t^{\varepsilon,s} = r(\varepsilon) N_{s,t}^{\{\varepsilon a_i\}}, \quad s < t. \tag{13}$$

The functionals $\{\phi^\varepsilon\}$ describe the local ("oscillatory") behavior of the process $X_\cdot$ near the point 0. Let us consider a specific example of the process $X_\cdot$ and use Theorem 1 in order to give the detailed description of such a behavior.

Let $X_\cdot$ be a skew Brownian motion with skewing parameter $q \in (-1, 1)$. It is a homogeneous Markov process with its transition probability density being equal to

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \left[e^{-\frac{(x-y)^2}{2t}} + q \operatorname{sign} y \cdot e^{-\frac{(|x|+|y|)^2}{2t}} \right].$$

This process was introduced in [2], Ch. 4.2, Problem 1, and can be described in different terms: in terms of its scale and speed functions ([2]), as the solution to an SDE with the delta function in the drift term ([4]), and as the simplest example of a generalized diffusion process ([5]). One of the possible constructions of the process is the following one (see [2]): take a Wiener process $X_\cdot^0$, consider the set of its excursions at the point 0, and then put on every excursion independently (both from $X_\cdot^0$ and other excursions) $X_\cdot = X_\cdot^0$ with probability $p_+ = \frac{1+q}{2}$ and $X_\cdot = -X_\cdot^0$ with probability $p_- = \frac{1-q}{2}$. This construction shows that 0 is the "point of asymmetry" for the skew Brownian motion and motivates the study of the local behavior of the process near this point.

We restrict our considerations, supposing that

$$M = 2n, \qquad a_1, \dots, a_{2n-1} > 0, \qquad a_2, \dots, a_{2n} < 0.$$

Theorem 3. *Let $r(\varepsilon)=\varepsilon$ in the given before definition (13). Then*

$$\phi_t^{\varepsilon,s} \xrightarrow{L_2} \Big[\frac{A_+}{p_+}+\frac{A_-}{p_-}\Big]^{-1} L_t^s, \quad \varepsilon\to 0,$$

where $A_+=a_1+a_3+\dots+a_{2n-1}, A_-=-a_2-a_4-\dots-a_{2n}$, and L_t^s is the symmetric local time of the process $X.$ at the point 0 defined as the mean square limit

$$L_t^s=\lim_{\Delta\to 0}\frac{1}{2\Delta}\int_s^t 1\!\!1_{X_r\in[-\Delta,\Delta]}\,dr.$$

Proof. One can easily verify that the family $\{\phi^\varepsilon\}$ is a QW-family (see Definition 1). In order to use Theorem 1, we need to verify conditions (1),(2). We consider the functions

$$V^\varepsilon(\lambda,x)\equiv\int_0^\infty e^{-\lambda t}f_t^\varepsilon(x)\,dt, \quad \lambda>0, x\in\Re,$$

and show that, for every $\Lambda_1\le\Lambda_2, \Lambda_{1,2}\in(0,+\infty)$, $V^\varepsilon\to\left[\frac{A_+}{p_+}+\frac{A_-}{p_-}\right]^{-1}V, \varepsilon\to 0$ uniformly on $[\Lambda_1,\Lambda_2]\times\Re$, where $V(\lambda,x)=\int_0^\infty e^{-\lambda t}f_t(x)\,dt$, $f_t(x)$ is the characteristic of L_t^0. Since f^ε and f are functions non-decreasing in t, this will imply (1),(2). The explicit expressions for the functions V^ε and V are given in the following lemma.

Lemma 3. *Denote, by τ_y, the moment of the first visit of the point $y\in\Re$ by the process $X.$ and put $v(\lambda,x,y)=E_x\exp[-\lambda\tau_y]$. Then*

$$V(\lambda,x)=v(\lambda,x,0)\cdot\frac{1}{\sqrt{2\lambda^3}}, \tag{14}$$

$$V^\varepsilon(\lambda,x)=\frac{\varepsilon v(\lambda,x,\varepsilon a_1)}{\lambda}\times\frac{v(\lambda,\varepsilon a_1,\varepsilon a_2)v(\lambda,\varepsilon a_2,\varepsilon a_3)\dots v(\lambda,\varepsilon a_{M-1},\varepsilon a_M)}{1-v(\lambda,\varepsilon a_1,\varepsilon a_2)v(\lambda,\varepsilon a_2,\varepsilon a_3)\dots v(\lambda,\varepsilon a_{M-1},\varepsilon a_M)v(\lambda,\varepsilon a_M,\varepsilon a_1)}. \tag{15}$$

Proof. Equality (14) is a consequence of the strong Feller property of X (note that the distribution of L_t, while X starts from 0, coincides with the distribution of the local time of the Wiener process at the point 0). Again, it follows from the strong Feller property that

$$V^\varepsilon(\lambda,x)=v(\lambda,x,\varepsilon a_1)V^\varepsilon(\lambda,\varepsilon a_1). \tag{16}$$

The calculation of $V^\varepsilon(\lambda,\varepsilon a_1)$ uses the standard renewal theory technique. Denoting, by θ, the first moment when the passing through the given template happens, we note that $X_\theta=\varepsilon a_M$ with probability 1. The distribution of θ is a convolution of the distributions of subsequent times of the first visits of the points $\varepsilon a_2,\varepsilon a_3,\dots,\varepsilon a_M$ by the process X. Therefore its Laplace transform is

$$\Theta(\lambda)\equiv E_{\varepsilon a_1}e^{-\lambda\theta}=v(\lambda,\varepsilon a_1,\varepsilon a_2)v(\lambda,\varepsilon a_2,\varepsilon a_3)\dots v(\lambda,\varepsilon a_{M-1},\varepsilon a_M).$$

Now, writing down the renewal equation at the moment θ, we obtain that

$$V^\varepsilon(\lambda,\varepsilon a_1)=\varepsilon\Theta(\lambda)+\Theta(\lambda)\cdot V^\varepsilon(\lambda,\varepsilon a_M).$$

Substituting, instead of $V^\varepsilon(\lambda,\varepsilon a_M)$, its expression (16) through $V^\varepsilon(\lambda,\varepsilon a_1)$ and then solving the linear equation for $V^\varepsilon(\lambda,\varepsilon a_1)$, we obtain (14). The lemma is proved.

Let us return to the proof of Theorem 3. For every fixed α,β with $\alpha\cdot\beta<0$, one has that $v(\lambda,\varepsilon\alpha,\varepsilon\beta)=v(\lambda,\varepsilon\alpha,0)v(\lambda,0,\varepsilon\beta)$. The distribution of the first visit of the process X to 0 is the same with the distribution of the first visit to 0 of the Brownian motion, which follows from the construction of X. Therefore, $v(\lambda,\varepsilon\alpha,0)=e^{-\varepsilon\sqrt{2\lambda}|\alpha|}$ (see [2], Ch. 1.7).

In order to calculate $v(\lambda, 0, \varepsilon\beta)$, consider the first moment θ when $|X| = \varepsilon|\beta|$. Due to the above-described excursion-based construction of the process X, one can say that the distribution of the moment θ is the same with the distribution of the first moment when $|W_\cdot| = \varepsilon|\beta|$ ($W_\cdot$ is a Wiener process.) The Laplace transformation of this distribution is $Q(\lambda) = 2 \cdot \Big[e^{\varepsilon\sqrt{2\lambda}|\beta|} + e^{-\varepsilon\sqrt{2\lambda}|\beta|}\Big]^{-1}$ (see [2], Ch. 1.7). Moreover, independently of θ, the variable X_θ takes the values $\varepsilon|\beta|$ or $-\varepsilon|\beta|$ with probabilities p_+ and p_-, respectively. Writing down the renewal equation at the moment θ, we obtain the equation

$$v(\lambda, 0, \varepsilon\beta) = p_\beta Q(\lambda) + p_{-\beta} Q(\lambda) v(\lambda, -\varepsilon\beta, 0) v(\lambda, 0, \varepsilon\beta),$$

where $p_y = \frac{1+q\cdot\operatorname{sign} y}{2}$, i.e.

$$v(\lambda, 0, \varepsilon\beta) = \frac{p_\beta Q(\lambda)}{1 - p_{-\beta} e^{-\varepsilon\sqrt{2\lambda}|\beta|} Q(\lambda)}.$$

Therefore, we have that

$$v(\lambda, \varepsilon\alpha, \varepsilon\beta) = 1 - \sqrt{2\lambda}\Big[|\alpha| + \frac{p_{-\beta}}{p_\beta}|\beta|\Big]\varepsilon + o(\varepsilon), \quad \varepsilon \to 0+,$$

uniformly for $\lambda \in [\Lambda_-, \Lambda_+]$. It is easy to verify that $v(\lambda, x, \varepsilon a_1) \to v(\lambda, x, 0)$ and $v(\lambda, \varepsilon a_i, \varepsilon a_j) \to 1$, $\varepsilon \to 0+$, uniformly for $\lambda \in [\Lambda_-, \Lambda_+]$, $x \in \Re$. Thus,

$$V^\varepsilon(\lambda, x) \to v(\lambda, x, 0) \cdot \frac{1}{\lambda \cdot \sqrt{2\lambda} \cdot C}, \quad \varepsilon \to 0+,$$

where

$$\begin{aligned} C &= (a_1 + \frac{p_+}{p_-}|a_2|) + (|a_2| + \frac{p_-}{p_+}a_3) + \cdots + (|a_{2n}| + \frac{p_-}{p_+}a_1) \\ &= A_+\Big(1 + \frac{p_-}{p_+}\Big) + A_-\Big(1 + \frac{p_+}{p_-}\Big) = \frac{A_+}{p_+} + \frac{A_-}{p_-}. \end{aligned}$$

The theorem is proved.

2.2. Number of intersections of a level by the diffusion. Let $\mathcal{X} = \Re$. For a given process $X_\cdot$, consider the sequence of functionals $\{\eta^n\}$,

$$\eta_t^{n,s} = \sum_{k: s < \frac{k}{n} \le t} 1\!\!1_{X_{\frac{k-1}{n}} \cdot X_{\frac{k}{n}} < 0}.$$

The limit behavior of the sequence $\{\eta^n\}$ as $n \to \infty$ essentially depends on the properties of the trajectories of the process $X_\cdot$. If these trajectories are smooth, then (under some additional non-degeneracy condition on the derivative) η^n tends to the number of intersections of the level 0 by the trajectory of $X_\cdot$. The same feature holds true for some L_2-differentiable stationary processes, for instance see [6], Ch.7 for the classical *Rice formula* for normal stationary processes. For diffusions, the situation is quite different, and typically $\eta^n \to +\infty$. Let us study thoroughly the specific example, when $X_\cdot$ is the skew Brownian motion. The following statement is a corollary of Theorem 1, [3], §6.

Proposition 1. *For every $t > 0$, the sequence $\{n^{-\frac{1}{2}}\eta_t^{n,s}\}$ converges in distribution to $\sqrt{\frac{2}{\pi}}(1 - q^2)L_t$.*

The technique developed in the previous section allows us to improve this result.

Theorem 4.

$$n^{-\frac{1}{2}}\eta_t^{n,s} \xrightarrow{L_2} \sqrt{\frac{2}{\pi}}(1-q^2)L_t^s, \quad n\to\infty.$$

Proof. We apply Theorem 2 (note that Theorem 1 cannot be used here since, for a fixed n, the functional η^n is not homogeneous). The family $\{\phi^n \equiv n^{-\frac{1}{2}}\eta^n\}$ satisfies conditions (9),(10), **VN**,**VD** with $\varepsilon = n^{-\frac{1}{2}}, S_\varepsilon = \{0, \frac{1}{n}, \frac{2}{n}, \dots\}$. One can easily verify also that the characteristic f of $\phi_\cdot \equiv \sqrt{\frac{2}{\pi}}(1-q^2)L_\cdot$ is continuous. Let us verify conditions (1),(2). Due to the strong Feller property of $X_\cdot$, in order to do this, it is sufficient to prove that, for every $T>0$,

$$\sup_{t\le T,\theta\in[0,\frac{1}{n})}\left|n^{-\frac{1}{2}}E\left[\eta_t^{n,\frac{1}{n}}\Big|X_\theta=0\right]-\frac{2(1-q^2)\sqrt{t}}{\pi}\right|\to 0, n\to+\infty \tag{17}$$

(we recall that $E_0L_t=\int_0^t\frac{1}{\sqrt{2\pi s}}\,ds=\sqrt{\frac{2t}{\pi}}$). We have

$$E[\eta_t^{n,\frac{1}{n}}|X_\theta=0]=\sum_{k=1}^{[tn]-1}\left[\int_{-\infty}^0 p(\frac{k}{n}-\theta,0,y)P_n^+(y)\,dy+\int_0^\infty p(\frac{k}{n}-\theta,0,y)P_n^-(y)\,dy\right], \tag{18}$$

where $p(\cdot,\cdot,\cdot)$ is the transition probability density of the process $X_\cdot$ and $\Phi_n^\pm(y)=P_y(\operatorname{sign}X_{\frac{1}{n}}=\pm1)$. The easy calculation gives

$$\begin{cases} P_n^+(y)=(1+q)\Phi(-y\sqrt{n}), & y<0,\\ P_n^-(y)=(1-q)\Phi(y\sqrt{n}), & y>0,\end{cases}\qquad \Phi(z)\equiv\int_z^\infty\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\,du.$$

Therefore, using the explicit expression for $p(\cdot,\cdot,\cdot)$, we obtain

$$n^{-\frac{1}{2}}E[\eta_t^{n,\frac{1}{n}}|X_\theta=0]=n^{-\frac{1}{2}}\sum_{k=1}^{[tn]-1}\int_{-\infty}^\infty\frac{(1-q)(1+q)}{\sqrt{2\pi(\frac{k}{n}-\theta)}}e^{-\frac{y^2}{2(\frac{k}{n}-\theta)}}\Phi(|y|\sqrt{n})\,dy=$$

$$=\frac{1-q^2}{n}\sum_{k=1}^{[tn]-1}\frac{1}{\sqrt{2\pi(\frac{k}{n}-\theta)}}\int_{-\infty}^{+\infty}e^{-\frac{w^2}{2(k-\theta n)}}\Phi(|w|)\,dw.$$

We have that

$$\int_{-\infty}^\infty\Phi(|w|)\,dw=\int_0^\infty\int_w^\infty\sqrt{\frac{2}{\pi}}e^{-\frac{u^2}{2}}\,du\,dw=\sqrt{\frac{2}{\pi}}\int_0^\infty ue^{-\frac{u^2}{2}}\,du=\sqrt{\frac{2}{\pi}}, \tag{19}$$

$$\frac{1}{\sqrt{2\pi(\frac{k}{n}-\theta)}}\int_{-\infty}^{+\infty}e^{-\frac{w^2}{2(k-\theta n)}}\Phi(|w|)\,dw\le\sqrt{n}\quad\text{for every}\quad k\ge1,\theta\in[0,\frac{1}{n}), \tag{20}$$

$$\sup_{t\le T,\theta\in[0,\frac{1}{n})}\left|\sum_{k=k_0}^{[tn]-1}\frac{1}{\sqrt{2\pi(\frac{k}{n}-\theta)}}-\int_0^t\frac{1}{\sqrt{2\pi s}}\,ds\right|\to0, n\to+\infty\quad\text{for every}\quad k_0>1. \tag{21}$$

For a given $\delta>0$, let us take $k_0>1$ such that

$$\int_{-\infty}^{+\infty}(1-e^{-\frac{w^2}{2(k_0-1)}})\Phi(|w|)\,dw<\delta.$$

Then (19)-(21) yield that

$$\limsup_{n\to+\infty}\sup_{t\le T,\theta\in[0,\frac{1}{n})}\left|n^{-\frac{1}{2}}E\left[\eta_t^{n,\frac{1}{n}}\Big|X_\theta=0\right]-\frac{2(1-q^2)\sqrt{t}}{\pi}\right|<\delta.$$

Since δ is arbitrary, this proves (17). The theorem is proved.

Remark. The functionals studied in subsections 2.1 and 2.2 can be considered as two possible answers to the question about how to construct the approximating aggregates for the number of intersections of the level by the diffusion or generalized diffusion process. In the first case, the level is made more "thick", and the time is discretized in the second case. For the skew Brownian motion, these two constructions give, after the appropriate normalization, the same (up to a constant) object, namely, the local time of the process. It should be mentioned that the situation can be essentially different for other processes. Let us give an example.

Let Y be a skew Brownian motion, and let ℓ be its symmetric local time at the point 0. We take $a > 0$ and define $\theta_t = t + a\ell_t$, $\sigma_t = [\theta^{-1}]_t \equiv \inf\{u | \theta_u \geq u\}$ and $X_t \equiv Y_{\sigma_t}, t \geq 0$. The process $X.$ is a Markov one (for more details see [3], §5) which spends a positive time at the point 0. Due to the latter fact, the point 0 is called *sticky*. The following proposition shows that two constructions described before give essentially different results for the process $X.$.

Proposition 2. *1) In the notations of subsection 2.1,*

$$\phi_t^{\varepsilon,s} \xrightarrow{L_2} \Big[\frac{A_+}{p_+} + \frac{A_-}{p_-}\Big]^{-1} \overset{\circ}{L}{}_t^s, \quad \varepsilon \to 0,$$

where

$$\overset{\circ}{L}{}_t^s = \lim_{\Delta\to 0} \frac{1}{2\Delta} \int_s^t 1\!\!1_{X_r \in [-\Delta, 0) \cup (0, \Delta]}\, dr.$$

2) In the notations of subsection 2.2, the sequence $\{\eta_t^{n,s}\}$ converges in distribution to some integer-valued random variable for every s, t.

Statement 2) was proved in [3], §6. One can prove statement 1) by either repeating the proof of Theorem 3 for the process with a sticky point or using the result of Theorem 3 and making a random time change.

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Kyiv 01033 Volodymyrska str. 64, Taras Shevchenko Kyiv National University

Kiev 01601 Tereshchenkivska str. 3, Institute of Mathematics, Ukrainian National Academy of Sciences

E-mail: kulik@imath.kiev.ua