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SINGULAR REDUCTION OF SYMMETRIES IN HAMILTONIAN MECHANICS AND CONTROL THEORY

We discuss the reduction of symmetries of dynamical systems, Hamiltonian systems, Hamiltonian systems with non-holonomic constraints, and non-linear control systems. We assume that the symmetry group of each system acts properly on the phase space of the system. Reduced system is described in the framework of theory of differential spaces.

Introduction. Existence of symmetries of dynamical systems usually simplifies the task of analyzing solutions of the systems. Reduction of symmetries leads to a dynamical system with a lower number of degrees of freedom.

If the action of the symmetry group G on the phase of the system P is not free and proper, the phase space $R = P/G$ of the reduced system need not be a manifold. If the action is proper, the reduced space phase of the system is a stratified space. It is a union of manifolds called strata [1].

If one is interested only in local properties of solutions, one can study the reduced dynamics stratum by stratum using standard techniques of differential geometry. However, if one wants to know asymptotic properties of solutions, one has to know global structure of the reduced phase space. In particular, one has to know how strata of the stratification are put together.

In 1983 Richard Cushman initiated a program of singular reduction in terms of the ring of smooth invariant functions on the phase space [2]. For a linear action of a compact group, invariant smooth functions are smooth functions of algebraic invariants [3]. Thus, in order to visualize the reduced phase space, it suffices to describe the set given by relations among algebraic invariants.

Sikorski's theory of differential spaces [4][1] provides a theoretical framework for Cushman's approach. It allows for a description of the differential structure of the orbit space as well as of its geometric structure. In particular, it allows for a discussion of ordinary differential equations directly on the reduced phase space [5].

The aim of this lecture is to describe a few examples in which one can get a complete description of the geometric structure of the reduced system. The general principle is to encode the invariant geometric structure of the original phase space P as an algebraic structure of the ring $C^\infty(P)^G$ of invariant functions. Since the ring of smooth functions $C^\infty(R)$ of the reduced phase space $R = P/G$ is isomorphic to $C^\infty(P)^G$, it inherits from $C^\infty(P)^G = C^\infty(R)^G$ its algebraic structure which, in turn, can be decoded as the geometric structure on the reduced space R. Since, invariant functions need not separate orbits of an improper action,

[1] In the introduction to his book, Sikorski wrote that his work had been based on some ideas of Postnikov. However, he did not give a reference.

this approach is applicable only if the action of G on P is proper.

1. Dynamical systems. We consider a dynamical system, given by a vector field X on a smooth manifold P. A symmetry of the system is a diffeomorphism of P preserving X. Let G be a connected Lie group. We denote by

$$\Phi : G \times P \to P : (g, p) \mapsto \Phi(g, p) \equiv \Phi_g(p) \equiv gp$$

the action of G on P, and assume that, for each $g \in G$, the diffeomorphism $\Phi_g : P \to P$ preserves X. In other words,

$$T\Phi_g \circ X = X \circ \Phi_g,$$

where $T\Phi_g$: $TP \to TP$ is the derived map.

We say that an action Φ is free if $\Phi(g, p) = p$ implies that g is the identity in G. Also, Φ is said to be proper if, for every convergent sequence (p_n) in P and a sequence (g_n) in G such that the sequence $(g_n p_n)$ is convergent, the sequence (g_n) has a convergent subsequence (g_{n_k}) and

$$\lim_{k\to\infty} (g_{n_k} p_{n_k}) = \left(\lim_{k\to\infty} g_{n_k}\right)\left(\lim_{k\to\infty} p_{n_k}\right).$$

1.1. Regular reduction. If the action Φ of G on P is free and proper, then the space $R = P/G$ of G-orbits on P is a manifold and the orbit map $\rho : P \to R$ is a fibration. The action Φ of G on P induces on P the structure of a (left) principal G-bundle over R. The G -invariance of the vector field X implies that X projects to a vector field $\bar{X}$ on R such that

$$T\rho \circ X = \bar{X} \circ \rho.$$

It should be noted that the ring $C^\infty(R)$ of smooth functions on R is isomorphic to the ring $C^\infty(P)^G$ of smooth G-invariant functions on P, with the isomorphism given by the pull-back by the projection map

$$\rho^* : C^\infty(R) \to C^\infty(P)^G : \bar{f} \mapsto \rho^*\bar{f} = \bar{f} \circ \rho. \tag{1}$$

The orbit space $R = P/G$ is often called the reduced phase space of the system. The passage from the dynamical system X on P to the dynamical system $\bar{X}$ on R, given by the vector field $\bar{X}$, is called the reduced called the reduction of symmetries. In order to solve equations of motion for X, we may first find integral curves of $\bar{X}$ and then lift them to integral curves of X. The second step is called reconstruction.

1.2. Singular reduction. Singular reduction deals with the situation when the action of the symmetry group is proper but not free. In this case, the orbit space $R = P/G$ is not a manifold but a stratified space [1]. For each $p \in P$ the isotropy group

$$G_p = \{g \in G \mid gp = p\}$$

of p is compact. For each compact subgroup $H \subseteq G$, the set

$$P_{(H)} = \{p \in P \mid G_p \text{ is congruent to } H\}$$

is a local manifold. Connected components of the projection of $P_{(H)}$ to R are smooth manifolds. They are strata of the stratification of R by orbit type.

We define the ring of smooth functions on R to be

$$C^\infty(R) = \{\bar{f} \in C^0(R) \mid \rho^* \bar{f} \in C^\infty(P)\},$$

where $\rho : P \to R$ is the orbit map. It satisfies the following conditions:

- The family of sets

  $$\{\bar{f}^{-1}((a,b)) \mid \bar{f} \in C^\infty(R), \text{ and } a, b \in \mathbb{R}\}$$

  is a sub-basis for the topology of R.

- For every $k \in \mathbb{N}$, every $\bar{f}_1, ..., \bar{f}_k \in C^\infty(R)$ and $F \in C^\infty(\mathbb{R}^k)$, the composition $F(f_1, ..., f_k)$ is in $C^\infty(R)$.

- If a function f on R is such that, for every $x \in R$, there exists an open neighbourhood U_x of x in R, and a function $f_x \in C^\infty(R)$ satisfying

  $$f_{|U_x} = f_{x|U_x},$$

  then f is in $C^\infty(R)$.

Definition. A topological space endowed with a subring of continuous functions satisfying the above conditions is called a differential space.

A homeomorphism $\varphi : R \to S$ of differential spaces is smooth if its pull-back φ^* maps $C^\infty(S)$ to $C^\infty(R)$. It is a diffeomorphism if it is invertible and $\varphi^{-1} : S \to R$ is smooth.

Since the action of G on P is proper, elements of $C^\infty(R)$ separate points in R. Hence, the quotient topology on R is Hausdorff. Moreover, each point of R has a neighbourhood diffeomorphic to a subset of $\mathbb{R}^n$ [6]. Hence, R is a subcartesian space in the sense of Aronszajn [7].

In order to see how the stratification structure of R is encoded in $C^\infty(R)$ we need to define an appropriate notion of a vector field on a differential space. A derivation of $C^\infty(R)$ is a linear map $\bar{X} : C^\infty(R) \to C^\infty(R)$ satisfying Leibniz's rule

$$\bar{X}(\bar{f}_1 \bar{f}_2) = (\bar{X}\bar{f}_1)\bar{f}_2 + (\bar{X}\bar{f}_2)\bar{f}_1$$

for all $\bar{f}_1, \bar{f}_2 \in C^\infty(R)$. A curve $\bar{c} : I \to R$, where I is an interval, is an integral curve of a derivation $\bar{X}$ if

$$\frac{d}{dt}\bar{f}(\bar{c}(t)) = (\bar{X}\bar{f})(\bar{c}(t))$$

for every $\bar{f} \in C^\infty(R)$ and $t \in I$. For a subcartesian differential space R, every derivation $\bar{X}$ of $C^\infty(R)$ and every $r \in R$, there exists a maximal integral curve of $\bar{X}$ through r [8]. A vector field on a subcartesian differential space R is defined as a derivation $\bar{X}$ of $C^\infty(R)$ such that translations along maximal integral curves of $\bar{X}$ give rise to a local one-parameter group $\exp(t\bar{X})$ of local diffeomorphisms of R. Let $\mathcal{X}(R)$ be the family of all vector fields on R. For each $r \in R$ the orbit of $\mathcal{X}(R)$, defined by

$$O_r = \\ = \{\exp(t_n\bar{X}_n)\circ ...\circ \exp(t_1\bar{X}_1)(r) \mid n \in \mathbb{N}, t_1, ..., t_n \in \mathbb{R}, \bar{X}_1, ..., \bar{X}_n \in \mathcal{X}(R)\} \tag{2}$$

is a manifold. Moreover, if R is a stratified space, orbits of $\mathcal{X}(R)$ coincide with strata of the stratification of R [5].

For each orbit O of $\mathcal{X}(R)$, the preimage $\rho^{-1}(O) \subseteq P$ is a manifold. Moreover, the restriction $\rho_O : \rho^{-1}(O) \to O$ of ρ to $\rho^{-1}(O)$ is a submersion.

We can now go back to the vector field X on P defining dynamics of our system. Since X is G-invariant, it preserves the ring $C^\infty(P)^G$ of G-invariant functions on P. Hence, it induces a derivation $\bar{X}$ of $C^\infty(R)$ such that, $(\bar{X}\bar{f}) \circ \rho = X(\bar{f} \circ \rho)$ for every $\bar{f} \in C^\infty(R)$, where $\rho : P \to R$ is the orbit map. Moreover, the local one-parameter group of local diffeomorphisms $\exp(tX)$ of P generated by X commute with the action of G and, hence, it induces a local one-parameter group of local diffeomorphisms of R generated by $\bar{X}$. Thus, $\bar{X}$ is a vector field on R and its integral curves are reduced evolutions. It should be noted that integral curves of $\bar{X}$ and every other vector field on R preserve the stratification of R.

For each stratum O of R, the restriction $\bar{X}_O$ of $\bar{X}$ to O is a vector field on O that is ρ_O related to the restriction $X_{\rho^{-1}(O)}$ of X to $\rho^{-1}(O)$. In other words,

$$T\rho_O \circ X_{\rho^{-1}(O)} = \bar{X}_O \circ \rho_O.$$

In many applications the dynamical system under considerations has an additional invariant geometric structure. Ideally, one would like to have this structure reproduced by reduction. In general, it is not possible. I shall discuss some examples in which the auxiliary structure is reproduced.

2. Hamiltonian systems. Let ω be a symplectic form on P. In other words, ω is a non-degenerate closed 2-form. For every $f \in C^\infty(P)$, there is a unique vector field X_f on P such that

$$X_f \lrcorner \omega = df,$$

where $\lrcorner$ denotes the left interior product (contraction). The vector field X_f is called the Hamiltonian vector field of f. A dynamical system on P is called Hamiltonian if there exists a function $h \in C^\infty(P)$ such that the vector field X, defining the dynamics, is the Hamiltonian vector field of h. In other words, $X = X_h$. The function h is called the Hamiltonian of the system. The triplet is called a Hamiltonian system. Evolutions of the Hamiltonian system (P, ω, h) are given by integral curves of X_h. In other words, a curve $c(t)$ in P is an evolution of the system if

$$\dot{c}(t) = X_h(c(t)), \tag{3}$$

where $\dot{c}(t)$ denotes the tangent vector of $c(t)$.

A symplectic form ω on P gives rise to a Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(P)$ defined as follows. For each $f_1, f_2 \in C^\infty(P)$,

$$\{f_1, f_2\} = X_{f_1} f_2 = \omega(X_{f_2}, X_{f_1}).$$

The Poisson bracket is bilinear, skew symmetric, satisfies the Jacobi identity

$$\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0$$

and

$$\{f_1 f_2, f_3\} = f_1\{f_2, f_3\} + f_2\{f_1, f_3\}$$

for all $f_1, f_2, f_3 \in C^\infty(P)$. The ring $C^\infty(P)$ endowed with the Poisson bracket is called the Poisson algebra of the symplectic manifold (P, ω). Evolution equations of our Hamiltonian system (3) can be written in the Poisson form

$$\frac{d}{dt} f(c(t)) = \{h, f\}(c(t)) \tag{4}$$

for every $f \in C^\infty(P)$.

Let a connected Lie group G be a symmetry group of the Hamiltonian system (P, ω, h). In other words, for each $g \in G$, the diffeomorphism Φ_g of P preserves the symplectic form ω and the Hamiltonian function h. This implies that Φ_g preserves the Poisson bracket on $C^\infty(P)$. In other words,

$$\Phi_g^*\{f_1, f_2\} = \{\Phi_g^* f_1, \Phi_g^* f_2\}$$

for all $f_1, f_2 \in C^\infty(P)$. If f_1 and f_2 are G-invariant, then $\Phi_g^*\{f_1, f_2\}$ is G-invariant. Hence, the ring $C^\infty(P)^G$ of G-invariant smooth functions of P is a Poisson subalgebra of $C^\infty(P)$. We can use the isomorphism (1) to pull-back the Poisson algebra structure on $C^\infty(P)^G$ to $C^\infty(R)$. The Poisson bracket on $C^\infty(R)$ is given by

$$\rho^*\{\bar{f}_1, \bar{f}_2\} = \{\rho^*\bar{f}_1, \rho^*\bar{f}_2\}$$

for every $\bar{f}_1, \bar{f}_2 \in C^\infty(R)$. Thus, the reduced space $R = P/G$ is a Poisson manifold.

Every $\bar{f} \in C^\infty(R)$ gives rise to a derivation $\bar{X}_{\bar{f}}$ of $C^\infty(R)$ such that

$$\bar{X}_{\bar{f}} \bar{f}' = \{\bar{f}, \bar{f}'\} \tag{5}$$

for every $\bar{f}' \in C^\infty(R)$. The derivation $\bar{X}_{\bar{f}}$ is a vector field on R, called the Hamiltonian (or Poisson) vector field of $\bar{f}$.

Since the Hamiltonian h is G-invariant, it follows that it pushes forward to a smooth function $\bar{h} \in C^\infty(R)$. In other words, $h = \rho^*\bar{h}$. Let $c(t)$ be an integral curve of X_h and $\bar{c}(t) = \rho \circ c(t)$ its projection to R. For every $\bar{f} \in C^\infty(R)$, equation (4) implies that

$$\begin{aligned}\frac{d}{dt}\bar{f}(\bar{c}(t)) &= \frac{d}{dt}\bar{f}(\rho \circ c(t)) = \frac{d}{dt}\bar{f} \circ \rho(c(t)) = \frac{d}{dt}\rho^*\bar{f}(c(t)) = \{h, \rho^*\bar{f}\}(c(t)) \\ &= \{\rho^*h, \rho^*\bar{f}\}(c(t)) = \rho^*\{\bar{h}, \bar{f}\}(c(t)) = \{\bar{h}, \bar{f}\}(\rho \circ c(t)) = \{\bar{h}, \bar{f}\}(\bar{c}(t)).\end{aligned}$$

This means that $\bar{c}(t)$ is an integral curve of the derivation $\bar{X}_{\bar{h}}$ of $C^\infty(R)$ given by

$$\bar{X}_{\bar{h}}\bar{f} = \{\bar{h}, \bar{f}\}$$

for every $\bar{f} \in C^\infty(R)$.

Let O be a stratum of R, i.e. an orbit of a family $\mathcal{X}(R)$ of all vector fields on R. Since The Poisson bracket of functions $f_1, f_2 \in C^\infty(R)$ at a point $r \in O$ depends only on the differentials at r of restrictions of f_1 and f_2 to O. Hence, $C^\infty(O)$ inherits the structure of a Poisson algebra. Hence O is a Poisson manifold. As in the preceding section, the restriction of $\bar{X}_{\bar{h}}$ to O is a vector field on O that is ρ_O-related to the restriction of X_h to $\rho^{-1}(O)$.

If the symmetry group G is not discrete, the symplectic form ω does not push forward to a symplectic form on R. However, every Poisson manifold is singularly foliated by symplectic manifolds [9]. Association to a symplectic manifold (P, ω) of the singular foliation of $R = P/G$ by symplectic manifolds is the essence of the optimal reduction of Ortega and Ratiu [10].

Let $\mathfrak{g}$ be the Lie algebra of the symmetry group G. An Ad^*-equivariant function $J : P \to \mathfrak{g}^*$ is called a momentum map for an action of G on (P, ω) if, for every $\xi \in \mathfrak{g}$, the action of the one-parameter subgroup $\exp(t\xi)$ of G is given by translation along integral curves of the Hamiltonian vector field X_{J_ξ}, where $J_\xi = \langle J \mid \xi \rangle$ is the evaluation of J on ξ. Assume that the action Φ admits an Ad^*-equivariant momentum map J. In this case, the symplectic leaves of the singular foliation of R can be described as follows.

For $p \in P$, let $\mu = J(p)$, and

$$G_\mu = \{g \in G \mid Ad^*_g \mu = \mu\}$$

be the isotropy group of μ. The symplectic leaf containing $\rho(p)$ can be identified with the intersection of the stratum of R through $\rho(p)$ with the connected component of $\rho(J^{-1}(\mu))$ containing $\rho(p)$. Note that $\rho(J^{-1}(\mu))$ can be identified with the space $J^{-1}(\mu)/G_\mu$ of G_μ-orbits in $J^{-1}(\mu)$. If the action of G is free, this description coincides with the Marsden-Weinstein reduction [11].

3. Non-holonomic constraints. Consider a mechanical system with configuration space Q, kinetic energy metric $k : TQ \times_Q TQ \to \mathbb{R}$, and potential energy $v : Q \to \mathbb{R}$. We assume that the motions of our system are constrained so that the velocity has to be in the constraint distribution P on Q. We assume further that the work of the reaction force of the constraints on virtual motions compatible with constraints vanishes.

In the following, we treat P as a submanifold of TQ. Let $\tau : P \to Q$ be the restriction to P of the tangent bundle projection $\tau_Q : TQ \to Q$, and

$$H = \{w \in TP \mid T\tau(w) \in P \subset TQ\}.$$

We say that a vector field Y is in H if $Y(u) \in H$ for each $u \in P$.

The pull-back of the canonical symplectic form ω_Q of T^*Q by the Legendre transformation $\mathcal{L} : TQ \to T^*Q$, corresponding to the Lagrangian $l(u) = \frac{1}{2}k(u,u) - v(\tau_Q(u))$ of the system, induces on each fibre of H_u a linear symplectic form ϖ_u. We say that (H, ϖ) is a symplectic distribution on P. For each $f \in C^\infty(P)$, the distributional Hamiltonian vector field of f is the unique vector field Y_f in H such that, for every $u \in D$ and $w \in H_u$,

$$\varpi(Y_f(u), w) = \langle df \mid w \rangle.$$

Let $\partial_H f$ denote the restriction of df to H. Then, we can write

$$Y_f \lrcorner \varpi = \partial_H f.$$

Motions of our system are given by integral curves of the distributional Hamiltonian vector field Y_h of the energy function

$$h(u) = \frac{1}{2}k(u,u) + v(\tau_Q(u)).$$

We may generalize this approach to non-holonomic constraints and consider an abstract distributional Hamiltonian systems (P, H, ϖ, h), where P is a manifold, (H, ϖ) is a symplectic distribution on P and $h \in C^\infty(P)$. This approach, described in [12], is a special case of a more general structure introduced by Bocharov and Vinogradov in 1977, [13].

For $f_1, f_2 \in C^\infty(P)$, the almost Poisson bracket of f_1 and f_2 is the derivative of f_2 in the direction of the vector field Y_{f_1}. In other words,

$$\{f_1, f_2\} = Y_{f_1} f_2.$$

The almost Poisson bracket on $C^\infty(P)$ is bilinear, skew-symmetric, and it satisfies Leibniz' rule

$$\{f_1, f_2 f_3\} = f_2\{f_1, f_3\} + \{f_1, f_2\} f_3$$

for all f_1, f_2, f_3. It satisfies the Jacobi identity if and only if P is an involutive distribution on Q [14].

The almost Poisson bracket was introduced by Van der Schaft and Maschke [14] in terms of coordinates. An abstract definition was given by Koon and Marsden, [15]. The term almost Poisson bracket was coined by Cantrijn, de León and de Diego [16].

Let G be a symmetry group of a distributional Hamiltonian system (P, H, ϖ, h), and Φ is the action of G on P. For each $g \in G$, the diffeomorphism Φ_g of P

preserves the distribution H, the symplectic form ϖ on H and the Hamiltonian h. Therefore, Φ_g preserves the almost Poisson bracket. In other words, for every $f_1, f_2 \in C^\infty(P)$,

$$\Phi_g^*\{f_1, f_2\} = \{\Phi_g^* f_1, \Phi_g^* f_2\}.$$

If f_1 and f_2 are G-invariant, then $\Phi_g^*\{f_1, f_2\}$ is G-invariant. Hence, the ring $C^\infty(P)^G$ of G-invariant smooth functions of P is stable under almost Poisson bracket. We can use the isomorphism (1) to pull-back the almost Poisson bracket on $C^\infty(P)^G$ to $C^\infty(R)$. Thus, the reduced space $R = P/G$ is a manifold with the ring $C^\infty(R)$ of smooth functions on R endowed with an almost Poisson bracket [15].

As in the case of Hamiltonian systems, the geometric structure of the reduced space R is encoded in the almost Poisson bracket on $C^\infty(R)$. Given $\bar{f} \in C^\infty(R)$, we denote by $\bar{Y}_{\bar{f}}$ the vector field on R such that

$$\bar{Y}_{\bar{f}} \bar{f}' = \{\bar{f}, \bar{f}'\} \tag{6}$$

for all $\bar{f}' \in C^\infty(R)$. The family $\{\bar{Y}_{\bar{f}} \mid \bar{f} \in C^\infty(R)\}$ of vector fields $Y_{\bar{f}}$ spans a generalized distribution $\bar{H}$ on R. For each stratum O of R, the restriction of $\bar{H}$ to O is a generalized distribution in the sense of Sussmann. The almost Poisson bracket on $C^\infty(R)$ gives rise to a symplectic form $\bar{\varpi}$ on $\bar{H}$ such that

$$\bar{\varpi}(\bar{Y}_{\bar{f}}, \bar{Y}_{\bar{f}'}) = \bar{Y}_{\bar{f}} \bar{f}\prime$$

for $\bar{f}, \bar{f}' \in C^\infty(R)$. Hence, $\bar{Y}_{\bar{f}}$, defined by equation (6), is distributional Hamiltonian vector field on R with respect to the symplectic form $\bar{\varpi}$ on $\bar{H}$. Moreover, the reduced dynamics is given by the distributional Hamiltonian vector field $\bar{Y}_{\bar{h}}$, where $h = \bar{h} \circ \rho$ is the energy function of our system. Thus, reduction of symmetries of a distributional Hamiltonian system (P, H, ϖ, h) gives a generalized almost distributional Hamiltonian system $(R, \bar{H}, \bar{\varpi}, \bar{h})$, where the term *generalized* refers to the fact that $\bar{H}$ need not have constant rank. An example in which $\bar{H}$ has variable rank is provided by Chaplygin skate [17].

4. Non-linear control systems. A smooth nonlinear control system, as defined by Brockett, [18], is a quadruple (B, M, π, φ) such that

1. (B, M, π) is a fibre bundle with total space B, base space M and projection $\pi : B \to M$, and
2. $\varphi : B \to TM$ is a bundle morphism such that, for each $x \in M$ and each $b \in B_x = \pi^{-1}(x)$, $\varphi(b) \in T_x M$.

The assumption that (B, M, π) is a fibre bundle implies that there exists a family $\Gamma(M, B)$ of smooth local sections σ of $\pi : B \to M$ such that M is covered by the domains of the sections $\sigma \in \Gamma(M, B)$. For each $\sigma \in \Gamma(M, B)$, the composition $X = \varphi \circ \sigma$ is a control vector field on M. In this way we obtain a family $\mathcal{D}$

$$\mathcal{D} = \{X = \varphi \circ \sigma \mid \sigma \in \Gamma(M, B)\} \tag{7}$$

of locally defined vector fields on M such that M is covered by the domains of $X \in \mathcal{D}$. A choice of the family $\Gamma(M, B)$ of sections leads to a description of the non-linear control system (B, M, π, φ) as a piecewise linear system given by a family of local vector fields on M.

An example of a control problem on M is the analysis of the structure of accessible sets of the family $\mathcal{D}$. For each $X \in \mathcal{D}$, we denote by $\exp(tX)$ the local one-parameter local group of diffeomorphisms of M generated by X. For every $x \in M$, the accessible set of $\mathcal{D}$ through x is

$$N_x = \{\exp(t_n X_n)\circ ...\circ \exp(t_1 X_1) \mid n \in \mathbb{N},\ t_1, ..., t_n \in \mathbb{R},\ X_1, ..., X_n \in \mathcal{D}\}. \quad (8)$$

It has been shown by Sussmann that N_x is a manifold immersed in M, [19]. The family of accessible sets of $\mathcal{D}$ defines on M the structure of a smooth foliation with singularities, [20].

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let

$$\Theta : G \times B \to B : (g, u) \mapsto \Theta(g, u) \equiv \Theta_g(u) \equiv gu,$$

and

$$\Phi : G \times M \to M : (g, x) \mapsto \Phi(g, x) \equiv \Phi_g(u) \equiv gx,$$

be left actions of G on B, and M, respectively. We say that G is a symmetry group of the control system (B, M, π, φ) if the map $\pi : B \to M$ intertwines the actions of G on B and M, and the map $\varphi : B \to TM$ intertwines the actions of G on B and TM. In other words, G is a symmetry if, for every $g \in G$,

$$\pi\circ \Theta_g = \Phi_g \circ \pi \text{ and } \varphi\circ \Theta_g = T\Phi_g\circ \varphi.$$

Clearly, if G is a symmetry group of a nonlinear control system (B, M, π, φ), then the action of G on B determines its actions on M and TM.

We consider here a special case in which all sections $\sigma \in \Gamma(M, B)$ intertwine the action Φ on M and the action Θ on B. In other words, we assume that

$$\sigma\circ \Phi_g = \Theta_g \circ \sigma \quad (9)$$

for every $\sigma \in \Gamma(M, B)$ and $g \in G$. This does not imply that the range of σ is G-invariant. On the other hand, if x and gx are in the domain of σ and $X = \varphi\circ \sigma$, then $T\Phi_g(X(x)) = X(gx)$. By definition, the domain of X is open. Hence, for every $x \in$ domain of X there exists a neighbourhood U of e in G such that $T\Phi_g(X(x)) = X(gx)$ for all $g \in U$. We shall refer to this property by saying that X is locally G-invariant. If the domain of σ were G-invariant, then X would be a G-invariant vector field. Thus, we are dealing with a control system on a manifold with symmetry in which all controls have same symmetry. Clearly, if G is a symmetry group of a nonlinear control system (B, M, π, φ), then the action of G on B determines its actions on M and TM.

4.1. Regular reduction. If the action of G on B is free and proper, then the actions Θ and Φ are free and proper. In this case, that orbit spaces $\bar{B} = B/G$

and $\bar{M} = M/G$ are quotient manifolds of B and M, respectively, with projection maps $\beta : B \to \bar{B}$ and $\mu : M \to \bar{M}$. Since the map $\pi : B \to M$ intertwines the actions Θ and Φ, it induces a map $\bar{\pi} : \bar{B} \to \bar{M}$ such that

$$\mu \circ \pi = \bar{\pi} \circ \beta.$$

Let $\psi = T\mu \circ \varphi : B \to T\bar{M}$. For every $g \in G$, and $u \in B$,

$$\psi(gu) = T\mu(\varphi(\Theta_g u)) = T\mu(T\Phi_g(\varphi(u)) = T\mu(\varphi(u)) = \psi(u).$$

Thus, ψ is constant on orbits of G, and it pushes forward to a smooth map $\bar{\varphi} : \bar{B} \to T\bar{M}$ such that

$$T\mu \circ \varphi = \bar{\varphi} \circ \beta.$$

The quadruple $(\bar{B}, \bar{M}, \bar{\pi}, \bar{\varphi})$ is a smooth nonlinear control system obtained from (B, M, π, φ) by reduction of symmetries.

4.2. Singular reduction. If the actions Θ and Φ are not free, the orbit spaces $\bar{B}$ and $\bar{M}$ need not be manifolds. If Θ and Φ are proper, then $\bar{B}$ and $\bar{M}$ are stratified spaces, As before, we define differential structures on $\bar{B}$ and $\bar{M}$ in terms of G-invariant smooth functions on B and M, respectively. More precisely,

$$C^\infty(\bar{B}) = \{h : \bar{B} \to \mathbb{R} \mid h \circ \beta \in C^\infty(B)\},$$

and

$$C^\infty(\bar{M}) = \{h : \bar{M} \to \mathbb{R} \mid h \circ \mu \in C^\infty(M)\}.$$

The spaces $\bar{B}$ and $\bar{M}$ endowed with these differential structures are subcartesian differential spaces.

In the category of differential spaces, the orbit maps $\beta : B \to \bar{B}$ and $\mu : M \to \to \bar{M}$ are smooth. As in the case of a free and proper action, we have a smooth projection $\bar{\pi} : \bar{B} \to \bar{M}$ such that

$$\mu \circ \pi = \bar{\pi} \circ \beta. \tag{10}$$

In order to describe a mapping $\bar{\varphi} : \bar{B} \to T\bar{M}$, we have to define what we mean here by the "tangent bundle space" of a subcartesian space. Different notions of tangent vectors, which are equivalent on a manifold, need not be equivalent in the case of a differential space. We choose here the notion of a Zariski tangent bundle $T\bar{M}$ defined as the union over $\bar{M}$ of all derivations of $C^\infty(\bar{M})$ at all points of $\bar{M}$. In other words, for $\bar{x} \in \bar{M}$, the Zariski tangent space $T_{\bar{x}}\bar{M}$ consists of linear maps $\bar{w} : C^\infty(\bar{M}) \to \mathbb{R}$ such that, for each $\bar{f}_1, \bar{f}_2 \in C^\infty(\bar{M})$

$$\bar{w}(\bar{f}_1 \bar{f}_2) = \bar{f}_1(\bar{x})\bar{w}(\bar{f}_2) + \bar{f}_2(\bar{x})\bar{w}(\bar{f}_1).$$

We define a projection map $\tau_{\bar{M}} : T\bar{M} \to \bar{M}$ such that $\tau_{\bar{M}}(\bar{w}) = \bar{x}$ if $\bar{w}$ is a derivation at $\bar{x}$. Each $\bar{f} \in C^\infty(\bar{M})$ gives rise to a function $d\bar{f}$ on $T\bar{M}$ such that $d\bar{f}(\bar{w}) = \bar{w}(\bar{f})$ for every $\bar{w} \in T\bar{M}$.

We are can now define a mapping $\bar{\varphi} : \bar{B} \to T\bar{M}$ as follows. If $u \in B$, and $x = \pi(u)$, then $\varphi(u) \in T_x M$ acts on $f \in C^\infty(M)$ by

$$\varphi(u) \cdot f = \frac{d}{dt} f(c(t))_{|t=0},$$

where $t \mapsto c(t)$ is a curve in M such that $c(0) = x$ and $\dot{c}(0) = \varphi(u)$. For every $g \in G$,

$$\varphi(\Theta_g u) \cdot f = T\Phi_g(\varphi(u)) \cdot f = \varphi \cdot \Phi_g^* f = \frac{d}{dt} f(\Phi_g(c(t)))_{|t=0}.$$

If f is G-invariant, then $f \circ \Phi_g = f$, and $\varphi(\Theta_g u) \cdot f = \varphi(u) \cdot f$ for every $g \in G$. In this case, $\varphi(u) \cdot f$ depends only on $\bar{u} = \beta(u) \in \bar{B}$. Since every G-invariant function on M is of the form $f = \bar{f} \circ \mu$, for a unique $\bar{f} \in C^\infty(\bar{M})$, we have a map $\bar{\varphi} : \bar{B} \to T\bar{M}$ such that

$$\bar{\varphi}(\bar{u}) \cdot \bar{f} = \beta(u) \cdot (\bar{f} \circ \mu) \tag{11}$$

for every $\bar{f} \in C^\infty(\bar{M})$, where u is any element of $\beta^{-1}(\bar{u})$.

The quadruple $(\bar{B}, \bar{M}, \bar{\pi}, \bar{\varphi})$ is a smooth nonlinear control system obtained from (B, M, π, φ) by singular reduction of symmetries. Here $\bar{B}$ and $\bar{M}$ are differential spaces and $\bar{\pi} : \bar{B} \to \bar{M}$ and $\bar{\varphi} : \bar{B} \to T\bar{M}$ are smooth maps of differential spaces, [21].

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